Multiple response optimisation: An approach from multiobjective stochastic programming

José A. Díaz-García a,*, Mahdi Bashiri b

a Universidad Autónoma Agraria Antonio Narro, Department of Statistics and Computation, 25315 Buenavista, Saltillo, Coahuila, Mexico
b Shahed University, Faculty of Engineering, Industrial Engineering Department, P.O. Box 18155/159, Teheran, Iran

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ABSTRACT
The multiresponse surface problem is modelled as a multiobjective stochastic optimisation, and diverse solutions are proposed. There are several crucial differences highlighted between this approach and the other proposed solutions. Finally, some particular solutions are applied and described in detail in a numerical example.

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1. Introduction

Many (and perhaps most) real-world design problems are in fact multiobjective optimisation problems in which the designer seeks to optimise simultaneously several performance attributes of a design, and often, an improvement in one objective only gained at the cost of deteriorations in others, and so a trade-off is necessary. Similar situations occur in the study of natural phenomena and in experimental trials.

Moreover, the response variables, the controllable variables and even some parameters involved in these studies may have a random (or stochastic) character.

A very useful statistical tool in the study of these designs, phenomena and experiments is the response surfaces methodology, in its multivariate version. This approach makes possible to determine an analytical relationship between the response and the control variables, through a process of continuous improvement and optimisation. Similarly, it allows us to obtain an approximate vector function (termed the multiresponse surface or predicted response vector) with a smaller amount of data and fewer experimental runs, see [1–4].

Although the response variables are random and in consequence the estimated multiresponse surface contains shape parameters that are estimated in the regression stage (i.e. they are random), the process was initially considered as deterministic optimisation, see [5] among others. Subsequently, this randomness or uncertainty was taken into account in the multiobjective optimisation process in different ways, and at different stages, see [6,1,7–10].

In general terms, thus, the multiresponse optimisation problem under uncertainty has been addressed as follows:

* Corresponding author. Tel.: +52 844 411 0293; fax: +52 844 411 0211.
E-mail addresses: jadiaz@uaaamx, jadiazg@ugr.es (J.A. Díaz-García), bashiri@shahed.ac.ir (M. Bashiri).

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1. It is considered as a deterministic multiobjective optimisation problem.
2. An equivalent deterministic univariate optimisation problem, such as goal programming, is proposed.
3. In the equivalent deterministic univariate optimisation problem, uncertainty is assumed.

However, there exists a well-grounded and documented theory about -Stochastic Optimisation- that addresses the following general problem:

$$\begin{align*}
\min_{x} & \begin{pmatrix}
h_1(x, \xi_1) \\
h_2(x, \xi_2) \\
\vdots \\
h_s(x, \xi_s)
\end{pmatrix} \\
\text{subject to} & \\
g_j(x, \delta_j) & \geq 0, \ j = 1, 2, \ldots, s,
\end{align*}$$ (1)

where $\mathbf{x}$ is $n$-dimensional, $\xi_i$ are $m_i$-dimensional, $i = 1, \ldots, r$ and $\delta_j, j = 1, \ldots, s$ are $l_j$-dimensional. If $\mathbf{x}$ or $\xi_i$ (for some $i$) or $\delta_j$ (for some $j$) are random, then (1) defines a multiobjective stochastic optimisation problem, see [11]. As shown in the following sections, the optimisation of a multiresponse surface can be proposed as a multiobjective stochastic optimisation problem. Moreover, most of the proposed approaches to solve this problem in the literature can be obtained as special cases of this proposed approach.

There are some related works on the multiple response optimisation for example, [12] worked on the multiple response optimisation in presence of outliers in the collected data. They proposed a method to decrease outlier effects on the problem. Amiri et al. [13] proposed an approach based on the multivariate process capability index in optimisation of multiple responses. The aggregation of the responses is done during the process capability calculation. Salmassia et al. [14] introduced an interactive approach for correlated multiple responses during the optimisation. Bashiri and Farshbaf [15] used an approach based on the response surface methodology in tuning of an artificial neural network. Gutiérrez and Lozano [16] tried to consider the efficiency during the optimisation of multiple responses.

In this work, the optimisation of a multiresponse surface is proposed as a multiobjective stochastic optimisation problem. Section 2 considers the notation and basic elements of the multiresponse surface, and several previous approaches are discussed. Section 3 proposes the optimisation of a multiresponse surface as a multiobjective stochastic optimisation problem and diverse solutions are proposed. Furthermore, practical considerations are provided with the aim to select the best method of multiresponse optimisation in each particular case. Several multiobjective stochastic solutions are studied in detail in Section 4. Finally, a real case from the literature is analysed in Section 5.

2. Multiresponse optimisation: definition and notation

A detailed discussion of multiresponse surface methodology may be found in [1, Chap. 7] and [6]. For convenience, the principal properties and usual notation are restated here.

Let $N$ be the number of experimental runs and $r$ be the number of response variables which can be measured for each setting of a group of $n$ coded variables (also termed factors) $x_1, x_2, \ldots, x_n$. We assume that the response variables can be modelled by a second order polynomial regression model in terms of $x_i$, $i = 1, \ldots, n$. Hence, the $k^{\text{th}}$ response model can be written as:

$$Y_k = \mathbf{X}_k \beta_k + \epsilon_k$$ (2)

where $Y_k$ is an $N \times 1$ vector of observations on the $k^{\text{th}}$ response, $\mathbf{X}_k$ is an $N \times p$ matrix of rank $p$ termed the design or regression matrix, $p = 1 - n + n(n + 1)/2$, $\beta_k$ is a $p \times 1$ vector of unknown constant parameters, and $\epsilon_k$ is a random error vector associated with the $k^{\text{th}}$ response. In the present case, it is assumed that $\mathbf{X}_i = \ldots = \mathbf{X}_r = \mathbf{X}$. Hence, (2) can be written as

$$\mathbf{Y} = \mathbf{X}\mathbf{B} + \mathbf{E}$$ (3)

where $\mathbf{Y} = \begin{bmatrix} Y_1, Y_2, \ldots, Y_r \end{bmatrix}'$, $\mathbf{B} = \begin{bmatrix} \beta_1, \beta_2, \ldots, \beta_r \end{bmatrix}'$ and $\mathbf{E} = \begin{bmatrix} \epsilon_1, \epsilon_2, \ldots, \epsilon_r \end{bmatrix}'$, such that $\mathbf{E} \sim \mathcal{N}_{N \times r}(\mathbf{0}, \mathbf{I}_N \otimes \mathbf{\Sigma})$ i.e. $\mathbf{E}$ has an $N \times r$ matrix multivariate normal distribution with $\mathbf{E}(\mathbf{E}) = \mathbf{0}$ and $\text{Cov}(\text{vec}\mathbf{E}) = \mathbf{I}_N \otimes \mathbf{\Sigma}$. The $\mathbf{\Sigma}$ is a $r \times r$ positive definite matrix. where

if $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r \end{bmatrix}$, then $\text{vec}\mathbf{A} = (\text{vec}\mathbf{A}_1, \text{vec}\mathbf{A}_2, \ldots, \text{vec}\mathbf{A}_r)'$ and $\otimes$ denotes the direct (or Kronecker) product of matrices, see [17, Theorem 3.2.2, p. 79]. In addition let

