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Note

A new lower bound on the double domination number of a graph

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ABSTRACT

A subset S of vertices of a graph G is a double dominating set of G if every vertex in $V(G)-S$ has at least two neighbors in S and every vertex of S has a neighbor in S. The double domination number $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. Chellali (2006) showed that if T is a nontrivial tree of order n, with ℓ leaves and s support vertices, then $\gamma_{\times 2}(T) \geq (2n+\ell-s+2)/3$. In this paper we generalize the above lower bound for any connected graph. We show that if G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles, ℓ leaves and s support vertices, then $\gamma_{\times 2}(G) \geq (2n+\ell-s+2)/3 - 2k/3$. We also characterize all graphs achieving equality for this new bound.

1. Introduction

For notation and terminology not given here we refer to [5]. Let $G=(V,E)$ be a graph with vertex set V of order n and edge set E. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of v is $\deg(v) = |N(v)|$. A vertex of degree one is referred as a *leaf* and its unique neighbor is called a *support vertex*. The set of all leaves of a graph G is denoted by $L(G)$, and the set of all support vertices of a graph G is denoted by $S(G)$. A *strong support vertex* is a support vertex adjacent to at least two leaves, while a *weak support vertex* is a support vertex adjacent to precisely one leaf. A *cactus graph* is a graph such that no pair of cycles have a common edge. A subset $S \subseteq V$ is a *dominating set* of G if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

A subset S of vertices of a graph G is a *double dominating set*, abbreviated DDS, of G if every vertex in $V(G)-S$ has at least two neighbors in S and every vertex of S has a neighbor in S, that is, $|N[v] \cap S| \geq 2$ for any vertex $v \in V(G)$. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. A double dominating set of G with minimum cardinality is called a $\gamma_{\times 2}(G)$-set. Double domination was introduced by Harary and Haynes [4] and further studied in, for example, [1–3,6].

Observation 1 (*Chellali [2]*). *Every DDS of a graph contains all its leaves and support vertices.*

Chellali [2] showed that if T is a tree of order n with ℓ leaves and s support vertices, then $\gamma_{\times 2}(G)$ is bounded below by $(2n+\ell-s+2)/3$. He then characterized trees achieving equality for the above bound. For this purpose he introduced a family of trees as follows. Let $\mathcal{G}_0$ be the class of all trees $T = T_k$ that can be obtained as follows. Let $T_1 = P_2 = uv$ and $A(T_1) = \{u, v\}$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

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Operation $\mathcal{O}_1$: Attach a vertex z by joining it to any support of T_i. Let $A(T_{i+1}) = A(T_i) \cup \{z\}$.

Operation $\mathcal{O}_2$: Attach a path $P_3 = abc$ by joining c to any vertex d of $A(T_i)$ with the condition that if d is a leaf of T_i then its support vertex is not strong in T_i. Let $A(T_{i+1}) = A(T_i) \cup \{a, b\}$.

Theorem 2 (*Chellali [2]*). *If T is a nontrivial tree of order n, with ℓ leaves and s support vertices, then $\gamma_{\times 2}(T) \geq (2n+\ell-s+2)/3$, with equality if and only if $T \in \mathcal{G}_0$.*

In this paper we generalize the above result. We present a lower bound for the double domination number of any connected graph G. We show that if G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles, ℓ leaves and s support vertices, then $\gamma_{\times 2}(G) \geq (2n + \ell - s + 2)/3 - 2k/3$, and we characterize all graphs achieving equality for this new bound.

For a graph G, we denote by $n(G)$, $s(G)$, $\ell(G)$ and $k(G)$, the order, the number of support vertices, the number of leaves and the number of cycles of G, respectively.

2. Families of graphs

For a graph G, a $\gamma_{\times 2}(G)$-set S is called a *special* $\gamma_{\times 2}(G)$*-set* if S contains at least two vertices x and y namely *special vertices* of S such that x and y are joined by a unique path of G, and if x or y is a leaf of G then its support vertex is not a strong vertex in G.

For any positive integer k, we define a sequence $H_0, H_1, \ldots, H_k$ of graphs as a *special sequence* as follows. Let H_0 be a tree obtained from a path $P_2 : xy$ and k 3-path $P^i : a^i b^i c^i$, $i = 1, 2, \ldots, k$, by joining x to each a^i, $i = 1, 2, \ldots, k$. For $i = 1, 2, \ldots, k$, we build the graph H_i, recursively, from H_{i-1} as follows. Let H_i be obtained from a H_{i-1} by adding a new vertex and joining it to both b^i and c^i.

Remark 3. It is easy to see that $H_0 \in \mathcal{G}_0$, and $S_0 = V(H_0) - \bigcup_{j=1}^{k}\{a^j\}$ is a $\gamma_{\times 2}(H_i)$-set for each $0 \leq i \leq k$. Moreover, for $i = 1, 2, \ldots, k$, b^i and c^i are two special vertices of H_{i-1}, and thus S_0 is a special $\gamma_{\times 2}(H_{i-1})$-set.

