Change point estimation of gamma regression profiles with a linear trend disturbance

Fatemeh Sogandi and Amirhossein Amiri*

Industrial Engineering Department,
Faculty of Engineering,
Shahed University,
Khalije Fars high way, Tehran
P.O. Box 18151-159, Iran
Email: f.sogandi@shahed.ac.ir
Email: amiri@shahed.ac.ir
*Corresponding author

Abstract: Estimating the real time of shifts in process would ease the identification of special causes and leads to saving in time and cost, so it is very important for process improvement. On the other hand, in many applications of statistical process control, quality of a process or product is summarised by a functional relationship referred to as profile. Although there are many real cases with gamma responses in profiles, little studies have been devoted to the estimation of the change point of this type of profiles. The main contribution of the paper is deriving a maximum likelihood estimator for change point estimation in gamma regression profiles with a linear trend disturbance. Also, a T2 control chart is used to detect the out-of-control signal in monitoring gamma regression profiles. The performance of the proposed estimator is evaluated through simulation studies. The results show the accurate and precise estimates of the change point by the proposed estimator. Afterwards, a confidence set estimator for the process change point based on the logarithm of the likelihood function is presented. Finally, the performance of the estimator is illustrated through a numerical example.

Keywords: change point estimation; maximum likelihood estimator; MLE; gamma regression profiles; phase 2; linear trend disturbance.

Reference to this paper should be made as follows: Sogandi, F. and Amiri, A. (2014) 'Change point estimation of gamma regression profiles with a linear trend disturbance', *Int. J. Quality Engineering and Technology*, Vol. 4, No. 4, pp.352–368.

Biographical notes: Fatemeh Sogandi is an MSc student of Industrial Engineering at Shahed University. Her research interests are statistical process control and design of experiments.

Amirhossein Amiri is an Associate Professor at Shahed University in Iran. He received his BS, MS, and PhD in Industrial Engineering from Khajeh Nasir University of Technology, Iran University of Science and Technology, and Tarbiat Modares University in Iran, respectively. He is now the Head of the Industrial Engineering Department at Shahed University in Iran and a member of the Iranian Statistical Association. His research interests are statistical quality control, profile monitoring, and Six Sigma. He has published many papers in the area of statistical process control in high quality international journals such as *Quality and Reliability Engineering International*, *Communications in Statistics*, *Computers and Industrial Engineering*, *Journal*

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of Statistical Computation and Simulation, *Soft Computing* and so on. He has also published a book with John Wiley and Sons in 2011 entitled *Statistical Analysis of Profile Monitoring*.

1 Introduction

Control charts are used to trigger the out-of-control signal when disturbances occur in the process. The corresponding disturbances should be fast recognised and rectified by implementation of corrective actions. Change point analysis is the process of effective identification of distributional shifts within time ordered observations. Hence, process improvement can be successfully achieved by quickly and accurately detecting special causes as early as possible. Samuel et al. (1998a) pioneered in suggestion of a maximum likelihood estimation (MLE) of the change point for normal process mean in which the type of the disturbance was step change. In another work, Samuel et al. (1998b) proposed a MLE to determine the time of a step change in the variance of normal processes. Ghazanfari et al. (2013) proposed a clustering approach to estimate the step change point in Shewhart type control charts.

Several researches have been done on the change point estimation of a linear drift. Perry and Pignatiello (2006) derived a MLE for the linear change in the mean of a normal process and compared the performance of their estimator with the MLE designed for step changes. They showed the MLE of the process change point designed for linear trends outperforms the MLE of the process change point designed for step changes when the real change type is a linear trend. Also, Perry et al. (2006) developed a MLE for linear trend disturbances to estimate the process change point in the Poisson rate parameter. Fahmy and Elsayed (2006) derived a MLE of the change point in normal process mean resulting from a linear trend disturbance and compared its performance with cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) change point estimators proposed by Page (1954) and Nishina (1992) respectively. Perry (2010) evaluated a MLE for the time of polynomial drift in the mean of autocorrelated processes. Noorossana and Heydari (2009) proposed a MLE of the process change point with considering a linear trend disturbance in the normal process variance. Moreover, Atashgar and Noorossana (2011) proposed a neural network-based change point estimator in order to identify the change point in a bivariate normal process considering a linear trend in the mean vector with constant covariance matrix. Afterwards, Zandi et al. (2011) applied a MLE for change point estimation of the process fraction non-conforming when a linear trend disturbance is present. Amiri and Khosravi (2012) and Niaki and Khedmati (2013) derived a MLE of the change point in a high-yield process when a linear trend disturbance occurs in the process fraction nonconforming.

On the other hand, there are many real cases that the relationship between a response variable and explanatory variable(s), which is named as profile, should be monitored over time. Profiles are usually classified according to the type of their relationship to simple linear profiles, polynomial profiles; generalised linear models (GLM) based profiles and so on. Although in recent years several studies have been devoted to the estimation of the change point of the process shifts for univariate and multivariate processes, few researches have been done on change point estimation in monitoring profiles. Most of the studies on change point estimation in the area of profile monitoring concerned with

normality assumption of the response variable. Zou et al. (2006) proposed a standardised likelihood ratio test (LRT) approach to estimate the step change point in monitoring simple linear profiles. Also, Zou et al. (2007) used LRT approach to estimate the real time of a step shift in phase 2 monitoring of general linear profiles. Mahmoud et al. (2007) proposed a change point method for linear profile data in phase 1. Eyvazian et al. (2011) considered the LRT method for step change point estimation in multivariate multiple linear regression profiles in phase 2. Recently, Kazemzadeh et al. (2014) proposed a change point estimator in multivariate linear profiles under linear drift.