- $\mathbf{x} = (x_1, x_2, \ldots, x_n)'$: The vector of controllable variables or factors. Formally, an $x_i$ variable is associated with each factor $A_i, B_i, \ldots$
- $\mathbf{\hat{B}} = \begin{bmatrix} \hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_r \end{bmatrix}$: The least squares estimator of $\mathbf{B}$ given by $\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$, from where $\hat{\beta}_k = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'y_k$, $k = 1, 2, \ldots, r$. Moreover, under the assumption that $\mathbf{E} \sim \mathcal{N}_{N \times r}(\mathbf{0}, \mathbf{I}_N \otimes \mathbf{\Sigma})$, then $\hat{\mathbf{B}} \sim \mathcal{N}_{r \times r}(\mathbf{B}, (\mathbf{X}'\mathbf{X})^{-1} \otimes \mathbf{\Sigma})$, with $\text{Cov}(\text{vec}\hat{\mathbf{B}}) = (\mathbf{X}'\mathbf{X})^{-1} \otimes \mathbf{\Sigma}$.
- $\mathbf{z}(\mathbf{x}) = (x_1, x_2, \ldots, x_n, x_1^2, x_2^2, \ldots, x_n^2, x_1x_2, x_1x_3, \ldots, x_{n-1}x_n)'$.
The response surface or predictor equation at the point \( \mathbf{x} \) for the \( k \)th response variable.

\[
\hat{Y}_k(\mathbf{x}) = \mathbf{z}'(\mathbf{x}) \hat{\beta}_k
\]

\[
= \tilde{\beta}_0 + \sum_{i=1}^{n} \tilde{\beta}_{i1} x_i + \sum_{i=1}^{n} \tilde{\beta}_{i2} x_i^2 + \sum_{i,j=1}^{n} \tilde{\beta}_{ij} x_i x_j
\]

The response surface or predictor vector at the point \( \mathbf{x} \).

\[
\hat{\mathbf{Y}}(\mathbf{x}) = \left[ \hat{Y}_1(\mathbf{x}), \hat{Y}_2(\mathbf{x}), \ldots, \hat{Y}_s(\mathbf{x}) \right]'
\]

\[
= \hat{\beta}' \mathbf{z}(\mathbf{x}): \text{ The multiresponse surface or predicted response vector at the point } \mathbf{x}.
\]

\[
\hat{\Sigma} = \left[ \hat{\beta} \mathbf{z}(\mathbf{x}) \right]' \left[ \hat{\beta} \mathbf{z}(\mathbf{x}) \right]^{-1} \mathbf{z}(\mathbf{x}) \Sigma.
\]

The estimator of the variance-covariance matrix \( \Sigma \) such that \( (N-p) \hat{\Sigma} \) has a Wishart distribution with \( (N-p) \) degrees of freedom and the parameter \( \Sigma \); this fact is denoted as \( (N-p) \hat{\Sigma} \sim W_{r, p, \Sigma} \).

Here, \( I_m \) denotes an identity matrix of order \( m \).

Finally, note that:

\[
E(\hat{\mathbf{Y}}(\mathbf{x})) = E(\hat{\beta}' \mathbf{z}(\mathbf{x})) = \mathbf{B}' \mathbf{z}(\mathbf{x})
\]

\[
\text{cov}(\hat{\mathbf{Y}}(\mathbf{x})) = \mathbf{z}'(\mathbf{x})(\mathbf{X \Sigma X}')^{-1} \mathbf{z}(\mathbf{x}) \Sigma.
\]

An unbiased estimator of \( \text{cov}(\hat{\mathbf{Y}}(\mathbf{x})) \) is given by:

\[
\hat{\text{cov}}(\hat{\mathbf{Y}}(\mathbf{x})) = \mathbf{z}'(\mathbf{x})(\mathbf{X \Sigma X}')^{-1} \mathbf{z}(\mathbf{x}) \hat{\Sigma}.
\]

In the following sections, we make use of multiresponse optimisation and multiobjective (or more general multicriteria) optimisation. For convenience, the concepts and notation required are listed below in terms of the estimated model of multiresponse surface optimisation. Definitions and detailed properties may be found in [5, 18–21, 11].

The multiresponse optimisation (MRO) problem in general is proposed as:

\[
\min_{\mathbf{x}} \hat{\mathbf{Y}}(\mathbf{x}) = \min_{\mathbf{x}} \left( \begin{array}{c}
\hat{Y}_1(\mathbf{x}) \\
\hat{Y}_2(\mathbf{x}) \\
\vdots \\
\hat{Y}_r(\mathbf{x})
\end{array} \right)
\]

subject to

\[
\mathbf{x} \in \mathbf{x},
\]

which is a deterministic nonlinear multiobjective optimisation problem, see [19, 18, 20]; and where \( \mathbf{x} \) denotes the experimental region, which in general is defined as a hypersphere

\[
\mathbf{x} = \left\{ \mathbf{x} | \mathbf{x}' \mathbf{x} \leq c^2, c \in \mathbb{R} \right\},
\]

where, in general, \( c \) is determined by the experimental design model used, see [1]. Alternatively, the experimental region is defined as a hypercube

\[
\mathbf{x} = \left\{ \mathbf{x} | l_i < x_i < u_i, \ i = 1, 2, \ldots, n \right\},
\]

where \( l = (l_1, l_2, \ldots, l_n) \), defines the vector of lower bounds of factors and \( \mathbf{u} = (u_1, u_2, \ldots, u_n)' \), define the vector of upper bounds of factors.

Alternatively (7) can be written as:

\[
\min_{\mathbf{x} \in \mathbf{x}} \hat{\mathbf{Y}}(\mathbf{x}).
\]

In the response surface methodology context, observe that in multiobjective optimisation problems, there rarely exists a point \( \mathbf{x}^* \) which is considered as an optimum, i.e. few cases satisfy the requirement that \( \hat{Y}_k(\mathbf{x}) \) is minimum for all \( k = 1, 2, \ldots, r \). From the viewpoint of multiobjective optimisation, this justifies the following notion of the Pareto point, which is more weakly defined than an optimum point:

We say that \( \hat{\mathbf{Y}}(\mathbf{x}) \) is a Pareto point of \( \hat{\mathbf{Y}}(\mathbf{x}) \), if there is no other point \( \hat{\mathbf{Y}}(\mathbf{x}) \) such that \( \hat{Y}_k(\mathbf{x}) \leq \hat{Y}_k(\mathbf{x}) \), i.e. for all \( k, \hat{Y}_k(\mathbf{x}) < \hat{Y}_k(\mathbf{x}) \) and \( \hat{Y}_k(\mathbf{x}) \neq \hat{Y}_k(\mathbf{x}) \).

Existence criteria for Pareto points in a multiobjective optimisation problem and the extension of scalar optimisation (Kuhn-Tucker's conditions) to the vectorial case are established in [19, 18, 20].

Methods for solving a multiobjective optimisation problem are based on the information possessed about a particular problem. There are three possible scenarios: when the investigator possesses either complete, partial or null information, see [18, 20, 19]. In a response surface methodology context, complete information means that the investigator understands
the population in such a way that it is possible to propose a value function reflecting the importance of each response variable, where.

