Preferred Robust Response Surface Design with Missing Observations Based on Integrated TOPSIS-AHP Method: An Application for a Nanolubrication Industry

Atefeh Ashuri
Industrial Engineering Department
Faculty of Engineering, Shahed University
Tehran, Iran
atefeh.Ashuri@gmail.com

Mahdi Bashiri
Industrial Engineering Department
Faculty of Engineering, Shahed University
Tehran, Iran
bashiri@shahed.ac.ir

Amirhossein Amiri
Industrial Engineering Department
Faculty of Engineering, Shahed University
Tehran, Iran
amiri@shahed.ac.ir

Abstract—Missing observations appear in experimental designs as a result of insufficient sampling, machine breakdown and high costs and errors in measurements. In many real applications in nanomanufacturing, experiments deal with a design with missing observations because the factors combination or molecular structure selected by a designer cannot be experimented simply. In a nano-scale environment, effects of design parameters on product characteristics cannot be ignored. So, selection of preferred robust response surface designs with missing observations is a critical decision for economic performance of nanomanufacturing because the design is insensitive to risk of missed values. In this paper, Box-behnken and Face-centered composite designs are studied because of their widely applications. For this purpose, eight robustness criteria including, D-efficiency, $t_{max}$, $t_{max} (1- \alpha)$ and their related sub-criteria are considered for evaluating the robustness of mentioned designs. Finally, the integrated TOPSIS-AHP methodology is applied to select the most suitable robust design. Numerical example illustrates applicability of the proposed approach.

Keywords—Robustness criteria; Risk of missed values; preferred robust response surface design; TOPSIS-AHP methodology; Nanomanufacturing

I. INTRODUCTION

Missing observation has attracted a lot of attention in the recent decades in experimental designs [1], [2], [3]. Missing observations appear in experimental designs as a result of insufficient sampling, high costs, and errors in measurements or during data acquisition [4]. Also, machine breakdown, illegible recording of response and damaged experimental resource are the common reasons, too [2]. Moreover, missing observations can be appeared in the nanomanufacturing industries while only some combinations of factors can be experimented successfully. As an example, the study of gas phase nano-scale lubrication experiments can be an application of missing observations in experimental designs, so it confirms the necessity of this study [2]. Hence, in many circumstances manufacturing industries face with this problem. To overcome with this problem, experimenters have used robust criteria to investigate the robustness of design. Some authors have investigated the robustness of design experiments where we deal with missing observations [4], [5], [6], and [7]. Ghosh investigated robustness of BIBD in the presence of missing observations [8]. MacEachern et al. studied a number of techniques for finding $t_{max}$ and applied it for evaluating the robustness of central composite designs and factorial experiments against...
The next considered criterion is $t_{\text{max}}^n$ that proposed by Ghosh [20]. Assume the following ordinary linear model:

$$y = X\beta + \epsilon$$

The next considered criterion is $t_{\text{max}}^n$ that proposed by Ghosh [20]. Assume the following ordinary linear model:
where $X$ is an $m \times n$ matrix that $n$ rows are of the form $(l, x_{i1}, \ldots, x_{ij})$ and $x_i = x_{i1}, \ldots, x_{ij}, x_{i+1}, \ldots, x_m$). For the second-order polynomial model, $p$ is defined as $p = (k+1)(k+2)/2$ that $k$ is the number of factors.

If the remaining design matrix $(n-t) \times p$ that is obtained after omitting observations is able to estimate all the parameters, we can conclude that design matrix $X$ is robust against missing observations. To check it, we apply the following Ghosh definition:

$$t_{\text{max}} = \max \left\{ t \mid t \leq t \leq n - p, \text{ and each} (n-t) \times p \text{ matrix yields } X_{it}^T X_{ij} = 0 \right\}.$$

To evaluate the robustness of design, we consider two criteria $t_{\text{max}}$ and $t_{\text{max}}/n$ where $n$ is the number of runs. The obtained results are shown in Table II.

