Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics 14 (2017) 112–117
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

Some notes on the isolate domination in graphs

Nader Jafari Rad

Department of Mathematics, Shahrood University of Technology, Shahrood, Iran

Received 13 July 2015; received in revised form 21 January 2017; accepted 23 January 2017
Available online 6 February 2017

Abstract

A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S is an *isolate dominating set* if the induced subgraph $G[S]$ has at least one isolated vertex. The *isolate domination number* $\gamma_0(G)$ is the minimum cardinality of an isolate dominating set of G. In this paper we study the complexity of the isolate domination in graphs, and obtain several bounds and characterizations on the isolate domination number, thus answering some open problems.

Keywords: Domination; Isolate domination; Total domination; Complexity

1. Introduction

For notation and graph theory terminology, we in general follow [1,2]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N(v)$. The degree of v is $\deg_G(v) = |N_G(v)|$. If the graph G is clear from the context, we simply write $N(v)$ and $\deg(v)$ rather than $N_G(v)$ and $\deg_G(v)$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A vertex of degree one is called a *leaf* and its unique neighbor a *support* vertex. A *pendant edge* is an edge which one of its end-points is a leaf. A *star* of order $n \geq 3$ is a tree that has precisely one vertex that is not a leaf. A *claw-free* graph is a graph with no induced subgraph isomorphic to a star of order 4. A *double-star* is a tree that has precisely two vertices that are not leaves. We refer $S(a, b)$ as a double-star which its central vertices have degree a and b, respectively. For a subset S of vertices of G we denote by $G[S]$ the subgraph of G *induced* by S. For two subsets of vertices X and Y of $V(G)$, we denote by $G[X, Y]$ the subgraph of G induced by $X \cup Y$. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between pairs of vertices of G. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle contained in G.

Peer review under responsibility of Kalasalingam University.
E-mail address: n.jafarirad@gmail.com.

http://dx.doi.org/10.1016/j.akcej.2017.01.003

A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. We refer a dominating set of cardinality $\gamma(G)$ as a $\gamma(G)$-set. A dominating set S in a graph with no isolated vertex is called a *total dominating set* of G if $G[S]$ has no isolated vertex. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. We refer a total dominating set of cardinality $\gamma_t(G)$ as a $\gamma_t(G)$-set. A total dominating set S in G is called an *efficient total dominating set* if the open neighborhoods of the vertices of S form a partition for $V(G)$. For a subset S of vertices of G, and a vertex $x \in S$, we say that a vertex $y \notin S$ is an *external private neighbor of x with respect to S* if $N(y) \cap S = \{x\}$. We denote by $epn(x, S)$ the set of all external private neighbors of x respect to S.

Hamid and Balamurugan [3] initiated the study of *isolate domination* in graphs. A dominating set S is an *isolate dominating set* if the induced subgraph $G[S]$ has at least one isolated vertex. The *isolate domination number* $\gamma_0(G)$ is the minimum cardinality of an isolate dominating set of G. The concept of isolate domination was further studied, for example in [4–9]. Hamid et al. [3] showed that for a cubic graph G, $\gamma(G) \le \gamma_0(G) \le \gamma(G) + 1$. They presented several bounds, and properties for the isolate domination number, and proposed the following problem(s).

Problems: (1) Characterize cubic graphs G with $\gamma_0(G) = \gamma(G) + 1$.
(2) Characterize graphs G with $\gamma_0(G) = \gamma(G)$, or $\gamma_0(G) = \frac{n}{2}$.
(3) Find bounds for $\gamma_0(G)$.

In this paper we first study the complexity of the isolate domination number in graph by showing that the decision problem for this variant is NP-complete, even when restricted to bipartite graphs. We then answer all of the above problems. We present several bounds, and characterizations for the isolate domination number in a graph.

In the following we state some known results that we need for the next. The *corona graph* of a graph G, denoted by GoK_1, is the graph obtained from G by adding a pendant edge to every vertex of G.