The value function is a function \( f : \mathbb{R}^n \to \mathbb{R} \) such that \( \min \tilde{Y}(\mathbf{x}') < \min \tilde{Y}(\mathbf{x}_1) \iff f(\tilde{Y}(\mathbf{x}')) < f(\tilde{Y}(\mathbf{x}_1)), \mathbf{x}' \neq \mathbf{x}_1. \)

In partial information, the investigator knows the main response variable of the study very well and this is sufficient support for the research. Finally, under null information, the researcher only possesses information about the estimators of the response surface parameter, and with this material an appropriate solution can be found too.

As can be observed, all the approaches proposed in the literature are particular cases of the models studied in multiobjective optimisation, and in particular of the \( \epsilon \)-constraint model and the value function model or a combination of the two. Accordingly, the equivalent nonlinear scalar optimisation problem of (7) is of the form:

\[
\begin{align*}
\min_{\mathbf{x}} & \quad f(\tilde{Y}(\mathbf{x})) \\
\text{subject to} & \quad \mathbf{x} \in \mathbb{X} \cap \Xi, \\
& \quad \bar{\mathbf{B}} \sim N_p, (\mathbf{B}^T \mathbf{X})^{-1} \otimes \Sigma \\
& \quad (N-p) \bar{\Sigma} \sim \mathcal{W}_\tau(N-p, \Sigma),
\end{align*}
\]

where \( \Xi \) is a subset generated by additional potential constraints, which derive from the particular technique used for establishing the equivalent deterministic scalar optimisation problem (8). In some particular cases of (8), a new fixed parameter may appear, such as a \( \mathbf{w} = (w_1, w_2, \ldots, w_r)' \), vector of response weights and/or \( \tau = (\tau_1, \tau_2, \ldots, \tau_r)' \), vector of target values for the response vector. Particular examples of this equivalent univariate objective optimisation are the use of goal programming, see [22], and of the \( \epsilon \)-constraint model, see [5], among many others.

When uncertainty is assumed in an MRO problem, in other words, when the MRO problem is considered as a stochastic program, different approaches have been proposed, see [8,7,9,6,1]. In particular some of these approaches can be formulated as:

\[
\begin{align*}
\min_{\mathbf{x}} & \quad f(\tilde{Y}(\mathbf{x})) \\
\text{subject to} & \quad \mathbf{x} \in \mathbb{X} \cap \Xi \\
& \quad \hat{\mathbf{B}} \sim N_p, (\mathbf{B}^T \mathbf{X})^{-1} \otimes \Sigma \\
& \quad (N-p) \hat{\Sigma} \sim \mathcal{W}_\tau(N-p, \Sigma),
\end{align*}
\]

where \( \hat{\mathbf{B}} \) and \( \hat{\Sigma} \) are independent.

In addition, it is sometimes assumed that \( \mathbf{w} \) and/or \( \tau \) are stochastic, and diverse strategies have been proposed to obtain particular values for these, including Group Decision Making, among others, see [6,10].

In this way, note that the approaches (7)–(9), are the corresponding formal statements of proposed approaches in the literature and have been described in the introduction. Moreover, these approaches are obtained as special cases of the approaches proposed here, and which are described in the next section.

Finally note that the concept of robustness can be considered. In this case assume that \( \tilde{Y}(\mathbf{x}) = (\tilde{Y}_1(\mathbf{x}), \ldots, \tilde{Y}_r(\mathbf{x}))' \) is the multiresponse surface or predicted response vector at the point \( \mathbf{x} \) for the variances, i.e., \( \tilde{Y}_k(\mathbf{x}) \) is the response surface at the point \( \mathbf{x} \) for the variance, corresponding to the \( k^{th} \) response variable \( \tilde{Y}_k \). This way, among other approaches, (7) can be formulated as:

\[
\begin{align*}
\min_{\mathbf{x}} & \quad \begin{pmatrix}
\tilde{Y}_1(\mathbf{x}) \\
\tilde{Y}_2(\mathbf{x}) \\
\vdots \\
\tilde{Y}_r(\mathbf{x}) \\
\tilde{Y}_1(\mathbf{x}) \\
\tilde{Y}_2(\mathbf{x}) \\
\vdots \\
\tilde{Y}_r(\mathbf{x})
\end{pmatrix} \\
\text{subject to} & \quad \mathbf{x} \in \mathbb{X},
\end{align*}
\]

then the following described techniques can be applied under minimal modifications.

3. Proposed approach

In the univariate case, [23] considered the problem as a stochastic optimisation program. The approach proposed in the present paper consists in extending this idea to multiresponse optimisation. Specifically, we propose the MRO as the following nonlinear multiobjective stochastic optimisation problem from the beginning:
\[
\begin{align*}
\min_{\mathbf{x}} & \left( \mathbf{Y}(\mathbf{x}, \mathbf{\hat{B}}) \right) \\
\text{subject to} & \\
\mathbf{x} & \in \mathbb{X} \\
\mathbf{\hat{B}} & \sim \mathcal{N}_{p,r}(\mathbf{B}, (\mathbf{XX})^{-1} \otimes \Sigma) \\
(N - p) \Sigma & \sim \mathcal{W}_p(N - p, \Sigma),
\end{align*}
\] (11)

where \( \mathbf{Y}(\mathbf{x}, \mathbf{\hat{B}}) \equiv \hat{\mathbf{Y}}(\mathbf{x}) \) and \( \hat{\mathbf{Y}}_k(\mathbf{x}, \mathbf{\hat{B}_k}) \equiv \hat{\mathbf{Y}}_k(\mathbf{x}) \).

The solution of (11) can be applied to any model (technique, method or solution) under multiobjective stochastic optimisation, which in general, is a multidimensional extension of stochastic optimisation models, see [21,23,11].

3.1. Multiobjective stochastic optimisation approaches

In this subsection we consider (11) under diverse multiobjective stochastic optimisation approaches. The properties of the solution obtained under the different approaches are described in detail by Kataoka [24], Stancu-Minasian [25] and Prékopa [11].

As shown below, each multiobjective stochastic optimisation approach can be proposed in several ways. In some cases, this possibility is a consequence of assuming that the response variables are correlated or not.