All of the above mentioned researches have concentrated on the change point estimation in profiles when the response variable follows normal distribution. However, in real word this assumption is often violated. In this case, usually the GLMs are used to model the profiles. The GLMs are including the family of exponential distributions including binomial, Poisson gamma distributions and so on. Zand et al. (2012) proposed a change point estimator in phase 1 monitoring of logistic regression profile. Sharafi et al. (2012a, 2013a) provided a MLE to identify the real time of a step and linear trend change in phase 2 monitoring of binary profiles. Sharafi et al. (2012b, 2013b) proposed a MLE approach for estimating the time of drift and step changes in Poisson regression profiles. Recently, Sogandi and Amiri (2014) proposed a step change point estimator in gamma regression profiles.

To the best of authors' knowledge, there is no research on the change point estimation of gamma regression profiles under a linear trend disturbance. As a matter of fact, non-normal processes commonly exist in industries. "Among those non-normal applications, the gamma process is one of the most important cases in industry" (Ramanayake, 2004). There are many real cases in which the gamma regression profiles applied for example when each observation represents the time interval between the occurrences of defects, the observations in each level of the explanatory variable (respective operational condition) follows gamma regression profile. Also, the linear trend can be occurred in the parameters of the gamma regression profile due to several reasons such as tool wear. Hence, in this paper, we propose a MLE method to estimate the change point in the parameters of gamma regression profiles under a linear trend disturbance.

The structure of the paper is as follows: Section 2 explains the gamma regression model. In Section 3, the change point model and estimation procedure of linear trend slope is presented. Afterwards, the performance of the proposed estimator is evaluated in Section 4 through simulation studies. In Section 5, a confidence set is defined and the set cardinality and coverage percentage criteria are computed to evaluate the performance of the confidence set estimator. The performance of the proposed estimator is illustrated through an illustrative example in Section 6. Conclusions and some recommendations for future researches are provided in the final section.

2 Gamma regression model

Gamma distribution has been frequently used in many statistical applications especially in reliability, survival analysis, income distribution modelling, queuing models, semiconductor manufacturing and so on. In reality, it has been reported that the gamma processes cover a wide class of applications (Yuehjen and Hou, 2011). Gamma

distribution is characterised by two parameters: shape parameter m and scale parameter λ. The probability density function of gamma distribution is:

$$y = \frac{\lambda^m}{\Gamma(m)} y^{m-1} e^{-\lambda y}. \tag{1}$$

Gamma distribution belongs to a larger class of distributions called the exponential family. Other distributions belonging to the exponential family are the normal, Poisson, exponential, and binomial distributions. A gamma quality characteristic can be related to an explanatory variable and forms a gamma regression profile. For example, waiting time between occurrences of defects is a gamma quality characteristic that can be related to an explanatory variable such as type of tools or materials.

The gamma regression profile can be modelled by a GLM. There are three components that comprise GLM: a random component, the random component is the outcome (Y) and follows a distribution from the exponential family. A systematic component which needs the x's to be combined in the model as a linear function and the link function that relates the mean of the response variable to linear combination of explanatory variables. Let assume there are p predictor variables for any of n independent experimental sets, which are shown by $\mathbf{x}_j = (x_{j1}, x_{j2},\ldots,x_{jp})^T$ in which $j = 1, 2, \ldots, n$. Assume y_j as a gamma distribution with parameters (m_j, λ_j). Then, the gamma regression profile is modelled by relationship between mean of a gamma random variable $\left(\frac{m_j}{\lambda_j}\right)$ and p predictor variables through a log link function. "Often, non-canonical links such as the identity link function, $m = \mathbf{x}\boldsymbol{\beta}$ and the log link function, $\log(m) = \mathbf{x}\boldsymbol{\beta}$, are used with gamma distributed data (McCullagh and Nelder, 1989). The identity link requires restrictions on $\boldsymbol{\beta}$; but the log link does not. The log link is probably the most commonly used for gamma regression" (Christensen, 1990). So the log link function is used in this paper and is as follows:

$$\log\left(\frac{m_j}{\lambda_j}\right) = \beta_1 x_{j1} + \beta_2 x_{j2} + \ldots + \beta_p x_{jp}, \tag{2}$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2,\ldots,\beta_p)^T$ is the regression parameter vector. We consider $x_{j1} \equiv 1$ for β_1 as the intercept of the model. The alternative equation which directly specify λ_j is as follows:

$$\lambda_j = \frac{m_j}{\exp\left(\mathbf{x}_j^T \boldsymbol{\beta}\right)}, \tag{3}$$

In the field, Albert and Anderson (1984) used the following likelihood function to approximate the model parameters:

$$l(\mathbf{y}, \boldsymbol{\lambda}) = \prod_{j=1}^{n} \frac{\lambda_j^{m_j}}{\Gamma\left(m_j\right)} y_j^{m_j - 1} e^{-\lambda_j y_j}. \tag{4}$$