We now introduce some families of graphs. Let $\mathcal{G}_0$ be the families of trees described in Section 1. For $i = 1, \ldots, k$, we construct a family $\mathcal{G}_i$ from $\mathcal{G}_{i-1}$, recursively, by the following Procedure.

- **Procedure A**: For each i with $1 \leq i \leq k$, let $\mathcal{G}_i$ be the family of all graphs G_i such that G_i can be obtained from a graph $G_{i-1} \in \mathcal{G}_{i-1}$ with a special $\gamma_{\times 2}(G_{i-1})$-set S_{i-1} by adding a new vertex and joining it to precisely two special vertices of S_{i-1}. Note that the existence of a graph $G_{i-1} \in \mathcal{G}_{i-1}$ with a special $\gamma_{\times 2}(G_{i-1})$-set S_{i-1} is guaranteed, since the graph H_{i-1} described in Remark 3 is one of such graphs.

The following observation follows from the definitions.

Observation 4. *For $k \geq 0$, every graph $G \in \mathcal{G}_k$ contains exactly k cycles.*

It is also worth noting that any graph $G \in \mathcal{G}_k$ for $k \geq 0$ is a cactus graph.

3. New lower bound

In this section we present our main result. We give a lower bound for the double domination number of a connected graph G in terms of the number of cycles of G, and then characterize all connected graphs achieving equality for the proposed bound.

Theorem 5. *If G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles, ℓ leaves and s support vertices, then $\gamma_{\times 2}(G) \geq (2n + \ell - s + 2)/3 - 2k/3$, with equality if and only if $G \in \mathcal{G}_k$.*

Proof. Let G be a connected graph of order n, with $k \geq 0$ cycles, ℓ leaves and s support vertices. We use an induction on k to show that $\gamma_{\times 2}(G) \geq (2n + \ell - s + 2)/3 - 2k/3$ with equality if and only if $G \in \mathcal{G}_k$. For the base step of the induction let $k = 0$. Then G is a tree, and the result follows by Theorem 2. Assume the result holds for all connected graphs G' of order n' with $0 < k' < k$ cycles l' leaves and s' support vertices (that is, $\gamma_{\times 2}(G') \geq (2n' + \ell' - s' + 2)/3 - 2k'/3$, with equality if and only if $G' \in \mathcal{G}_{k'}$). Now consider the connected graph G of order n with $k \geq 1$ cycles.

We first show that $\gamma_{\times 2}(G) \geq (2n + \ell - s + 2)/3 - 2k/3$. Let S be a $\gamma_{\times 2}(G)$-set, and $C = u_1 u_2 \ldots u_r u_1$ be a cycle of G. If $\{u_1, u_2, \ldots, u_r\} \subseteq S$, then S is a double dominating set of the graph $G' = G - u_1u_2$, and thus by the inductive hypothesis, $|S| \geq \gamma_{\times 2}(G') \geq (2n + \ell(G') - s(G') + 2)/3 - 2k(G')/3$. Clearly $l - s \leq \ell(G') - s(G')$ and $k(G') \leq k - 1$. Thus $\gamma_{\times 2}(G) \geq (2n + \ell - s + 2)/3 - 2(k-1)/3 > (2n + \ell - s + 2)/3 - 2k/3$. Next assume that $u_j \notin S$ for some $1 \leq j \leq r$. By Observation 1, u_j is not a support vertex of G. Let $G'_1, G'_2, \ldots, G'_w$ be the components of $G - u_j$. Clearly $S \cap V(G'_i)$ is a double dominating set for G'_i, for each $1 \leq i \leq w$. Furthermore, $n(G'_i) \geq 2$, for each $1 \leq i \leq w$, since u_j is not a support vertex of G. Thus, by the inductive hypothesis, $|S| \geq \Sigma_{i=1}^{w}((2n(G'_i)+\ell(G'_i)-s(G'_i)+2)/3-2k(G'_i)/3)$. Observe that $\ell - s \leq \Sigma_{i=1}^{w}\ell(G'_i)-s(G'_i)$ and $k - 1 \geq \Sigma_{i=1}^{w}k(G'_i)$. Thus $|S| \geq (2(n-1) + \ell - s + 2w)/3 - 2(k-1)/3 \geq (2n + \ell - s + 2)/3 - 2k/3$, and consequently $\gamma_{\times 2}(G) \geq (2n+\ell-s+2)/3-2k/3$. (Note that it is easy to see that if $w \geq 2$, then we have $\gamma_{\times 2}(G) > (2n+\ell-s+2)/3-2k/3$).