**TABLE II. $t_{\text{max}}$ VALUES FOR BB AND FCC DESIGNS**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
<th>$t_{\text{max}}$</th>
<th>$t_{\text{max}}/n$</th>
<th>$k$</th>
<th>$n$</th>
<th>$t_{\text{max}}$</th>
<th>$t_{\text{max}}/n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>3</td>
<td>16.7</td>
<td>4</td>
<td>28</td>
<td>3</td>
<td>10.7</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>4</td>
<td>10.7</td>
<td>5</td>
<td>30</td>
<td>3</td>
<td>10.0</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>4</td>
<td>10.0</td>
<td>6</td>
<td>48</td>
<td>3</td>
<td>6.3</td>
</tr>
</tbody>
</table>

The third criterion is $t_{\text{max}}(1-\alpha)$ proposed by Tanco et al. [3]. This criterion is defined as the maximum number of observations that can be missing and we can estimate the parameters of model with a high probability. This criterion is defined as:

$$t_{\text{max}}(1-\alpha) = \max \left\{ t \mid t \leq n - p, \text{ and p (Model is not estimable) } \right\},$$

where the probability of model is not estimable computed as

$$P(\text{Model is not estimable}) = \sum_{t=1}^{n} I[X_{ij}^T X_{ij} = 0] / n,$$

where $\binom{n}{k}$ is the total number of combinations that observation can be missed and indicator $I$ counts the total number of combinations that $I[X_{ij}^T X_{ij} = 0]$ and the model is not estimable.

To compare robustness of designs using this criterion, $t_{\text{max}}(0.95)$ and $t_{\text{max}}(0.99)$ were computed which are defined as the maximum number of observations which are allowed to be missing and we can estimate the model with probability of 95% and 99%, respectively. The obtained results are summarized in the following table.

**TABLE III. $t_{\text{max}}(0.99)$ VALUES FOR BB AND FCC DESIGNS**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
<th>$t_{\text{max}(0.99)}$</th>
<th>$t_{\text{max}(0.99)/n}$</th>
<th>$k$</th>
<th>$n$</th>
<th>$t_{\text{max}(0.99)}$</th>
<th>$t_{\text{max}(0.99)/n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>3</td>
<td>16.6</td>
<td>4</td>
<td>28</td>
<td>3</td>
<td>17.8</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>5</td>
<td>13.3</td>
<td>5</td>
<td>30</td>
<td>4</td>
<td>17.8</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>4</td>
<td>13.3</td>
<td>6</td>
<td>48</td>
<td>8</td>
<td>16.6</td>
</tr>
</tbody>
</table>

**TABLE IV. $t_{\text{max}}(0.95)$ VALUES FOR BB AND FCC DESIGNS**

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
<th>$t_{\text{max}(0.95)}$</th>
<th>$t_{\text{max}(0.95)/n}$</th>
<th>$k$</th>
<th>$n$</th>
<th>$t_{\text{max}(0.95)}$</th>
<th>$t_{\text{max}(0.95)/n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>3</td>
<td>22.2</td>
<td>4</td>
<td>28</td>
<td>7</td>
<td>25.0</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
<td>5</td>
<td>16.6</td>
<td>5</td>
<td>30</td>
<td>5</td>
<td>16.6</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>5</td>
<td>16.6</td>
<td>6</td>
<td>48</td>
<td>10</td>
<td>20.8</td>
</tr>
</tbody>
</table>

To evaluate the best design by TOPSIS method, we should determine the relative importance of different robust criteria. AHP provides such a procedure. The TOPSIS-AHP method includes the following steps:

Step1. Assume a decision matrix having $n$ criteria and $m$ alternatives. The decision matrix is shown as

$$D = \begin{bmatrix}
    x_{11} & x_{12} & \ldots & \ldots & x_{1m} \\
    x_{21} & x_{22} & \ldots & \ldots & x_{2m} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    x_{m1} & x_{m2} & \ldots & x_{mm} \\
\end{bmatrix}$$

where element $x_{ij}$ indicates the performance of the $i$th alternative with respect to $j$th criterion.

Step2. Compute the normalized decision matrix. The normalized value $r_{ij}$ can be given as

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}, \quad i = 1, \ldots, m \quad j = 1, \ldots, n$$

where $x_{ij}$ is the performance of the $i$th alternative with respect to $j$th criterion.

In this section, we introduce TOPSIS method applied to distinguish more preferred robust designs. To select the best design by TOPSIS method, we should determine the relative importance of different robust criteria.
Step 3. Determine the relative importance of alternative. In the AHP method, the pair comparisons are redone and values from 1 to 9 are assigned to introduce the relative importance of the alternatives. Table V indicates the comparison scale used in the weighting of two criteria [15].