On the other hand, equation (5) can be concluded from equation (3):

$$\boldsymbol{\lambda} = \left(\lambda_1, \lambda_2, \ldots, \lambda_n\right) = \left(\frac{m_1}{\exp\left(\mathbf{x}_1^T \boldsymbol{\beta}\right)}, \frac{m_2}{\exp\left(\mathbf{x}_2^T \boldsymbol{\beta}\right)}, \ldots, \frac{m_n}{\exp\left(\mathbf{x}_n^T \boldsymbol{\beta}\right)}\right)^T, \tag{5}$$

and also:

$$\mathbf{y} = \left(y_1, y_2, \ldots, y_n\right). \tag{6}$$

3 Maximum likelihood change point estimator

The shape parameter is a critical parameter in type of distribution. Hence, change in the shape parameter leads to change in type of distribution. Notice that the family of gamma distribution includes the chi-square, exponential and Erlang distributions. The gamma distribution is an Erlang distribution when m is a positive integer, and becomes an exponential distribution with parameter λ and $m = 1$. Also, when $\lambda = 1/2$, it is a chi-square distribution with the degrees of freedom equals to $2m$. Hence, "typical change point problems considered a change in the scale parameter while assuming that the shape parameter is unchanged" (Hsu, 1979). Based on the aforementioned explanations, we considered a constant shape parameter and supposed that the change only occurs in the scale parameter.

The mass probability function of the gamma observations in the gamma regression profile when the process is in-control is as follows:

$$f\left(y_{ij}\right) = \frac{e^{-\lambda_j y_{ij}} \left(\lambda_j\right)^{m_j} y_{ij}^{m_j - 1}}{(m_j - 1)!}, \tag{7}$$

where y_{ij} is the value taken by the response variable for the jth value of the predictor variable in the ith profile. We consider a linear trend disturbance in the scale parameter for gamma regression profile in phase 2. After an indefinite amount of time elapses, in an unknown profile in τ that named as the process change point, the scale parameter changes to an unknown out-of-control state as $\boldsymbol{\lambda}' = \boldsymbol{\lambda} + \beta(i - \tau)$ for profiles $i = \tau + 1, \tau + 2, \ldots, T$ where $\boldsymbol{\lambda}' > \boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}$ are n-dimensional out-of-control and in-control scale vectors and β is the magnitude or slope of the linear trend disturbance. Hence, in the likelihood function for $i = 1, 2, \ldots, \tau$, the scale parameter of process is equal to its known in-control value of $\boldsymbol{\lambda}$. While for profiles $i = \tau + 1, \tau + 2, \ldots, T$, it becomes equal to some unknown parameter $\boldsymbol{\lambda}'$ where T is the last profile sampled. There are two unknown parameters in the model including τ and β, which represent the last profile taken from an in-control process and magnitude of the linear trend disturbance, respectively. Based on the aforementioned explanations, the likelihood function for gamma regression profile is given by:

$$l(\tau, \mathbf{y}_{ij}) = \frac{\prod_{i=1}^{\tau}\prod_{j=1}^{n}(\lambda_j)^{m_j}\prod_{i=1}^{\tau}\prod_{j=1}^{n} y_{ij}^{m_j-1} e^{-\sum_{i=1}^{\tau}\sum_{j=1}^{n}\lambda_j y_{ij}}}{\prod_{i=1}^{\tau}\prod_{j=1}^{n}(m_j-1)!} \times \frac{\prod_{i=\tau+1}^{T}\prod_{j=1}^{n}(\lambda'_j)^{m_j}\prod_{i=\tau+1}^{T}\prod_{j=1}^{n} y_{ij}^{m_j-1} e^{-\sum_{i=\tau+1}^{T}\sum_{j=1}^{n}\lambda'_j y_{ij}}}{\prod_{i=\tau+1}^{T}\prod_{j=1}^{n}(m_j-1)!}. \tag{8}$$

The MLE of τ is the value of τ that maximises the likelihood function in equation (8) or, equivalently, its logarithm. So it is better to take the natural logarithm of the likelihood function which is shown in equation (9).

$$\begin{aligned}\ln l(\tau, y_{ij}) = &\sum_{i=1}^{\tau}\sum_{j=1}^{n} m_j \ln(\lambda_j) - \sum_{i=1}^{\tau}\sum_{j=1}^{n}\ln((m_j-1)!) + \sum_{i=1}^{\tau}\sum_{j=1}^{n}(m_j-1)\ln(y_{ij}) \\ &-\sum_{i=1}^{\tau}\sum_{j=1}^{n} y_{ij}\lambda_j + \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}(m_j-1)\ln(y_{ij}) + \sum_{i=\tau+1}^{T}\sum_{j=1}^{n} m_j \ln(\lambda'_j) \\ &-\sum_{i=\tau+1}^{T}\sum_{j=1}^{n} y_{ij}\lambda'_j - \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}\ln((m_j-1)!).\end{aligned} \tag{9}$$