We next show that $\gamma_{\times 2}(G) = (2n+\ell-s+2)/3-2k/3$ if and only if $G \in \mathcal{G}_k$. Assume that $\gamma_{\times 2}(G) = (2n+\ell-s+2)/3-2k/3$. Let S be a $\gamma_{\times 2}(G)$-set, and $C = u_1u_2\ldots u_r u_1$ be a cycle of G. According to the first part of the proof, we have $\{u_1, u_2, \ldots, u_r\} \nsubseteq S$. Thus there is an integer j with $1 \leq j \leq r$ such that $u_j \notin S$. Suppose that $\deg_G(u_j) \geq 3$. Clearly u_j is not a support vertex in G by Observation 1. Let $G' = G - u_j$. According to the first part of the proof G' contains only one connected component. Notice that $S \cap G'$ is a double dominating set of the graph G'. By the inductive hypothesis, $|S| \geq (2n(G')+\ell(G')-s(G')+2)/3-2k(G')/3$. Clearly $l-s \leq l(G')-s(G')$ and $k-2 \geq k(G')$. Thus $|S| \geq (2(n-1)+\ell-s+2)/3-2(k-2)/3 \geq (2n+\ell-s)/3-2(k-2)/3$ and so $\gamma_{\times 2}(G) \geq (2n+\ell-s+2)/3-2k/3+2/3$, a contradiction. Thus $\deg_G(u_j) = 2$. Evidently, $\{u_{j-1}, u_{j+1}\} \subseteq S$, since S is double dominating set of G and $u_j \notin S$. Let $G' = G - u_j$. Clearly S is a double dominating set of the graph G'. Then by the inductive hypothesis, $|S| \geq (2n(G')+\ell(G')-s(G')+2)/3-2k(G')/3$. Suppose that $|S| > (2n(G')+\ell(G')-s(G')+2)/3-2k(G')/3$. Clearly $\ell-s \leq (\ell(G')-s(G'))$ and $k(G') \leq k-1$. Then $|S| > (2(n-1)+\ell-s+2)/3-2(k-1)/3 \geq (2n+\ell-s+2)/3-2k/3$ and so $\gamma_{\times 2}(G) > (2n+\ell-s+2)/3-2k/3$, a contradiction. Thus $|S| = (2n(G')+\ell(G')-s(G')+2)/3-2k(G')/3$. By the inductive hypothesis, $G' \in \mathcal{G}_{k(G')}$. Suppose that u_{j-1} and u_{j+1} are joined by at least two paths of G'. Then clearly $k(G') \leq k-2$. Now $|S| \geq (2n(G')+\ell(G')-s(G')+2)/3-2(k-2)/3$ and so $\gamma_{\times 2}(G) \geq (2n+\ell-s+2)/3-2k/3+2/3$, a contradiction. Thus u_{j-1} and u_{j+1} are joined by a unique path of G'. Thus clearly $k(G') = k-1$, and so $G' \in \mathcal{G}_{k-1}$. If $\{u_{j-1}, u_{j+1}\} \cap L(G') = \emptyset$, then u_{j-1} and u_{j+1} are two special vertices of S and so S is a special $\gamma_{\times 2}(G')$-set. Thus G is obtained from G' and u_j by Procedure A, and consequently $G \in \mathcal{G}_k$, as desired. Thus assume that $\{u_{j-1}, u_{j+1}\} \cap L(G') \neq \emptyset$. Assume that $u_{j+1} \in L(G')$. Then clearly u_{j+2} is a support vertex of G'. Suppose that u_{j+2} is a strong support vertex of G'. Then clearly $\ell - s < \ell(G') - s(G')$ and so $|S| = (2n(G')+\ell(G')-s(G')+2)/3-2(k-1)/3 > (2(n-1)+\ell-s+2)/3-2(k-1)/3 = (2n+\ell-s+2)/3-2k/3$. Thus $\gamma_{\times 2}(G) > (2n+\ell-s+2)/3-2k/3$, a contradiction. We deduce that u_{j+2} is not a strong support vertex of G'. Similarly u_{j-2} is not a strong support vertex of G' if $u_{j-1} \in L(G')$. Thus u_{j-1} and u_{j+1} are two special vertices of S and so S is a special $\gamma_{\times 2}(G')$-set. Consequently, G is obtained from G' by adding the vertex u_j according to the Procedure A. Consequently, $G \in \mathcal{G}_k$.

For the converse let $G \in \mathcal{G}_k$. Thus G is obtained from a graph $G' \in \mathcal{G}_{k-1}$, by the Procedure A. Let S' be the special $\gamma_{\times 2}(G')$-set that used to produce G. Notice that G' contains exactly $k-1$ cycles by Observation 4. By the inductive hypothesis $|S'| = (2n(G')+\ell(G')-s(G')+2)/3-2(k-1)/3$. Clearly $\ell-s = \ell(G')-s(G')$ and thus $|S'| = (2n+\ell-s+2)/3-2k/3$. Evidently, S' is a double dominating set of G. Since by Observation 4, G contains exactly k cycles, by the first part of the proof, $\gamma_{\times 2}(G) = (2n+\ell-s+2)/3-2k/3$. ■

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