Theorem 1 (*[1]*)**.** *For a graph G of order n with no isolated vertex, $\gamma(G) \le \frac{n}{2}$, with equality if and only if each component of G is a C_4 or the corona HoK_1 for any connected graph H.*

Lemma 2 (*[3]*)**.** *For paths and cycles of order n, $\gamma_0(P_n) = \gamma_0(C_n) = \lceil \frac{n}{3} \rceil$.*

Theorem 3 (*[2]*)**.** *If G is a graph of order n and with no isolated vertex then $\gamma_t(G) \ge \frac{n}{\Delta(G)}$.*

2. Complexity

In this section we show that the decision problem for the isolate domination is NP-complete, even when restricted to bipartite graphs. We use a transformation from the 3-SAT problem. A *truth assignment* for a set U of Boolean variables is a mapping $t : U \to \{T, F\}$. A variable u is said to be *true* (or *false*) under t if $t(u) = T$ (or $t(u) = F$). If u is a variable in U, then u and $\overline{u}$ are *literals* over U. The literal u is true under t if and only if the variable u is true under t, and the literal $\overline{u}$ is true if and only if the variable u is false. A *clause* over U is a set of literals over U, and it is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathcal{C}$ of clauses over U is satisfiable if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathcal{C}$. Such a truth assignment is called a satisfying truth assignment for $\mathcal{C}$. The 3-SAT problem is specified as follows.

3-SAT problem:

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.

Question: Is there a truth assignment for U that satisfies all the clauses in C?

Note that the 3-SAT problem was proven to be NP-complete in [10]. Consider the following decision problem.

Isolate dominating set (IDS):

Instance: A graph $G = (V, E)$ and a positive integer k.

Question: Does G have an isolate dominating set of size at most k?

Theorem 4. *The isolate domination problem is NP-complete for bipartite graphs.*

Proof. It is clear that the isolate domination problem belongs to NP, since it is easy to verify a "yes" instance of the isolate domination problem in polynomial time. We show the NP-hardness of the isolate domination problem by transforming the 3-SAT to it in polynomial time. Let $U = \{u_1, u_2, \ldots, u_n\}$, and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of the 3-SAT problem. We construct a graph G and an integer k such that $\mathcal{C}$ is satisfiable if and only if $\gamma_0(G) \le k$. The graph G is constructed as follows. For $i = 1, 2, \ldots, n$, let H_i be a 6-cycle $u_i v_i \overline{u_i} d_i b_i a_i u_i$ being consecutive vertices. For $i = 1, 2, \ldots, n$, corresponding to each variable $u_i \in U$, associate the graph H_i. Corresponding to each clause $C_j = \{x_j, y_j, z_j\} \in \mathcal{C}$, associate a single vertex c_j and add the edge-set $E_j = \{c_j x_j, c_j y_j, c_j z_j\}$ for $j = 1, 2, \ldots, m$. Let G be the resulted graph. Clearly G is a bipartite graph.

We show that $\mathcal{C}$ is satisfiable if and only if $\gamma_0(G) \le 2n$. Assume that $\mathcal{C}$ is satisfiable. Let $t: U \to \{T, F\}$ be a satisfying truth assignment for $\mathcal{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then we put the vertices u_i and d_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and a_i in D. Then it can be easily checked that D is an isolate dominating set for G of cardinality $2n$. Conversely, assume that S is an isolate dominating set for G of cardinality at most $2n$. Clearly $|V(H_i) \cap S| \ge 2$ for $i = 1, 2, \ldots, n$. Since $|S| \le 2n$, we obtain $|V(H_i) \cap S| = 2$ for $i = 1, 2, \ldots, n$, and $S \cap \{c_1, \ldots, c_m\} = \emptyset$. If $\{u_i, \overline{u_i}\} \subseteq S$ for some $i \in \{1, 2, \ldots, n\}$ then b_i is not dominated by S, a contradiction. Thus $|\{u_i, \overline{u_i}\} \cap S| \le 1$ for each $i = 1, 2, \ldots, n$. If $\{u_i, \overline{u_i}\} \cap S = \emptyset$ for some $i \in \{1, 2, \ldots, n\}$ then $\{b_i, v_i\} \subseteq S$, and then we replace $\{b_i, v_i\}$ by $\{u_i, d_i\}$. Thus we may assume that $|\{u_i, \overline{u_i}\} \cap S| = 1$ for each $i = 1, 2, \ldots, n$. Clearly for each c_j, $(j = 1, 2, \ldots, m)$ there is $i \in \{1, 2, \ldots, n\}$ such that c_j is dominated by some vertex in $S \cap V(H_i)$. Let $t: U \to \{T, F\}$ be a mapping defined by $t(u_i) = T$ if $u_i \in S$, and $t(u_i) = F$ if $\overline{u_i} \in S$. For each $j \in \{1, 2, \ldots, m\}$, there is an integer $i \in \{1, 2, \ldots, n\}$ such that c_j is dominated by $S \cap \{u_i, \overline{u_i}\}$. Assume that $u_i \in S$ and c_j is dominated by u_i. By the construction of G, the literal u_i is in the clause C_j. Then $t(u_i) = T$, which implies that the clause C_j is satisfied by t. Next assume that $\overline{u_i} \in S$ and c_j is dominated by $\overline{u_i}$. By the construction of G, the literal $\overline{u_i}$ is in the clause C_j. Then $t(u_i) = F$. Thus t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j. Hence $\mathcal{C}$ is satisfiable.