3.1.1. Multiobjective expected value solution: multiobjective E-model

Point \( \mathbf{x} \in \mathbb{X} \) is the expected value solution to (11) if it is an efficient solution in the Pareto sense to the following deterministic multiobjective optimisation problem:

\[
\min_{\mathbf{x} \in \mathbb{X}} \left( \mathbb{E} \left( \hat{\mathbf{Y}}(\mathbf{x}, \mathbf{\hat{B}}) \right) \right).
\] (12)

3.1.2. Multiobjective minimum variance solution: multiobjective V-model

The \( \mathbf{x} \in \mathbb{X} \) point is the minimum variance solution to problem (11) if it is an efficient solution in the Pareto sense of the deterministic multiobjective optimisation problem:

\[
\min_{\mathbf{x} \in \mathbb{X}} \left( \begin{array}{c}
\text{Var} \left( \hat{\mathbf{Y}}_1(\mathbf{x}) \right) \\
\text{Var} \left( \hat{\mathbf{Y}}_2(\mathbf{x}) \right) \\
\vdots \\
\text{Var} \left( \hat{\mathbf{Y}}_r(\mathbf{x}) \right)
\end{array} \right).
\] (13)

This efficient solution is adequate if it is assumed that the response variables are Uncorrelated. However, if the response variables are assumed to be correlated a better one is:

\[
\min_{\mathbf{x} \in \mathbb{X}} \text{Cov} \left( \hat{\mathbf{Y}}(\mathbf{x}, \mathbf{\hat{B}}) \right).
\] (14)

3.1.3. Multiobjective expected value standard deviation solution: multiobjective modified E-model

Point \( \mathbf{x} \in \mathbb{X} \) is an expected value standard deviation solution to the problem (11) if it is an efficient solution in the Pareto sense of the mixed deterministic multiobjective-matrix optimisation problem:

\[
\min_{\mathbf{x} \in \mathbb{X}} \left( \begin{array}{c}
\mathbb{E} \left( \hat{\mathbf{Y}}(\mathbf{x}, \mathbf{\hat{B}}) \right) \\
\left( \text{Cov} \left( \hat{\mathbf{Y}}(\mathbf{x}, \mathbf{\hat{B}}) \right) \right)^{1/2}
\end{array} \right),
\] (15)

where \( \left( \mathbf{A}^{1/2} \right)^2 = \mathbf{A} \), see [17, Appendix].

We now define the concept of the efficient solution of multiobjective minimum risk of joint aspiration level \( \tau = (z_1, z_2, \ldots, z_r) \) and the efficient solution with a joint probability \( \delta \). The two solutions are obtained by applying the multivariate versions of minimum risk and the Kataoka criteria, respectively, referred to in the literature as criteria of maximum probability or satisfying criteria, due to the fact that, as shown below, in both cases the criteria to be used provide, in one way or another, “good” solutions in terms of probability, see [24].

3.1.4. Multiobjective minimum risk solution of joint aspiration level \( \tau \): multiobjective modified P-model

Point \( \mathbf{x} \in \mathbb{X} \) is a minimum risk solution of joint aspiration level \( \tau \) to problem (11) if it constitutes an efficient solution in the Pareto sense of the multiobjective stochastic optimisation problem:
\[
\min_{x \in X} \left\{ \begin{array}{l}
\mathbb{P}(\tilde{Y}_1(x) \leq \tau_1) \\
\mathbb{P}(\tilde{Y}_2(x) \leq \tau_2) \\
\vdots \\
\mathbb{P}(\tilde{Y}_r(x) \leq \tau_r) 
\end{array} \right. 
\]

(16)

It is also possible to consider the following alternative multiobjective \( \mathbb{P} \)-model:

\[
\min_{x \in X} \left\{ \begin{array}{l}
\mathbb{P}(\tilde{Y}_1(x) \leq \tau_1) \\
\mathbb{P}(\tilde{Y}_2(x) \leq \tau_2) \\
\vdots \\
\mathbb{P}(\tilde{Y}_r(x) \leq \tau_r) 
\end{array} \right. 
\]

(17)

Again, (17) is more adequate if the response variables are correlated. However (17) is considerably more complicated to solve than (16). When \( r = 2 \), Prékopa [26] proposed an algorithm for a similar problem (probabilistic constrained programming), which can be applied to solve (17).

3.1.5. Multiobjective Kataoka solution with probability \( \delta \)

Point \( x \in X \) is a multiobjective Kataoka solution with probability \( \delta \) (fixed) to problem (11) if it is an efficient solution in the Pareto sense of the multiobjective optimisation problem:

\[
\min_{x, \tau} \tau \\
\text{subject to} \\
\mathbb{P}(\tilde{Y}_k(x) \leq \tau_k) = \delta, \; k = 1, 2, \ldots, r \\
x \in X. 
\]

(18)

Alternatively (18) can be proposed as:

\[
\min_{x, \tau} \tau \\
\text{subject to} \\
\mathbb{P}(\tilde{Y}_1(x) \leq \tau_1) \\
\mathbb{P}(\tilde{Y}_2(x) \leq \tau_2) \\
\vdots \\
\mathbb{P}(\tilde{Y}_r(x) \leq \tau_r) \\
x \in X. 
\]

(19)

Note that (18) and (19) are multiobjective probabilistic constrained programming, see [27,25,11].

Many other approaches can be used to solve (11). For example, [25] proposed a stochastic version of the sequential technique, termed the Lexicographic method, for solving (13) and (16) or direct application to (11); among many other options. In all cases, observe that:

\[
\mathbb{E}(\tilde{Y}(x, \bar{B})) = Y(x, \bar{B}) \text{ and } \text{Cov}(\tilde{Y}(x, \bar{B})) = z'(x)(XX)^{-1}z(x)\Sigma,
\]

in general are unknown. Then, from a practical point of view, and having the final expression of the equivalent deterministic problem of (11), \( \mathbb{E}(\tilde{Y}(x, \bar{B})) \) and \( \text{Cov}(\tilde{Y}(x, \bar{B})) \) should be replaced by their corresponding estimators

\[
\mathbb{E}(\tilde{Y}(x, \bar{B})) = \bar{Y}(x, \bar{B}) \quad \text{and} \quad \text{Cov}(\tilde{Y}(x, \bar{B})) = z'(x)(XX)^{-1}z(x)\bar{\Sigma}.
\]

Remark 1. In the vast literature on stochastic optimisation and multicriteria optimisation there are recommendations on the application of different techniques from both fields optimisation, see [11,25,19], among many others. Next are particularised some of these recommendations in the context of the MRO.
First, note that if the response surface models to be optimised are estimated models, then formally the optimisation problem you have is a stochastic multiobjective optimisation problem described in (11) (also see (1)) and not a deterministic multiobjective optimisation problem established in (7), as some authors in the literature have considered. We do not mean to consider the problem as deterministic wrong, but to emphasize that the problem presented (under estimated models), formally is a stochastic multiobjective optimisation problem. This first point gives leads to, brings forward to, originates the observation that the deterministic approaches should not be compared with approaches under a stochastic context, because formally are solving different problems.

Now assume that you actually have that a multiresponse optimisation problem is a stochastic multiobjective optimisation problem, then two considerations must be taken into account:

1. For the actual decision on which stochastic solution use, can be used - in addition to economical or feasibility considerations - note that, the multivariate E-model being based on an expectation (i.e. long term average), implies that the solution that it offers would be reasonable if the system is pretended to be in operation for a long period of time; the multivariate E-model solution would tend to give preference to sets of conditions with high predicted mean and low prediction error; whereas the modified multivariate V-model solution would emphasize error protection with a reasonable predicted mean. Since the multivariate P-model approach does not require strict optimisation of the response, is flexible and can be used jointly with other sets of restrictions.

2. Now for example, to decide whether to apply the method (16) or the method (17), the decision maker must determine whether the response variables under study are dependent or not, and in terms of the response the corresponding method can be selected. Note if the decision maker can support tests of independence to make such a decision.

Note, these two qualifications to dispel the comparison between these different multiobjective stochastic optimisation methods, because formally applying different stochastic multiobjective optimisation methods are providing solutions to different problems MRO.