Then by using functional form of $\lambda'_j = \lambda_j + \beta(i - \tau)$ for $j = 1, 2,\ldots,n$, simplifying them and integrating known terms in constant term as C, equation (9) changes to equation (10):

$$\ln l(\tau, \beta \mid y) = \sum_{j=1}^{n}\sum_{i=\tau+1}^{T} m_j \ln(\lambda_j + \beta(i-\tau)) - \sum_{j=1}^{n}\sum_{i=\tau+1}^{T}(\lambda_j + \beta(i-\tau))y_{ij} + C. \tag{10}$$

The aim is to find the values of β and τ that maximise equation (10). To determine unknown parameters of the likelihood function logarithm, we should take the partial derivative from equation (10) with respect to β and τ. The partial derivative of equation (10) with respect to β is given by:

$$\frac{\partial \ln l(\tau, \beta \mid y)}{\partial \beta} = \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}(i-\tau)y_{ij} + \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}\frac{m_j(i-\tau)}{\lambda + \beta(i-\tau)}. \tag{11}$$

As seen in equation (11), there is no closed-form solution for β. To provide an estimate of β for each τ without need to an explicit closed-form expression, we applied Newton's method to solve it at each potential change point value (Polyak, 2007). Hence, the slope parameter β can be estimated iteratively for any potential value of τ, in which the $\hat{\beta}_\tau$ at $k + 1$ iteration can be obtained by:

$$\hat{\beta}_{\tau,k+1} = \hat{\beta}_{\tau,k} - \frac{-\sum_{i=\tau+1}^{T}\sum_{j=1}^{n}(i-\tau)y_{ij} + \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}\frac{m_j(i-\tau)}{\lambda + \hat{\beta}_{\tau,k}(i-\tau)}}{\sum_{j=1}^{n}\sum_{i=\tau+1}^{T}\frac{-m_j(i-\tau)^2}{\left(\lambda_j + \hat{\beta}_{\tau,k}(i-\tau)\right)^2}}, \tag{12}$$

where $\hat{\beta}_{\tau,0} = 0$ and the convergence threshold in the Newton method to estimate $\hat{\beta}_{\tau,k+1}$ is considered very small. It should be noted that since the scale parameter is always greater than zero, the condition $\hat{\beta}_{\tau} < \frac{\lambda_j' - \lambda_j}{(i-\tau)}$, for j = 1, 2,…, n should be satisfied. As a result, change point estimator that is denoted as $\hat{\tau}$ can be obtained as follows:

$$\begin{aligned}\hat{\tau} = \arg\max_{1\le t\le T-1}\Bigg\{&\sum_{i=1}^{\tau}\sum_{j=1}^{n} m_j \ln\left(\lambda_j\right) + \sum_{i=1}^{\tau}\sum_{j=1}^{n}\left(m_j - 1\right)\ln\left(y_{ij}\right) - \sum_{i=1}^{\tau}\sum_{j=1}^{n} y_{ij}\lambda_j \\ &+ \sum_{j=1}^{n}\sum_{i=\tau+1}^{T} m_j \ln\left(\lambda_j + \hat{\beta}(\iota-\tau)\right) + \sum_{i=\tau+1}^{T}\sum_{j=1}^{n}(m_j - 1)\ln\left(y_{ij}\right) \\ &- \sum_{j=1}^{n}\sum_{i=\tau+1}^{T}\left(\lambda_j + \hat{\beta}_{\tau}(i-\tau)\right)y_{ij}\Bigg\}.\end{aligned} \tag{13}$$

Yeh et al. (2009) introduced five Hotelling T^2 control charts to monitor binary profiles in phase 1 that any of these T^2 control charts uses a different method to estimate the mean vector and covariance matrix. They showed that the performance of T^2 control chart which estimates the covariance matrix by averaging the covariance estimates of each given sample is better than the other methods in detecting both step and drift shifts in the parameters. So we use this T^2 control chart to monitor a gamma regression profile and estimate change point in phase 2. This control chart is applied in phase 2 with the assumption that the mean vector and covariance matrix are known. The T^2 statistic for profile i^{th} (i = 1, 2,…, T) in phase 2 is defined as:

$$T_i^2 = \left(\hat{\boldsymbol{\beta}}_i - \hat{\boldsymbol{\beta}}_0\right)^{\mathrm{T}} \sum^{-1}\left(\hat{\boldsymbol{\beta}}_i - \hat{\boldsymbol{\beta}}_0\right), \tag{14}$$

where $\boldsymbol{\beta}_0$ and $\sum$ are the mean vector and covariance matrix of gamma regression parameters, respectively and $\sum$ is defined by equation (15). Also, when the process is in-control, the upper control limit for the proposed control chart is equal to $\chi^2_{2,\alpha}$ which is the percentile of the chi-square distribution with two degrees of freedom and $\mathbf{W}$ = diag(var(γ_{i1}), var(γ_{i2}),…,var(γ_{in})) is a $p \times p$ diagonal matrix.

$$\sum = \left(\mathbf{X}^{\mathbf{T}}\mathbf{W}\mathbf{X}\right)^{-1}. \tag{15}$$

When the T^2 control chart is applied to monitor a process, as long as the plotted points fall below the upper control limit, the process is assumed to be in-control. However, when a point exceeds the upper control limit, the control chart signals a change in the scale parameter of the process, and the process is assumed to be out-of-control. In these situations, the most important problem is that there is usually a considerable time lag

between the alarm time and the real time at which the change has happened. Thus, whenever the T^2 control chart signals an out-of-control alarm, the real time of a change can be estimated using equation (13).