Since the construction of the p-reinforcement instance is straightforward from a 3-SAT instance, the size of the p-reinforcement instance is bounded above by a polynomial function of the size of 3-SAT instance. It follows that this is a polynomial transformation, as desired. □

3. On graphs G with $\gamma_0(G) = \gamma(G)$

In this section we derive some properties of graphs G with $\gamma_0(G) = \gamma(G)$. We begin with the following obvious observation.

Observation 5. *For a graph G, $\gamma_0(G) = \gamma(G)$ if and only if there is a $\gamma(G)$-set S such that $G[S]$ has some isolated vertex.*

The following follows immediately from Observation 5.

Corollary 6. *For a graph G, $\gamma_0(G) \ne \gamma(G)$ if and only if every minimum dominating set is a total dominating set.*

We say that $\gamma(G)$ and $\gamma_t(G)$ are *strongly equal*, denoted by $\gamma(G) \equiv \gamma_t(G)$ if every $\gamma(G)$-set is also a $\gamma_t(G)$-set. The strong equality between two domination parameters was first introduced by Haynes et al. [11]. From Corollary 6 we obtain the following.

Theorem 7. *For a graph G, $\gamma_0(G) = \gamma(G)$ if and only if $\gamma(G) \not\equiv \gamma_t(G)$.*

Note that in [11] all trees T with $\gamma(T) \equiv \gamma_t(T)$ was constructively characterized. In the next proposition we state some properties of graphs G with $\gamma_0(G) \ne \gamma(G)$. The proof is straightforward, and thus is omitted.

Proposition 8. *If $\gamma_0(G) \ne \gamma(G)$, then $\gamma(G) = \gamma_t(G) \le \frac{n}{3}$, and $|epn(x, S)| \ge 2$ for any $\gamma(G)$-set S, and any vertex $x \in S$.*

The following follows immediately from Proposition 8.

Proposition 9. *If $diam(G) = 2$ then $\gamma_0(G) = \gamma(G)$.*

Theorem 10. *For any claw-free graph G, $\gamma_0(G) = \gamma(G)$.*

Proof. Let G be a claw-free graph. Suppose that $\gamma_0(G) \neq \gamma(G)$. Let S be a $\gamma(G)$-set. By Proposition 8, $\gamma(G) = \gamma_t(G)$, and $|epn(u, S)| \geq 2$ for any vertex $u \in S$. Let x be a vertex of S with $\deg_{G[S]}(x) = 1$, and $y \in N(x) \cap S$. If $G[epn(x, S)]$ is a complete graph then $(S - \{x\}) \cup \{w\}$ is a $\gamma(G)$-set which is not a total dominating set, where $w \in epn(x, S)$. This contradicts Corollary 6. Thus $G[epn(x, S)]$ is not a complete graph. Since y is adjacent to no vertex of $epn(x, S)$, and $|epn(x, S)| \geq 2$, we find that G contains a claw, a contradiction. □

As noted, it is shown in [3] that for a cubic graph G, $\gamma(G) \leq \gamma_0(G) \leq \gamma(G) + 1$. In the following we answer Problem (1).