Finally, similar considerations are set forth in the choice of the different techniques of multiobjective optimisation solution, because these techniques depend on the amount of information that the decision maker have on different response variables (complete, partial or null information), resulting in that each potential solution of MRO through stochastic multiobjective optimisation formally solves a different problem and as a result, in general all possible solutions should not be compared.

4. Equivalent deterministic programs

In this section we study several particular equivalent deterministic programs from (14) in detail.

4.1. Multiobjective V-model

Taking into account the final comment in Section 3, our intention is to solve the matrix optimisation problem:

$$\min_{x \in X} \text{Cov} \left( \hat{Y}(x, \hat{B}) \right).$$ (20)

For the sake of convenience, in this section we denote \( \text{Cov} \left( \hat{Y}(x, \hat{B}) \right) \) as \( \text{Cov}(\hat{Y}(x)) \). Obviously, the difficulty in expressing the problem in this way lies in defining the meaning of the minimum of a matrix function. The idea of minimising a matrix function, and in particular a matrix of variance–covariance, has been studied with respect to various areas of statistical theory. For example, when regression estimators are determined for a multivariate general linear model, this is done by minimising the determinant or the trace of sums of squares and cross-products matrix of the error, see [28]. Similarly, the choice or comparison of experimental design models is done by minimising a function of the variance–covariance matrix of treatment estimators, see [1,29].

Fortunately, it is possible to reduce the nonlinear matrix minimisation problem (20) to a univariate nonlinear minimisation problem by taking into account the following considerations. Observe that the procedure described here is just one of various possible options, see [18,20].

Assume that \( \text{Cov}(\hat{Y}(x)) \) is a positive definite matrix for all \( x \), denoting it as \( \text{Cov}(\hat{Y}(x)) > 0 \). Now, let \( x_1 \) and \( x_2 \) be two possible values of the vector \( x \) and let \( B = \text{Cov}(\hat{Y}(x_1)) - \text{Cov}(\hat{Y}(x_2)) \). Then we say that

$$\text{Cov}(\hat{Y}(x_1)) < \text{Cov}(\hat{Y}(x_2)) \iff B < 0,$$ (21)

i.e. if the matrix \( B \) is a negative definite matrix. Moreover, note that \( \text{Cov}(\hat{Y}(x_1)) \) and \( \text{Cov}(\hat{Y}(x_2)) \), are diagonalizable: then, let \( D_{x_1} \) and \( D_{x_2} \) be the diagonal matrixes associated with \( \text{Cov}(\hat{Y}(x_1)) \) and \( \text{Cov}(\hat{Y}(x_2)) \), respectively; with \( D_{x_1} = \text{diag}(x_1, \ldots, x_n) \), \( x_1 > \cdots > x_n > 0 \) and \( D_{x_2} = \text{diag}(y_1, \ldots, y_n) \), \( y_1 > \cdots > y_n > 0 \), where \( x_j \) and \( y_j \) denote the eigenvalues of \( \text{Cov}(\hat{Y}(x_1)) \) and \( \text{Cov}(\hat{Y}(x_2)) \), respectively. Thus, expression (21) can alternatively be presented as:
\[ \text{Cov}(\bar{Y}(x_1)) < \text{Cov}(\bar{Y}(x_2)) \iff D_{x_1} - D_{x_2} < 0, \]

i.e.
\[ \text{Cov}(\bar{Y}(x_1)) < \text{Cov}(\bar{Y}(x_2)) \iff \gamma_i - \gamma_j < 0, \quad (22) \]

and \( \text{Cov}(\bar{Y}(x_1)) \neq \text{Cov}(\bar{Y}(x_2)) \); which defines a weak Pareto order. Then, there exists a function \( g : S \rightarrow \mathbb{R} \), such that
\[ \text{Cov}(\bar{Y}(x_1)) < \text{Cov}(\bar{Y}(x_2)) \iff g\left(\text{Cov}(\bar{Y}(x_1))\right) < g\left(\text{Cov}(\bar{Y}(x_2))\right). \quad (23) \]

where \( \text{Cov}(\bar{Y}(x)) \in S \subset \mathbb{R}^{n(n-1)/2} \) and \( S \) is the set of positive definite matrices. From \( (23), (19,18,20) \) prove that the non-linear matrix minimisation problem \( (20) \) is reduced in the following scalar non-linear minimisation problem:
\[ \min_{x \in \mathbb{R}} g\left(\text{Cov}(\bar{Y}(x))\right). \quad (24) \]

Unfortunately, the function \( g(\cdot) \) is not unique. For example, in other statistical contexts we can find the following commonly used functions \( g(\cdot) \), see \([28]\):

1. The trace of the matrix \( \text{Cov}(\bar{Y}(x)) \):
\[ g(\text{Cov}(\bar{Y}(x))) = \text{tr}\left(\text{Cov}(\bar{Y}(x))\right) = z'(x)(XX)^{-1}z(x) \sum_{j=1}^{r} \sigma_{jj}. \]

2. The determinant of the matrix \( \text{Cov}(\bar{Y}(x)) \):
\[ g(\text{Cov}(\bar{Y}(x))) = \left|\text{Cov}(\bar{Y}(x))\right| = \left|z'(x)(XX)^{-1}z(x)\right| \cdot |\Sigma|. \]

3. The sum of all the elements of the matrix \( \text{Cov}(\bar{Y}(x)) \):
\[ g(\text{Cov}(\bar{Y}(x))) = z'(x)(XX)^{-1}z(x) \sum_{j=1}^{r} \sigma_{jj}. \]

4. \( g(\text{Cov}(\bar{Y}(x))) = \lambda_{\text{max}}(\text{Cov}(\bar{Y}(x))) \), where \( \lambda_{\text{max}} \) is the maximum eigenvalue of the matrix \( \text{Cov}(\bar{Y}(x)) \).

5. \( g(\text{Cov}(\bar{Y}(x))) = \lambda_{\text{min}}(\text{Cov}(\bar{Y}(x))) \), where \( \lambda_{\text{min}} \) is the minimum eigenvalue of the matrix \( \text{Cov}(\bar{Y}(x)) \).

6. \( g(\text{Cov}(\bar{Y}(x))) = \lambda_j(\text{Cov}(\bar{Y}(x))) \), where \( \lambda_j \) is the \( j \)-th eigenvalue of the matrix \( \text{Cov}(\bar{Y}(x)) \), among others.