Table V. Definition of Scale Values in Pair-Wise Comparison Matrix

<table>
<thead>
<tr>
<th>Relative importance</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Equal importance</td>
</tr>
<tr>
<td>3</td>
<td>Weak importance</td>
</tr>
<tr>
<td>5</td>
<td>Strong importance</td>
</tr>
<tr>
<td>7</td>
<td>Very strong importance</td>
</tr>
<tr>
<td>9</td>
<td>Absolute importance</td>
</tr>
<tr>
<td>2, 4, 6, 8</td>
<td>Intermediate values</td>
</tr>
</tbody>
</table>

Then, the matrix of the comparison of criteria is as follows:

\[
A = \begin{bmatrix}
1 & a_{12} & \ldots & a_{1n} \\
a_{21} & 1 & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & 1
\end{bmatrix}
\]

The every element represents the relative importance of criterion \( i \) with respect to criterion \( j \). So, we conclude that \( a_{ij} = 1/a_{ji} \).

\[
W_j = \frac{\left( \prod_{k=1}^{n} a_{kj} \right)^{1/n}}{\sum_{k=1}^{n} \left( \prod_{j=1}^{n} a_{kj} \right)^{1/n}}, \quad i, j = 1, 2, \ldots, n
\]

A consistency ratio (CR) is used to determine inconsistency and expressed as

\[
CR = \frac{CI}{RI},
\]

where the consistency index (CI) can be calculated by

\[
CI = \frac{\lambda_{\text{max}} - n}{n - 1},
\]

where \( \lambda_{\text{max}} \) is the largest eigenvalue [21] and the RI is obtained based on Table VI that shows the random index used in decision making. If the value of CR is less than 0.1, we conclude that this model is validated.

Step 4: The weighted normalized matrix \( V_j \) is computed as

\[
V_j = W_j r_j
\]

where \( W_j \) is obtained from AHP method in the previous step.

Step 5. Obtain positive idealsolutions \( V^+ \) and negative ideal solutions \( V^- \) that are calculated as

\[
V^+ = \left\{ \left( \sum_{j=1}^{m} V_{ij} / \sum_{j=1}^{m} \right) / j=1,2,\ldots,m \right\} = \{v^1, v^2, \ldots, v^m\}
\]

\[
V^- = \left\{ \left( \sum_{j=1}^{m} V_{ij} / \sum_{j=1}^{m} \right) / j=1,2,\ldots,m \right\} = \{v^1, v^2, \ldots, v^m\},
\]

where \( J = (j=1,2,\ldots,n) \) is related to the beneficial criteria and \( J' = (j=1,2,\ldots,n) \) is related to the non-beneficial criteria.

Step 6. Calculate the separation distance of each alternative. The separation of each alternative from the positive ideal solution is defined as

\[
S_i^+ = \sqrt{\sum_{j=1}^{m} (V_{ij} - V^+)^2}, \quad i = 1, 2, \ldots, m
\]

Similarly, calculate the separation distance of each alternative from the negative ideal solution as follows:

\[
S_i^- = \sqrt{\sum_{j=1}^{m} (V_{ij} - V^-)^2}, \quad i = 1, 2, \ldots, m
\]

Step 7. Calculate the relative closeness to ideal solutions

\[
C_i = \frac{S_i^-}{S_i^- + S_i^+},
\]

Step 8. Rank alternatives based on the value of \( C_i \) in descending order.

IV. Evaluation of Robust Designs by the Proposed Method

We use the combined TOPSIS-AHP method for selecting the most suitable robustness design. In order to select preferred design in terms of robustness against missing observations which is very important in nanolubrication industries, we consider eight efficient criteria including average D-efficiency, worst D-efficiency, \( t_{\text{max}}, t_{\text{max}}(0.99), t_{\text{max}}(0.95), t_{\text{max}} / n, t_{\text{max}}(0.99) / n \).
and $t_{\text{max}}(0.95)/n$. These criteria are explained in details in section 2. The hierarchical structure of decision making is demonstrated in the following figure.

![Hierarchical Structure of Decision Making](image)

At first, we construct decision matrix where each element expresses the performance of considered alternative with respect to available criterion.

### TABLE VII. DECISION MATRIX OF ROBUSTNESS CRITERIA

<table>
<thead>
<tr>
<th>$k$</th>
<th>AVE-D</th>
<th>$t_{\text{max}}$</th>
<th>$t_{\text{max}}(0.99)$</th>
<th>$t_{\text{max}}(0.95)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.4440</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0.4643</td>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>0.3000</td>
<td>4</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>0.4167</td>
<td>8</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Then, the normalization of decision matrix is obtained based on Equation (7) and expressed as:

### TABLE VIII. NORMALIZATION OF DECISION MATRIX

<table>
<thead>
<tr>
<th>$k$</th>
<th>AVE-D</th>
<th>$t_{\text{max}}$</th>
<th>$t_{\text{max}}(0.99)$</th>
<th>$t_{\text{max}}(0.95)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.3602</td>
<td>0.2122</td>
<td>0.375</td>
<td>0.6031</td>
</tr>
<tr>
<td>4</td>
<td>0.3767</td>
<td>0.3514</td>
<td>0.375</td>
<td>0.3864</td>
</tr>
<tr>
<td>5</td>
<td>0.2434</td>
<td>0.0363</td>
<td>0.375</td>
<td>0.3611</td>
</tr>
<tr>
<td>6</td>
<td>0.3381</td>
<td>0.3333</td>
<td>0.125</td>
<td>0.2275</td>
</tr>
</tbody>
</table>

The pair-wise comparison matrix is demonstrated in Table IV to show the relative importance of each criterion.