Theorem 11. *For any cubic graph G, $\gamma_0(G) = \gamma(G)$.*

Proof. Let G be a cubic graph of order n. Suppose that $\gamma_0(G) \neq \gamma(G)$. Then $\gamma_0(G) = \gamma(G)+1$. Let S be a $\gamma(G)$-set. By Theorem 3 and Proposition 8, $\gamma(G) = \gamma_t(G) = |S| = \frac{n}{3}$. By Theorem 7, $S = \frac{\gamma(G)}{2}K_2$, and S is an efficient total dominating set. Thus $|epn(x, S)| = 2$ for any $x \in S$. For each $x \in S$, let $epn(x, S) = \{x^1, x^2\}$. Clearly $x^1 \notin N(x^2)$ for any $x \in S$, since any minimum set is a total dominating set. Let $x_1 \in S$, and $x_2^2 \in N(x_1^2)$. Then x_2^2 is adjacent to a unique vertex $x_2 \in S$. Let $S_1 = (S - \{x_1, x_2\}) \cup \{x_1^1, x_2^2\}$. Clearly $G[S_1]$ has some isolated vertex. Assume that S_1 is not an isolate dominating set for G. Then x_2^1 is not dominated by S_1. Let $x_3^1 \in N(x_2^1)$. Then x_3^1 is dominated by a vertex $x_3 \in S_1$. Let $S_2 = (S_1 - \{x_3\}) \cup \{x_3^1\}$. Assume that S_2 is not an isolate dominating set. Then x_3^2 is not dominated by S_2. Let $x_4^2 \in N(x_3^2)$. Then x_4^2 is dominated by a vertex $x_4 \in S_2$. Let $S_3 = (S_2 - \{x_4\}) \cup \{x_4^2\}$. Continuing this process, we obtain a set S_k such that $|S_k| = |S|$ and S_k is an isolate dominating set. Consequently, $\gamma_0(G) = \gamma(G)$. □

4. Bounds

In this section we provide several bounds and characterization for the isolate domination number of a graph.

Theorem 12. *For any graph G, $\gamma_0(G) \leq \gamma(G) - 1 + p_0$, where $p_0 = \min\{|epn(x, D)| : D$ is a $\gamma(G)$-set, $x \in D$ and $epn(x, D) \neq \emptyset\}$.*

Proof. Let D be a $\gamma(G)$-set. Clearly $p_0 \geq 1$. The result is obvious if $G[D]$ has an isolated vertex. Thus assume that $G[D]$ has no isolated vertex. Let $x_0 \in D$ be a vertex such that $0 < |epn(x_0, D)| \leq |epn(x, D)|$ for any vertex $x \in D$ with $epn(x, D) \neq \emptyset$. Let X be a minimum independent dominating set for $epn(x_0, D)$. Then $(D - \{x_0\}) \cup X$ is an isolate dominating set for G, as desired. □

It is clear that in the above proof, $|epn(x_0, D)| \leq \lfloor\frac{n-\gamma(G)}{\gamma(G)}\rfloor = \lfloor\frac{n}{\gamma(G)}\rfloor - 1$. Thus we obtain the following.

Corollary 13. *For any graph G of order n, $\gamma_0(G) \leq \max\{\gamma(G), \gamma(G) - 2 + \lfloor\frac{n}{\gamma(G)}\rfloor\}$.*

Furthermore, since $|epn(x_0, D)| \leq \Delta(G) - 1$, for any connected graph G of order $n \geq 3$, $\gamma_0(G) \leq \gamma(G) + \Delta(G) - 2$. We next answer the second part of Problem (2).

It can be easily seen that if each component of G is a C_4 or the corona HoK_1 for any connected graph H, then $\gamma_0(G) = \gamma(G)$. Let $\mathcal{E}_n$ be the class of all bipartite graphs G of even order n such that G can be obtained from a double-star $S(\frac{n}{2}, \frac{n}{2})$ by adding edges between X and Y such that $\gamma_0(G[X, Y]) \geq \frac{n}{2} - 1$, where X is the set of all leaves of a partite set of $S(\frac{n}{2}, \frac{n}{2})$, and Y is the set of all leaves of the other partite set of $S(\frac{n}{2}, \frac{n}{2})$.

Theorem 14. *If G is a graph of order $n \geq 2$ and with no isolated vertex, then $\gamma_0(G) \leq \frac{n}{2}$. Equality holds if and only if $G \in \mathcal{E}n$ or each component of G is a C_4 or the corona HoK_1 for any connected graph H.*

Proof. Let G be a graph of order n. If $\gamma_0(G) = \gamma(G)$ then clearly by Theorem 1, $\gamma_0(G) = \gamma(G) \leq \frac{n}{2}$ with equality if and only if each component of G is a C_4 or the corona HoK_1 for any connected graph H. Thus assume that $\gamma_0(G) > \gamma(G)$. Clearly $\gamma(G) \geq 2$.