But observe that:
\[ \lambda_j(\text{Cov}(\bar{Y}(x))) = \lambda_j(z'(x)(XX)^{-1}z(x) \Sigma) = z'(x)(XX)^{-1}z(x) \lambda_j(\Sigma). \]

Hence we can conclude that as a consequence of the structure of the covariance matrix \( \text{Cov}(\bar{Y}(x)) \), for all the particular definitions of the function \( g \) considered above, the scalar non-linear minimisation problem \( (24) \) has a unique solution given by the solution of the non-linear minimisation problem:
\[ \min_{x \in \mathbb{R}} z'(x)(XX)^{-1}z(x). \quad (25) \]

4.2. Multiobjective P-model

Proceeding as in \([23]\), the equivalent multiobjective deterministic problem to \( (11) \) via the \( P \)-model \( (16) \) is
\[ \min_{x \in \mathbb{R}} \begin{pmatrix} z_1(x) \sigma_{x_1} \sqrt{z_1(x)(XX)^{-1}z(x)} \\ \cdots \\ z_n(x) \sigma_{x_n} \sqrt{z_n(x)(XX)^{-1}z(x)} \end{pmatrix}. \quad (26) \]
4.3. Multiobjective Kataoka model

From [23], the equivalent multiobjective deterministic problem to (11) via the Kataoka model (18) is given by

\[
\min_{\mathbf{x}, \mathbf{z}} \begin{cases} 
z'(\mathbf{x}) \bar{\mathbf{f}}_1 + \Phi^{-1}(\delta) \sqrt{\bar{\sigma}_{11}} z'(\mathbf{x})(\mathbf{XX})^{-1}z(\mathbf{x}) \\
z'(\mathbf{x}) \bar{\mathbf{f}}_2 + \Phi^{-1}(\delta) \sqrt{\bar{\sigma}_{22}} z'(\mathbf{x})(\mathbf{XX})^{-1}z(\mathbf{x}) \\
\vdots \\
z'(\mathbf{x}) \bar{\mathbf{f}}_r + \Phi^{-1}(\delta) \sqrt{\bar{\sigma}_{rr}} z'(\mathbf{x})(\mathbf{XX})^{-1}z(\mathbf{x}) 
\end{cases}
\]  

(27)

where \( \Phi \) denotes the distribution function of the standard Normal distribution.

Similar equivalent multiobjective deterministic problems to (11) are obtained by applying the other stochastic solutions described in Section 3. Note that, if each stochastic solution is combined with each multiobjective optimisation technique, an infinite number of possible solutions to (11) are obtained. For example, note that the function of value \( f(\cdot) \) may take an infinite number of forms. One of these particular forms is the weighting method. Under this approach, problem (27) can be re-stated as:

\[
\min_{\mathbf{x} \in \mathcal{X}} \sum_{k=1}^{r} w_k \left( z'(\mathbf{x}) \bar{\mathbf{f}}_k + \Phi^{-1}(\delta) \sqrt{\bar{\sigma}_{kk}} z'(\mathbf{x})(\mathbf{XX})^{-1}z(\mathbf{x}) \right) 
\]  

(28)

such that \( \sum_{k=1}^{r} w_k = 1, \ w_k \geq 0 \ \forall \ k = 1, 2, \ldots, r \); where \( w_k \) weights the importance of each characteristic. The solution \( \mathbf{x} \in \mathcal{X} \) of (28) can be termed the multiobjective Kataoka solution with probability \( \delta \) to problem (11), via the weighting method.

Similarly, \( \mathbf{x} \in \mathcal{X} \) is the multiobjective Kataoka solution with probability \( \delta \) to the problem (11), via goal programming if \( \mathbf{x} \in \mathcal{X} \) is:

\[
\min_{\mathbf{x} \in \mathcal{X}} \sum_{k=1}^{r} w_k (d^*_k + d_k^*) 
\]

subject to

\[
z'(\mathbf{x}) \bar{\mathbf{f}}_k + \Phi^{-1}(\delta) \sqrt{\bar{\sigma}_{kk}} z'(\mathbf{x})(\mathbf{XX})^{-1}z(\mathbf{x}) - d^*_k + d_k^* = \tau_k, \ k = 1, 2, \ldots, r
\]

where

\[
d^*_k = \frac{1}{2} \left( z'(\mathbf{x}) \bar{\mathbf{f}}_k + \Phi^{-1}(\delta) \sqrt{\bar{\sigma}_{kk}} z'(\mathbf{x})(\mathbf{XX})^{-1}z(\mathbf{x}) - \tau_k \right),
\]

\[
d_k^* = \frac{1}{2} \left( z'(\mathbf{x}) \bar{\mathbf{f}}_k + \Phi^{-1}(\delta) \sqrt{\bar{\sigma}_{kk}} z'(\mathbf{x})(\mathbf{XX})^{-1}z(\mathbf{x}) - \tau_k \right),
\]

5. Application

A real case from the literature is analysed by applying several approaches and particular solutions from multiobjective stochastic optimisation.

In the following example, taken from [30], there are two response variables \( \mathbf{Y} = (Y_1, Y_2)' \) with 4 replicates and three control (decision) variables \( \mathbf{x} = (x_1, x_2, x_3)' \). The experimental data are shown in Table 1. It is assumed that the targets of the responses are \( \tau = (\tau_1, \tau_2) = (103, 73)' \).

Eqs. (30) and (31) are response surfaces of second-order for \( Y_1 \) and \( Y_2 \).

\[
\hat{Y}_1(\mathbf{x}) = 104.86 - 3.147x_1 - 0.142x_2 - 0.199x_3 + 2.379x_1x_2 - 0.35x_1x_3 - 0.106x_2x_3 
\]

(30)

\[
\hat{Y}_2(\mathbf{x}) = 70.45 - 0.348x_1 + 3.59x_2 + 0.28x_3 + 0.323x_1x_2 - 0.45x_1x_3 + 0.614x_2x_3 
\]

(31)

From which the multiresponse optimisation problem is given as:

\[
\min_{\mathbf{x} \in \mathcal{X}} \hat{\mathbf{Y}}(\mathbf{x}) = \min_{\mathbf{x} \in \mathcal{X}} \begin{pmatrix} \hat{Y}_1(\mathbf{x}) \\ \hat{Y}_2(\mathbf{x}) \end{pmatrix},
\]

(32)

where \( \mathbf{x} = \{x_i \in [-1, 1], \ i = 1, 2, 3\} \).
Table 1
Experimental data for the numerical example.

<table>
<thead>
<tr>
<th>Replicate</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
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<tbody>
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<td>8</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>104.45</td>
<td>105.03</td>
<td>99.79</td>
<td>104.92</td>
<td>76.90</td>
<td>77.03</td>
<td>67.99</td>
<td>75.77</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>104.12</td>
<td>104.80</td>
<td>104.20</td>
<td>104.34</td>
<td>72.99</td>
<td>74.25</td>
<td>73.94</td>
<td>73.28</td>
</tr>
<tr>
<td>6</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>98.73</td>
<td>99.16</td>
<td>102.84</td>
<td>94.24</td>
<td>67.10</td>
<td>63.61</td>
<td>68.65</td>
<td>62.42</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>100.19</td>
<td>99.63</td>
<td>100.27</td>
<td>100.60</td>
<td>67.03</td>
<td>66.18</td>
<td>66.58</td>
<td>67.94</td>
</tr>
<tr>
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<td>-1</td>
<td>-1</td>
<td>101.15</td>
<td>106.96</td>
<td>107.62</td>
<td>103.44</td>
<td>71.68</td>
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<td>77.50</td>
<td>76.37</td>
</tr>
<tr>
<td>3</td>
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<td>-1</td>
<td>-1</td>
<td>106.08</td>
<td>105.64</td>
<td>105.67</td>
<td>105.39</td>
<td>72.94</td>
<td>72.85</td>
<td>72.58</td>
<td>72.38</td>
</tr>
<tr>
<td>5</td>
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<td>-1</td>
<td>-1</td>
<td>113.52</td>
<td>111.12</td>
<td>112.85</td>
<td>106.67</td>
<td>68.29</td>
<td>68.47</td>
<td>68.96</td>
<td>64.71</td>
</tr>
<tr>
<td>1</td>
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<td>-1</td>
<td>-1</td>
<td>109.90</td>
<td>109.76</td>
<td>110.70</td>
<td>109.77</td>
<td>67.70</td>
<td>67.24</td>
<td>67.96</td>
<td>66.93</td>
</tr>
</tbody>
</table>