4 Performance of the MLE estimator

In this section, we use the Monte Carlo simulation to make performance comparisons between the estimators derived for step change and linear trend through an example. We assume that number of explanatory variables is equal to 2 for easy representation of formulation, so the link function is simplified as:

$$\log\left(\frac{m_j}{\lambda_j}\right) = \beta_1 x_{j1} + \beta_2 x_{j2}, \tag{16}$$

where $j = 1, 2,\ldots,9$ ($n = 9$) and we set the design matrix $\mathbf{X}$:

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & & 1 & 1 \\ \log(10) & \log(15) & \cdots & \log(45) & \log(50) \end{bmatrix}.$$

We assume that the in-control parameters of gamma regression profile is estimated using a historical dataset in phase 1 and are equal to $(-4, 2)^T$. Moreover, assume α is equal to 0.005, so the upper control limit for the control chart is equal to $\chi^2_{2,0.005} = 10.59$ and m_j = 30 for all j = 1,…,9 is considered. The inverse of covariance matrix of the gamma regression parameters computed by using equation (15) is as follows:

$$\sum^{-1} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}^{-1} = \begin{pmatrix} 2.0472 & -0.5433 \\ -0.5433 & 0.1446 \end{pmatrix}.$$

Now, assume a linear trend occurs in the parameters of gamma regression profile and the vector of scale parameter changes from $\boldsymbol{\lambda}$ to $\boldsymbol{\lambda}' = \boldsymbol{\lambda} + \beta(i - \tau)$ for profiles $i = \tau + 1, \tau + 2,\ldots,T$ where $\boldsymbol{\lambda}' > \boldsymbol{\lambda}$. Moreover, the σ_1 and σ_2 obtained from equation (15) are equal to 1.4308 and 0.38026, respectively and the convergence parameter in the Newton method to estimate β is considered equal to $\varepsilon = 0.0005$. The change point is simulated at profile 50. The independent observations is produced by gamma distribution with scale parameter $\boldsymbol{\lambda}$ and from profile 51th, observations are simulated from the out-of-control process with $\boldsymbol{\lambda}'$ until the T^2 control chart alarms for occurrence of an assignable cause. At this time, the change point estimator in equation (13) is used and the real time of the process change is determined. This procedure is repeated 10,000 times under different linear trend disturbances.

The mean and the corresponding standard error of change point estimator under different linear shifts in the scale parameter of the gamma regression profile or slope of the linear trend disturbance are summarised in Table 1. In each simulation run, E(T) is the expected value of the number of samples taken until the first alarm happens so E(T)=ARL + 50. This table show E(T), for given shifts. When estimate of τ is perfect that value of $\bar{\hat{\tau}}$ will be 50, indicating the last observation before the change occurs in the process. For example, we conclude from the results of simulation for the shift equal to 0.03, the expected number of samples taken until the signal is E(T) = 57.98. For this case,

the average of the change point estimates is 50.37, which is close to the actual change point of $\tau = 50$ as shown in Table 1. Moreover, the standard error of the change point estimator is 0.04.

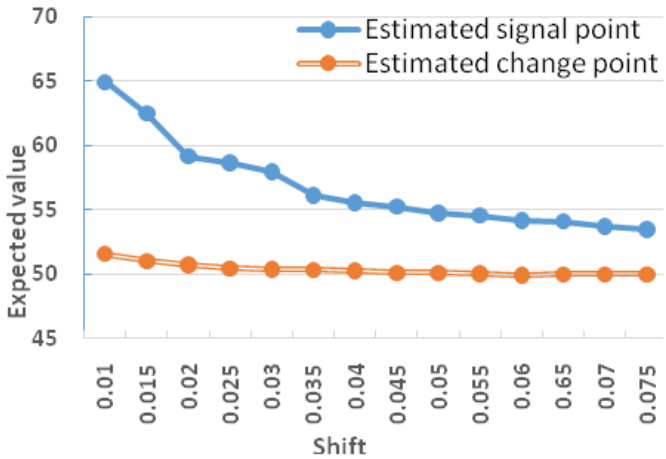
Hence, the proposed change point estimator performs satisfactory under all slopes of the linear trend disturbance even in small slopes. Furthermore, as the magnitude of the slope increases, the performance of the proposed estimator improves significantly. In other words, the proposed method works well and provides adequately accurate and reliable estimates of the real change point. Also, the performance of the proposed change point estimator designed for linear trend disturbances is compared to the change point estimator proposed in the presence of a step change.