First assume that $\gamma(G) > 2$. If $\gamma_0(G) > \frac{n}{2}$ then by Corollary 13, $\frac{n}{2} < \gamma_0(G) \leq \gamma(G) - 2 + \frac{n}{\gamma(G)}$, and a simple calculation implies that $\gamma(G) > \frac{n}{2}$, a contradiction. Thus $\gamma_0(G) \leq \frac{n}{2}$. Assume that equality holds. Let S be a $\gamma(G)$-set. By Proposition 8, $|S| = \gamma(G) = \gamma_t(G) \leq \frac{n}{3}$. By Proposition 8, $|epn(x, S)| \geq 2$. Assume that there is a vertex $x \in S$ such that $|epn(x, S)| = 2$. Then $(S - \{x\}) \cup epn(x, S)$ is an isolate dominating set for G, implying that $\frac{n}{2} = \gamma_0(G) \le |S| + 1 \le \frac{n}{3} + 1$. This implies that $n \le 6$. Since by Proposition 8, $|epn(x, S)| \ge 2$ for any vertex $x \in S$, we obtain $|S| = \gamma(G) = 2$, a contradiction. Thus assume that $|epn(x, S)| \ge 3$ for any $x \in S$. Then $|S| = \gamma(G) = \gamma_t(G) \le \frac{n}{4}$. If $p_0 \le \frac{n}{4}$ then by Theorem 12, $\frac{n}{2} = \gamma_0(G) \le \gamma(G) - 1 + p_0 \le \frac{n}{2} - 1$, a contradiction. Thus $p_0 \ge \frac{n}{4} + 1$. Thus $|epn(x, S)| \ge \frac{n}{4} + 1$ for any $x \in S$. This implies that $|S| \le 3$, and thus $|S| = 3$. Let $S = \{x, y, z\}$. Without loss of generality assume that $|epn(x, S)| \le |epn(y, S)| \le |epn(z, S)|$. Let X be an independent dominating set for $epn(x, S)$. Then $|X| \le |epn(x, S)| \le \frac{n-3}{3}$. Now $X \cup \{y, z\}$ is an isolate dominating set for G, implying that $\frac{n}{2} \le 2 + \frac{n-3}{3}$. This inequality implies that $n \le 6$. This is a contradiction, since $|epn(u, S)| \ge 3$ for any $u \in S$.

Next assume that $\gamma(G) = 2$. By Proposition 8, $\gamma_t(G) = 2$. Let $S = \{x, y\}$ be a $\gamma(G)$-set. Let X be a minimum independent dominating set in $G[epn(x, S)]$, and Y be a minimum independent dominating set in $G[epn(y, S)]$. Assume that $|X| \le |Y|$. Then $\{y\} \cup X$ is an isolate dominating set of cardinality at most $\frac{n}{2}$, and so $\gamma_0(G) \le \frac{n}{2}$. Assume that $\gamma_0(G) = \frac{n}{2}$. Then $|X| = |Y| = \frac{n-2}{2}$, $X = N(x) - \{y\}$ and $Y = N(y) - \{x\}$. Assume that $\gamma_0(G[X, Y]) < \frac{n}{2} - 1$. Let D be a $\gamma_0(G[X, Y])$-set. If $D \cap X \ne \emptyset$ and $D \cap Y \ne \emptyset$ then D is an isolate dominating set for G, a contradiction. Thus assume that $D \cap Y = \emptyset$. Then $(D \cap X) \cup \{y\}$ is an isolate dominating set for G, a contradiction. Thus $\gamma_0(G[X, Y]) \ge \frac{n}{2} - 1$. Consequently $G \in \mathcal{E}_n$. Conversely it is straightforward to see that $\gamma_0(G) = \frac{n}{2}$ if $G \in \mathcal{E}_n$. □

Theorem 15. *If G is a graph of order n with $\delta(G) \ge 2$ and $g(G) \ge 5$ then $\gamma_0(G) \le \left\lceil \frac{n - \lfloor \frac{g(G)}{3} \rfloor}{2} \right\rceil$. This bound is sharp.*

Proof. Let H be a graph obtained from G by removing the vertices and edges of a shortest cycle C. We show that any vertex of $V(H)$ has at most one neighbors in C. Let $v \in V(H)$. Suppose that $a, b \in N_G(v) \cap V(C)$. Clearly $d_G(a, b) \ge 3$ and $d_C(a, b)] \ge 3$, since $g(G) \ge 5$. Let $aa_1a_2 \ldots a_{k-1}b$ be the path on C from a to b. Then replacing this path with avb produce a cycle C' in G of length less than $g(G)$, a contradiction. Thus any vertex of $V(H)$ has at most one neighbors in C. This implies that H has no isolated vertex, since $\delta(G) \ge 2$. By Theorem 14, $\gamma_0(H) \le \lceil \frac{n-g(G)}{2} \rceil$. Let D_1 be a $\gamma_0(H)$-set. If D_1 is an isolate dominating set for G, then $\gamma_0(G) \le \lceil \frac{n-g(G)}{2} \rceil \le \lceil \frac{n-\lfloor \frac{g(G)}{3} \rfloor}{2} \rceil$, as desired. Thus assume that D_1 is not an isolate dominating set for G. Let D_2 be a $\gamma_0(C)$-set. Clearly by Lemma 2, $\gamma_0(C) = \lceil \frac{g(G)}{3} \rceil$. Furthermore, for any $\gamma_0(C)$-set S, and any vertex $x \in S$, there is a vertex $y \in S$ such that $d(x, y) \le 3$. If $D_1 \cup D_2$ is an isolate dominating set for G then