### TABLE IX. PAIR-WISE COMPARISON MATRIX

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/2</td>
<td>1/3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2/3</td>
<td>1</td>
<td>10</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>3/2</td>
<td>1/5</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

To obtain weights of the robust criteria, we use Equation (9) in the previous section. The final results are shown in the following table:

### TABLE X. DESIGN SELECTION

<table>
<thead>
<tr>
<th>Design</th>
<th>Average $d$-efficiency</th>
<th>Worst $d$-efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.3602</td>
<td>0.6031</td>
</tr>
<tr>
<td>#2</td>
<td>0.3767</td>
<td>0.3864</td>
</tr>
<tr>
<td>#3</td>
<td>0.2434</td>
<td>0.3611</td>
</tr>
</tbody>
</table>
The largest eigenvalue of the pairwise comparison matrix is equal to 8. The value of random consistency index is obtained from table and it is equal to 1.48. Inconsistency ratio is computed near to zero. Because the value of RC is less than 0.1, we can conclude comparisons are consistent.

In the next step, the weighted normalized matrix is calculated and summarized in Table XI.

Table XI. Normalized Weighted Matrix

<table>
<thead>
<tr>
<th>k</th>
<th>AVE-D</th>
<th>t_{max}</th>
<th>t_{ave}(0.99)</th>
<th>t_{ave}(0.95)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.0435</td>
<td>0.0256</td>
<td>0.0905</td>
<td>0.2184</td>
</tr>
<tr>
<td>4</td>
<td>0.0455</td>
<td>0.0424</td>
<td>0.0905</td>
<td>0.1399</td>
</tr>
<tr>
<td>5</td>
<td>0.0294</td>
<td>0.0044</td>
<td>0.0905</td>
<td>0.1308</td>
</tr>
<tr>
<td>6</td>
<td>0.0408</td>
<td>0.0402</td>
<td>0.0905</td>
<td>0.0824</td>
</tr>
<tr>
<td>3</td>
<td>0.0367</td>
<td>0.0311</td>
<td>0.0302</td>
<td>0.0824</td>
</tr>
<tr>
<td>4</td>
<td>0.0455</td>
<td>0.0518</td>
<td>0.0905</td>
<td>0.1399</td>
</tr>
<tr>
<td>5</td>
<td>0.0512</td>
<td>0.0622</td>
<td>0.0905</td>
<td>0.0888</td>
</tr>
<tr>
<td>6</td>
<td>0.0452</td>
<td>0.0544</td>
<td>0.0905</td>
<td>0.0754</td>
</tr>
</tbody>
</table>

The relative closeness to ideal solution is calculated as follows:

Table XIV. Relative Closeness to Ideal Solution

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>0.5387</td>
<td>0.5290</td>
<td>0.5393</td>
<td>0.5209</td>
</tr>
<tr>
<td>n</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>C</td>
<td>0.4488</td>
<td>0.4715</td>
<td>0.9049</td>
<td>0.9413</td>
</tr>
</tbody>
</table>

Based on the relative closeness values, the nanolubrican processes can select the robustness design for experimental design with the following priorities:

\[ C_8 > C_7 > C_3 > C_1 > C_2 > C_4 > C_6 > C_5 \]
V. CONCLUSION AND FUTURE RESEARCH

As mentioned before missing observations in experimentations can be appeared in many industrial applications such as nanomanufacturing. Hence, the selection of a preferred design with missing observations in nanomanufacturing leads to the effective economic decision. We considered an integrated TPOSIS-AHP methodology to select robust design. We considered Box-behnken and Face-centered composite designs due to their several applications. To evaluate robustness of the Box-behnken and Face-centered composite designs, eight efficient robustness criteria were considered in the proposed decision making approach. The results show that the preferred design can be selected by the proposed approach. The analysis confirmed that the Face-centered composite design with the factor number of \( k=6 \) can be selected as preferred robust design with missing observations. Proposing the other robust criteria for evaluating the other experimental designs in the presence of missing data is proposed as a future research.

REFERENCES