Table 1 The averages and standard errors of the change point estimators over range of β values with 10,000 simulations runs when $p = 2$ and $\tau = 50$

β	$E(T)$	$\bar{\hat{\tau}}_{lt}$	$se(\bar{\hat{\tau}}_{lt})$	$\bar{\hat{\tau}}_{sc}$	$se(\bar{\hat{\tau}}_{sc})$
0.01	64.94	51.57	0.061	51.93	0.073
0.015	62.48	51.04	0.052	51.78	0.064
0.02	59.16	50.73	0.044	51.35	0.056
0.025	58.63	50.45	0.041	50.88	0.049
0.03	57.98	50.37	0.037	50.76	0.041
0.035	56.11	50.32	0.034	50.45	0.04
0.04	55.56	50.24	0.032	50.27	0.038
0.045	55.25	50.1	0.032	50.24	0.036
0.05	54.75	50.09	0.031	50.22	0.034
0.055	54.55	50.01	0.028	50.2	0.031
0.06	54.16	49.91	0.025	50.18	0.028
0.65	54.08	49.99	0.023	50.1	0.027
0.07	53.72	50	0.017	50.08	0.022
0.075	53.5	50	0.015	50.05	0.024

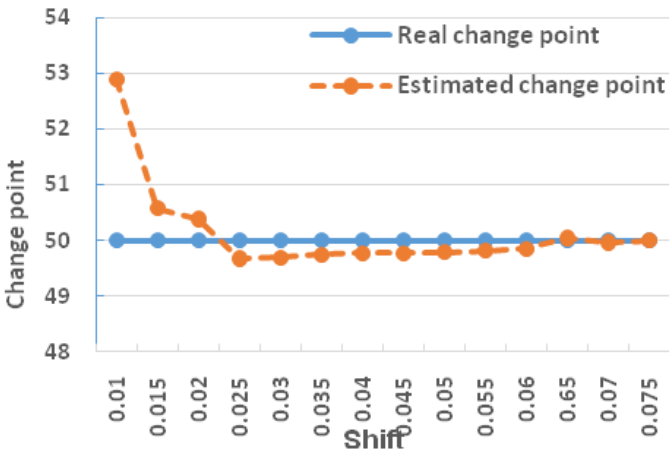
The linear trend and step change point estimators are denoted by subscripts *lt* and *sc*, respectively. As shown in Table 1, the proposed linear trend estimator outperforms the step change point estimator under all β values considered in the paper. With comparison the performance of the two change point estimators in terms of mean square error (MSE), one can conclude that the proposed estimator $\hat{\tau}_{lt}$ performs better than the $\hat{\tau}_{sc}$ under linear trend shifts. Because the MSE of $\hat{\tau}_{lt}$ is smaller than the MSE of $\hat{\tau}_{sc}$ under different linear trends.

Figure 1 Expected time of a signal with T^2 control chart and average of change point estimates $E(\hat{\tau})$ (see online version for colours)



In order to illustrate the benefit of the proposed change point estimator, $\mathrm{E}(\hat{\tau})$ is compared to E(T) in Figure 1. It can be easily seen from Figure 1 that if one only relies on E(T) and searches for the special cause around it, most probably, will not be able to find the assignable cause. However, the change point estimator $\hat{\tau}_{lt}$, on average, directs one accurately to the actual change point and enables one to find the assignable cause more effectively.

Figure 2 The performance of the MLE estimator under different shifts slopes of the linear trend disturbance (see online version for colours)



In addition, underestimation and or overestimation of the exact change point by the proposed linear trend change point estimator is negligible. Figure 2 shows a perfect convergence of the estimated change point to the exact change point over a range of slope of the linear trend disturbance except in very small β values. Moreover, as the magnitude of the slope of the linear trend disturbance increases, the performance of the linear trend change point estimator improves significantly. Hence, it can be concluded that the performance of the proposed estimator is sufficient under different shifts.

Table 2 Estimated precision performances over range of β values for $\hat{\tau}_{lt}$ and $\hat{\tau}_{sc}$ (shown in parenthesis) with 10,000 simulations runs when $p = 2$ and $\tau = 50$

β	*Pr 0*	*Pr 1*	*Pr2*	*Pr 3*	*Pr 4*	*Pr 5*	*Pr 6*	*Pr 7*
0.01	0.09	0.24	0.37	0.48	0.58	0.67	0.79	0.85
	(0.01)	(0.1)	(0.12)	(0.25)	(0.34)	(0.42)	(0.57)	(0.64)
0.015	0.11	0.31	0.49	0.57	0.69	0.74	0.88	0.97
	(0.02)	(0.15)	(0.19)	(0.28)	(0.39)	(0.44)	(0.53)	(0.62)
0.02	0.14	0.36	0.54	0.69	0.8	0.88	0.96	1
	(0.04)	(0.18)	(0.29)	(0.37)	(0.43)	(0.56)	(0.66)	(0.71)
0.025	0.18	0.4	0.61	0.75	0.86	0.95	0.98	1
	(0.06)	(0.2)	(0.38)	(0.46)	(0.55)	(0.61)	(0.72)	(0.79)
0.03	0.2	0.46	0.67	0.82	0.91	0.96	0.99	1
	(0.09)	(0.27)	(0.44)	(0.58)	(0.64)	(0.76)	(0.81)	(0.85)
0.035	0.21	0.52	0.74	0.87	0.94	0.97	1	
	(0.11)	(0.32)	(0.49)	(0.6)	(0.82)	(0.89)	(0.91)	(0.91)
0.04	0.23	0.55	0.78	0.9	0.95	0.98	1	
	(0.19)	(0.39)	(0.53)	(0.53)	(0.88)	(0.92)	(0.96)	(0.99)
0.045	0.25	0.61	0.8	0.91	0.96	0.99	1	
	(0.23)	(0.45)	(0.59)	(0.69)	(0.89)	(0.92)	(0.97)	(0.99)
0.05	0.27	0.63	0.84	0.93	0.97	1		
	(0.24)	(0.5)	(0.61)	(0.75)	(0.9)	(0.93)	(0.98)	(0.99)
0.055	0.3	0.67	0.86	0.95	0.98	1		
	(0.29)	(0.51)	(0.68)	(0.8)	(0.89)	(0.95)	(0.97)	(1)
0.06	0.33	0.7	0.89	0.96	0.98	1		
	(0.3)	(0.62)	(0.71)	(0.92)	(0.93)	(1)		
0.65	0.39	0.75	0.94	0.98	1			
	(0.34)	(0.69)	(0.72)	(0.92)	(0.96)	(1)		
0.07	0.45	0.79	0.94	0.97	1			
	(0.41)	(0.7)	(0.8)	(0.94)	(1)			
0.075	0.61	0.82	0.96	1				
	(0.57)	(0.78)	(0.9)	(0.97)	(1)			