Now, assume that the importance of each response variable must be assessed from the decision makers’ viewpoint. For the purposes of this example, consider \( \mathbf{w} = (w_1, w_2)' = (0.285, 0.715)' \).\(^1\)

From (30) and (31)

\[
\hat{\boldsymbol{\beta}} = \begin{bmatrix}
\hat{\beta}_0 \\
\hat{\beta}_1 \\
\hat{\beta}_2 \\
\hat{\beta}_{12} \\
\hat{\beta}_{13} \\
\hat{\beta}_{23}
\end{bmatrix} = \begin{bmatrix}
104.86 \\
-3.147 \\
-0.142 \\
-0.199 \\
2.379 \\
-0.35 \\
-0.106
\end{bmatrix} = \begin{bmatrix}
70.45 \\
-0.348 \\
3.59 \\
0.28 \\
0.323 \\
-0.45 \\
0.614
\end{bmatrix}
\]

Also,

\[
\hat{\Sigma} = \begin{bmatrix}
4.190 & 3.546 \\
3.546 & 4.666
\end{bmatrix}
\]

From where

\[
\hat{\text{Cov}}(\text{vec}\hat{\boldsymbol{\beta}}) = \hat{\Sigma} \otimes (\mathbf{X}'\mathbf{X})^{-1}
\]

\[
= \begin{bmatrix}
4.190 & 3.546 \\
3.546 & 4.666
\end{bmatrix} \otimes 0.03125 \mathbf{I}_2.
\]

In particular, \( \hat{\text{Cov}}(\hat{\beta}_1) = 0.131 \mathbf{I}_2 \) and \( \hat{\text{Cov}}(\hat{\beta}_2) = 0.145 \mathbf{I}_2 \). Therefore, the estimator of the covariance matrix of response surfaces according to Eq. (6) is

\[
\hat{\text{Cov}}(\mathbf{Y}(\mathbf{x})) = (1 + x_1^2 + x_2^2 + x_3^2 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) \begin{bmatrix}
0.131 & 0.111 \\
0.111 & 0.145
\end{bmatrix}
\]

Next, we propose diverse multiobjective stochastic solutions and their deterministic equivalent corresponding programs:

- **Equivalent multiobjective V-model**

\[
\min_{\mathbf{x} \in \mathbb{R}^3} \begin{bmatrix}
\text{Var}(\tilde{Y}_1(\mathbf{x}, \hat{\beta}_1)) \\
\text{Cov}(\tilde{Y}_1(\mathbf{x}, \hat{\beta}_1), \tilde{Y}_2(\mathbf{x}, \hat{\beta}_2)) \\
\text{Var}(\tilde{Y}_2(\mathbf{x}, \hat{\beta}_2))
\end{bmatrix},
\]

and its corresponding deterministic equivalent program is

\[
\min_{\mathbf{x} \in \mathbb{R}^3} \mathbf{z}(\mathbf{x}).
\]

- **Equivalent deterministic multiobjective modified E-model**

\[
\min_{\mathbf{x} \in \mathbb{R}^3} \begin{bmatrix}
\tilde{\mathbf{Y}}(\mathbf{x}, \hat{\beta}) \\
\left(\text{Cov}(\tilde{\mathbf{Y}}(\mathbf{x}, \hat{\beta}))\right)^{1/2}
\end{bmatrix},
\]

\(^1\) The importance of each response variable must be assessed from the decision makers’ viewpoint. In [10] Table 4 shows the hypothetical importance of two response variables from the viewpoint of twenty decision makers, whose values are those adopted in the present example.
where
\[
\begin{pmatrix}
\hat{Y}_1(x, \hat{b}_1) \\
\hat{Y}_2(x, \hat{b}_2)
\end{pmatrix}
\]
and \((\text{Cov}(\hat{Y}(x, \hat{b})))^{1/2}\) is
\[
\begin{pmatrix}
\text{Var}(\hat{Y}_1(x, \hat{b}_1)) & \text{Cov}(\hat{Y}_1(x, \hat{b}_1), \hat{Y}_2(x, \hat{b}_2)) \\
\text{Cov}(\hat{Y}_1(x, \hat{b}_1), \hat{Y}_2(x, \hat{b}_2)) & \text{Var}(\hat{Y}_2(x, \hat{b}_2))
\end{pmatrix}^{1/2}.
\]
In this case the deterministic equivalent program can be stated as (among many other options, including lexicographic or \(\epsilon\)-constraint models)
\[
\min_{x \in X} f(\hat{Y}(x, \hat{b})) + r_2 \cdot g\left((\text{Cov}(\hat{Y}(x, \hat{b})))^{1/2}\right),
\]
where \(r_j \geq 0, j = 1, 2\) are constants such that \(r_1 + r_2 = 1\) (in general), whose values indicate the relative importance of the expectation and matrix covariance of \(\hat{Y}(x, \hat{b})\), and \(f\) and \(g\) are value functions.

In particular, using (25), a deterministic equivalent program via the weighting method is
\[
\min_{x \in X} \left( w_1 \cdot z'(x)\hat{b}_1 + w_2 \cdot z'(x)\hat{b}_2 \right) + r_2 \cdot z'(x)(X'X)^{-1}z(x)
\]
and assuming that the primary objective function is \(g\), via the \(\epsilon\)-constraint model we have
\[
\min_{x \in X} z'(x)(X'X)^{-1}z(x)
\]
subject to
\[
\begin{align*}
z'(x)\hat{b}_1 &= \tau_1 \\
z'(x)\hat{b}_2 &= \tau_2.
\end{align*}
\]

* Equivalent deterministic multiobjective P-model
\[
\min_{x \in X} \begin{pmatrix}
\frac{\tau_1 - z'(x)\hat{b}_1}{\sqrt{\sigma_{11} z'(x)(X'X)^{-1}z(x)}} \\
\frac{\tau_2 - z'(x)\hat{b}_2}{\sqrt{\sigma_{22} z'(x)(X'X)^{-1}z(x)}}
\end{pmatrix}.
\]
In this case the deterministic equivalent program via the weighting method is
\[
\min_{x \in X} \begin{pmatrix}
\frac{\tau_1 - z'(x)\hat{b}_1}{\sqrt{\sigma_{11} z'(x)(X'X)^{-1}z(x)}} + w_1 \left( \frac{\tau_2 - z'(x)\hat{b}_2}{\sqrt{\sigma_{22} z'(x)(X'X)^{-1}z(x)}} \right)
\end{pmatrix},
\]
and assuming that \(\hat{Y}_2(x, \hat{b}_2)\) is the primary objective function, the deterministic equivalent program via the \(\epsilon\)-constraint method is
\[
\min_{x \in X} \frac{\tau_2 - z'(x)\hat{b}_2}{\sqrt{\sigma_{22} z'(x)(X'X)^{-1}z(x)}}
\]
subject to
\[
\begin{align*}
\frac{\tau_1 - z'(x)\hat{b}_1}{\sqrt{\sigma_{11} z'(x)(X'X)^{-1}z(x)}} &= \tau_1.
\end{align*}
\]

* Equivalent deterministic multiobjective Kataoka model
\[
\min_{x \in X} \begin{pmatrix}
z'(x)\hat{b}_1 + \Phi^{-1}(\delta) \cdot \sqrt{\sigma_{11} z'(x)(X'X)^{-1}z(x)} \\
z'(x)\hat{b}_2 + \Phi^{-1}(\delta) \cdot \sqrt{\sigma_{22} z'(x)(X'X)^{-1}z(x)}
\end{pmatrix},
\]
Table 2
Comparison of the results of the proposed model and those derived by other methods.