$$\gamma_0(G) \le |D_1| + |D_2| \le \left\lceil \frac{n - g(G)}{2} \right\rceil + \left\lceil \frac{g(G)}{3} \right\rceil \le \left\lceil \frac{n - \lfloor \frac{g(G)}{3} \rfloor}{2} \right\rceil .$$

Thus assume that $D_1 \cup D_2$ is not an isolate dominating set for G. Then any vertex of D_2 is adjacent to a vertex of D_1 in G. Let D_3 be obtained from D_2 by rotating D_2 on C, clock-wise, such that any vertex is replaced by its next successive vertex on C. If $D_1 \cup D_3$ is an isolate dominating set for G then, as before, $\gamma_0(G) \le \lceil \frac{n-\lfloor \frac{g(G)}{3} \rfloor}{2} \rceil$, since $|D_2| = |D_3|$. Thus assume that $D_1 \cup D_3$ is not an isolate dominating set for G. Then any vertex of D_3 is adjacent to a vertex of D_1 in G. Now let D_4 be obtained from D_3 by rotating D_3 on C, clock-wise, such that any vertex is replaced by its next successive vertex on C. Then $D_1 \cup D_4$ is an isolate dominating set for G, since D_1 is not an isolate dominating set for G. This completes the proof of the bound. To see the sharpness consider cycles of length at least five. □

We finish by providing a *Nordhaus–Gaddum type* inequality for the isolate domination number.

Theorem 16. *For any graph G of order n, $\gamma_0(G) + \gamma_0(\overline{G}) \le n + 1$, with equality if and only if $G = K_n$ or $G = \overline{K_n}$.*

Proof. If G has an isolated vertex then $\gamma_0(\overline{G}) = 1$, and so $\gamma_0(G) + \gamma_0(\overline{G}) \le n + 1$, since $\gamma_0(G) \le n$. Thus assume that G has no isolated vertex, and similarly $\overline{G}$ has no isolated vertex. Now Theorem 14 implies that $\gamma_0(G) + \gamma_0(\overline{G}) \le \frac{n}{2} + \frac{n}{2} = n$. Assume that equality holds. Then by the above proof, G or $\overline{G}$ have some isolated vertices. Assume that G has some isolated vertex. Then clearly $\gamma_0(\overline{G}) = 1$. This means that $\gamma_0(G) = n$. We conclude that $G = K_n$. The converse is obvious. □

References

[1] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.
[2] M.A. Henning, A. Yeo, Total Domination in Graphs, in: Springer Monographs in Mathematics, 2013.
[3] I. Sahul Hamid, S. Balamurugam, Isolate domination in graphs, Arab J. Math. Sci. 22 (2015) 232–241.
[4] B.H. Arriola, Isolate domination in the join and corona of graphs, Appl. Math. Sci. 9 (2015) 1543–1549.
[5] I.S. Hamid, S. Balamurugan, Isolate domination number and maximum degree, Bull. Internat. Math. Virtual Inst. 3 (2013) 127–133.
[6] I. Sahul Hamid, S. Balamurugam, A. Navaneethakrishnan, A note on isolate domination, Electron. J. Graph Theory Appl. 4 (2016) 94–100.
[7] I.S. Hamid, S. Balamurugan, Isolate domination in unicyclic graphs, Internat. J. Math. Soft Comput. 3 (2013) 79–83.
[8] I.S. Hamid, S. Balamurugan, Extended chain of domination parameters in graphs, ISRN Combin. (2013) http://dx.doi.org/10.1155/2013/792743.
[9] I.S. Hamid, S. Balamurugan, ertex criticality with respect to isolate domination, VDiscrete Math. Algorithm. Appl. 07 (2015) 1550010. 9 pages. http://dx.doi.org/10.1142/S179383091550010X.
[10] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.
[11] T.W. Haynes, M.A. Henning, P.J. Slater, Strong equality of domination parameters in trees, Discrete Math. 260 (1–3) (2003) 77–87.