The results of 10,000 simulation runs in Table 2 demonstrate that the estimates by the proposed linear trend and step change point estimators (shown in parenthesis) lie within a specified tolerance of the real change point value under a linear drift. Assume that

precision i under the given shifts are percent of results which distance of the estimated change point from the exact change point is i or less than i that called as $Pr\ i$. In other words, a measure of the precision of the change point estimator can be examined by constructing a frequency distribution for $P(|\hat{\tau}-\tau| \leqslant i)$ where i is equal to 0, 1,...,7. Hence, results demonstrate that the estimates lie within a specified tolerance. For example, under the slope of the linear trend equal to 0.03, in 67% of situations distance of change point estimator from the exact change point is less than 2. Moreover, in this case in 20% of the simulation runs, the estimator correctly identifies the real time of the change. Also, the percentage of those simulation trials identifying the change point correctly are 9%, 11%, 14%, 18%, 20%, 21%, 23%, 25%, 27%, 30%, 33%, 39%, 45%, 61% and 91% for the slopes equal to 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05, 0.055, 0.06, 0.065, 0.07 and 0.075, respectively. Clearly, the probability of exact estimation by increasing magnitude of the shift in slope of linear trend disturbance increases. Hence, as the magnitude of the linear trend disturbance increases, the performance of the estimator improves significantly and the estimated probabilities get closer to 1. The results show the suitable precision of the proposed estimator in estimating the real time of the linear trend disturbance in gamma regression profile. Thus, it can be concluded that the performance of the change point estimator is sufficient under different shifts. Afterwards, similar Monte Carlo simulations are conducted to estimate precision performances for step change point estimator under the same linear trend disturbances and the results are shown in parenthesis. The results of Table 2 indicate that the precision of $\hat{\tau}_{lt}$ is better than $\hat{\tau}_{sc}$ since all of the estimated probabilities for the proposed change point estimator are greater than the corresponding values for $\hat{\tau}_{sc}$.

5 Cardinality and coverage percentage of confidence set estimator

We consider constructing confidence set on the process change point. Such a set would provide a window of possible change points that covers the true process change point with a given level of confidence and enhances the identification chance of special cause. In reality confidence set for assumed D and β values is a window consisting of possible change points whose log likelihoods fall within the log-likelihood of the estimated change point (τ) minus assumed reference value, D. The number of included points in the confidence set is referred to as cardinality. The probability of the coverage is also estimated as a fraction of the cardinality of the confidence set to the number of samples taken until seeing an out-of-control signal (T). Box and Cox (1964) suggested constructing confidence regions on parameter estimates using the likelihood function by a confidence set of the form:

$$CS = \{t : \log_e L(t) > \log_e L(\hat{\tau}) - D\}, \tag{17}$$

where $\log_e L(\hat{\tau})$ is the maximum of the log likelihood function evaluated over all possible change points t. If the value of the log likelihood function at time t, $\log_e L(t)$, exceeds the maximum of the log likelihood function minus a reference value D, then t is included in the confidence set. We use critical values of D between 1 and 6 and β values equal to 0.01, 0.015, 0.02, 0.025, 0.03 and 0.035 to compute the cardinality and coverage percentage of confidence set estimator. Figure 3 provides a surface plot showing the

relationship among cardinality, coverage, β and D for the confidence set estimator. For example, if $\beta = 0.015$, $D = 6$, the confidence set obtained using equation (17) will yield an expected cardinality of approximately 38.64. In addition, 67.1 percent of possible change points contain the estimated change point.

Figure 3 Surface plot obtained from confidence set estimator showing estimated relationships between set cardinality, coverage, and β value D (see online version for colours)

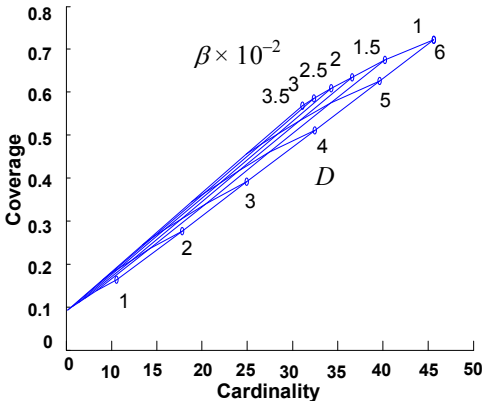
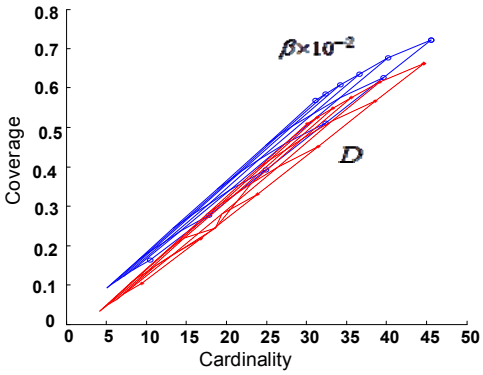