<table>
<thead>
<tr>
<th>Stochastic programming method</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$F(x)$</th>
<th>$\bar{Y}_1(x)$</th>
<th>$\bar{Y}_2(x)$</th>
<th>$\text{Var} (\bar{Y}_1(x))$</th>
<th>$\text{Var} (\bar{Y}_2(x))$</th>
<th>$\text{Cov} (\bar{Y}_1(x), \bar{Y}_2(x))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chiao and Hamada [7]</td>
<td>1.000</td>
<td>1.000</td>
<td>-1.000</td>
<td>-</td>
<td>104.612</td>
<td>73.574</td>
<td>0.917</td>
<td>1.021</td>
<td>0.776</td>
</tr>
<tr>
<td>Hejazi et al. [10]</td>
<td>Distance Based$^b$</td>
<td>0.953</td>
<td>0.709</td>
<td>0.407</td>
<td>103.247</td>
<td>73.000</td>
<td>0.428</td>
<td>0.476</td>
<td>0.362</td>
</tr>
<tr>
<td></td>
<td>Robust E-model</td>
<td>1.000</td>
<td>0.707</td>
<td>0.483</td>
<td>103.332</td>
<td>73.000</td>
<td>0.470</td>
<td>0.523</td>
<td>0.397</td>
</tr>
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<td></td>
<td>Lexicographic (First)</td>
<td>1.000</td>
<td>0.707</td>
<td>0.483</td>
<td>103.000</td>
<td>73.000</td>
<td>0.470</td>
<td>0.523</td>
<td>0.397</td>
</tr>
<tr>
<td></td>
<td>$\tilde{Y}(F(x_i))$</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>104.865</td>
<td>70.453</td>
<td>0.131</td>
<td>0.146</td>
<td>0.111</td>
</tr>
<tr>
<td></td>
<td>Modified V-model</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>104.865</td>
<td>70.453</td>
<td>0.131</td>
<td>0.146</td>
<td>0.111</td>
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<tr>
<td>Multiobjective stochastic approaches</td>
<td>V-model</td>
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<td>0.000</td>
<td>0.000</td>
<td>104.865</td>
<td>70.453</td>
<td>0.131</td>
<td>0.146</td>
<td>0.111</td>
</tr>
<tr>
<td></td>
<td>Modified E-model$^d$</td>
<td>0.522</td>
<td>-1.000</td>
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<td>39.588</td>
<td>102.100</td>
<td>66.449</td>
<td>0.336</td>
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<td>Modified E-model($\epsilon$-constraint)</td>
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<td>0.707</td>
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<td>1.511</td>
<td>103.019</td>
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<td>P-model (Weighting method)</td>
<td>-0.349</td>
<td>1.000</td>
<td>0.548</td>
<td>-2.672</td>
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<td>74.630</td>
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<td>F-model ($\epsilon$-constraint)</td>
<td>0.910</td>
<td>-0.658</td>
<td>0.000</td>
<td>8.799</td>
<td>100.672</td>
<td>67.577</td>
<td>0.343</td>
<td>0.382</td>
</tr>
<tr>
<td></td>
<td>Kataoka$^e$ ($\epsilon$-constraint)</td>
<td>1.000</td>
<td>-1.000</td>
<td>1.000</td>
<td>74.989</td>
<td>99.039</td>
<td>65.405</td>
<td>0.917</td>
<td>1.021</td>
</tr>
<tr>
<td></td>
<td>Goal Programming</td>
<td>0.541</td>
<td>-1.000</td>
<td>0.851</td>
<td>67.296</td>
<td>101.780</td>
<td>66.006</td>
<td>0.556</td>
<td>0.619</td>
</tr>
</tbody>
</table>

$^a$ Objective function.
$^b$ Squared Euclidean distance.
$^c$ Where weights $w$ are considered deterministic.
$^d$ $\tau_1 = \tau_2 = 0.5$.
$^e$ $\delta = 0.95$.

The deterministic equivalent program via the weighting method is

$$
\min_{x \in X} \left\{ z'(x) \tilde{p}_1 + \Phi^{-1}(\delta) \sqrt{\sigma_{11}} z'(x)(xx')^{-1}z(x) \right\} + w_2 \left\{ z'(x) \tilde{p}_2 + \Phi^{-1}(\delta) \sqrt{\sigma_{22}} z'(x)(xx')^{-1}z(x) \right\}
$$

Now assuming that $\tilde{Y}_1(x, \tilde{p}_2)$ is the primary objective function, the deterministic equivalent program via the $\epsilon$-constraint method is

$$
\min_{x \in X} \left\{ z'(x) \tilde{p}_1 + \Phi^{-1}(\delta) \sqrt{\sigma_{11}} z'(x)(xx')^{-1}z(x) \right\}
$$

subject to

$$
z'(x) \tilde{p}_2 + \Phi^{-1}(\delta) \sqrt{\sigma_{22}} z'(x)(xx')^{-1}z(x) = \tau_1.
$$

Table 2 shows the solution of (7) by diverse multiobjective stochastic methods and other methods described in the literature. The results were computed using the commercial software Hyper LINGO/PC, Ver. 6.0, see [31]. The default optimisation methods used by LINGO to solve the nonlinear optimisation program is Generalised Reduced Gradient (GRG), see [32].

Note that while all of these optimisation techniques essentially provide the solution to the same practical problem, i.e. that of obtaining the critical value of the variables $x$, from a mathematical point of view and more precisely from the standpoint of mathematical programming, problem (9) as solved by Chiao and Hamada [7] and Hejazi et al. [10] and problem (11) examined in the present paper are not the same. Therefore, unless a reasonable basis for comparison is proposed, Table 2 should be taken simply as an example of different approaches and different solutions to the practical problem. From the latter, experts, researchers or decision makers can select the most suitable method for solving their own problem in terms of the particular context.

6. Conclusions

It is important to emphasise that even a comparison between the diverse techniques proposed in this paper must be made with appropriate reservations, since the solutions discussed refers to different decision-making criteria. For example, Table 2 shows that the methods designed to minimise the variance actually obtain a lower variance than the other solutions, but perhaps the optimum response variables are a little further from the target. Likewise, the minimum risk models provide more conservative solutions.
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References