Figure 4 reveals the surface obtained from the confidence set estimator derived for step changes with dashed surface superimposed on the surface obtained from confidence set estimator for linear trends. Note that the scales of β and D are similar to what shown in Figure 3. As shown in Figure 4, for any given value of D, the confidence set estimator derived for linear trends will produce more coverage percentage than that offered by the estimator derived for step changes. Usually, the confidence set estimator derived for step changes will give confidence sets of smaller set cardinality and less coverage than the confidence set estimator derived for linear trends. So, in real world applications, if a linear trend occurs and step change point estimator is used, we may face the problem of estimating the actual change point.

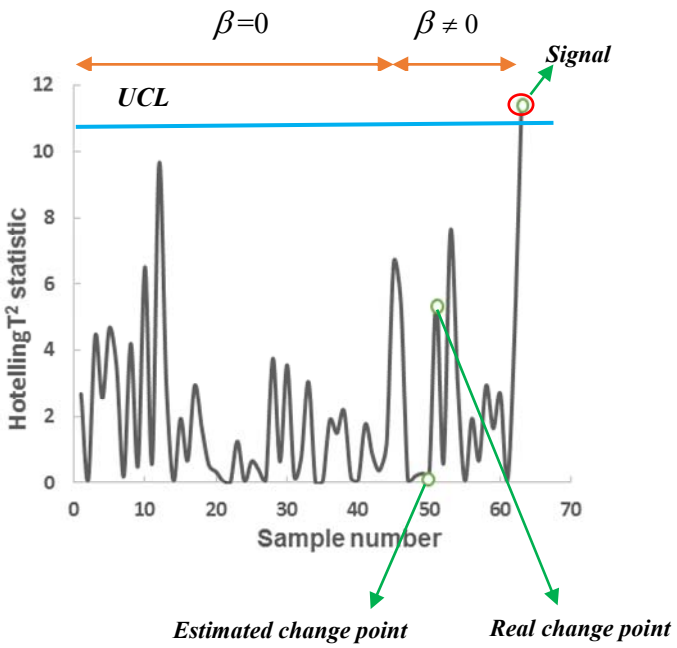
Figure 4 Surface plot obtained from confidence set estimator for step changes (dashed-lines) superimposed on the surface obtained from confidence set estimator for linear trends (see online version for colours)



6 An illustrative example

In this section we present a numerical example to show the performance of the proposed change point estimator so, we set a gamma regression model with one predictor variable and $m = m_j = 30$ for $j = 1, 2,\ldots, 9$. The explanatory variable, the mean and covariance matrix of the regression model parameters are the same as before. Here, the upper control limit for the proposed control chart is set equal to $\chi^2_{2,\,0.005} = 10.59$, which is the 0.005 percentile point of the chi-square distribution with 2 degrees of freedom. Also, the convergence threshold in the Newton method to estimate the slope of the linear trend disturbance is considered equal to 0.0005. Assume that a linear trend disturbance occurs in the scale parameter of gamma regression profile in the 50th profile. Then, the subgroups coming from the out-of-control process are plotted on T^2 control chart until the control chart issues a genuine alarm signal. At this time, the change point estimator is calculated by using equation (13). By using the simulated data, the number of samples taken until the signal is 63 and the change point estimator is 50.37 which is close to exact change point 50. Figure 4 shows the T^2 control chart used for monitoring the process as well as the actual and estimated change points. As shown in Figure 4 and approved by our simulation studies, the proposed estimator has a sufficient performance in estimating the real time of a change in the parameters of the gamma regression profile.

Figure 5 T^2 control chart with a linear trend disturbance in the gamma regression profile (see online version for colours)



7 Conclusions and future researches

In this paper, a maximum likelihood estimator for identifying the time of linear trend change in a gamma regression profile was proposed. We used the Newton method to estimate the slope parameter β at each potential change point value. Afterwards, the Monte Carlo simulation was applied to compare the performance of the proposed estimator under different slopes of linear trend disturbances. Also, cardinality and coverage percent of a confidence set estimator was analysed through simulation studies. The performance of the proposed estimator was also shown through an illustrative example. Simulation results showed that the proposed change point estimator performs well over all the slopes of the linear trend disturbance. Also, the accuracy and precision of the proposed estimator is better than the step change point estimator under drift in the scale parameters of the gamma regression profile. Moreover, the results showed that the performance of the proposed confidence set estimator is better than the confidence set estimator derived for step change using measures of cardinality and coverage percentage.

Developing a change point estimator for the other change types such as monotonic in gamma regression profile could be a fruitful area for future researches. Also, investigating the other methods such as clustering and artificial neural network for linear trend disturbance could be considered as future researches.

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