The Hamiltonian Connectivity of Alphabet Supergrid Graphs*

Ruo-Wei Hung[1,*], Fatemeh Keshavarz-Kohjerdi[2], Chuan-Bi Lin[3], and Jong-Shin Chen[3]

***Abstract*—The Hamiltonian path problem on general graphs is well-known to be NP-complete. In the past, we have proved it to be also NP-complete for supergrid graphs. A graph is called Hamiltonian connected if there exists a Hamiltonian path between any two distinct vertices in it. Determining whether a supergrid graph is Hamiltonian connected is clear to be NP-complete. Recently, we proved the Hamiltonian connectivity of some special supergrid graphs, including rectangular, triangular, parallelogram, and trapezoid. In this paper, we will study the Hamiltonian connectivity of alphabet supergrid graphs. There are 26 types of alphabet supergrid graphs in which every capital letter is represented by a type of alphabet supergrid graphs. We will prove *L*-, *C*-, *F*-, *E*-, *N*-, and *Y*-alphabet supergrid graphs to be Hamiltonian connected. The Hamiltonian connectivity of the other alphabet supergrid graphs can be verified similarly. The Hamiltonian connected property of alphabet supergrid graphs can be applied to compute the minimum stitching trace of computerized embroidery machines during the sewing process.**

***Index Terms*—Hamiltonian connectivity, alphabet supergrid graphs, shaped supergrid graphs, computerized embroidery machines.**

I. Introduction

A *Hamiltonian path* (resp., *cycle*) of a graph is a simple path (resp., cycle) in which each vertex of the graph appears exactly once. The *Hamiltonian path* (resp., *cycle*) *problem* is to determine whether or not a graph contains a Hamiltonian path (resp., cycle). A graph G is said to be *Hamiltonian* if it contains a Hamiltonian cycle, and is called *Hamiltonian connected* if for each pair of distinct vertices u and v of G, there exists a Hamiltonian path between u and v in G. The Hamiltonian path and cycle problems have numerous applications in different areas, including establishing transport routes, production launching, the on-line optimization of flexible manufacturing systems [1], computing the perceptual boundaries of dot patterns [40], pattern recognition [2], [42], [45], DNA physical mapping [14], fault-tolerant routing for 3D network-on-chip architectures [9], etc. It is well known that the Hamiltonian path and cycle problems are NP-complete for general graphs [11], [28]. The same holds true for bipartite graphs [35], split graphs [12], circle graphs [8], undirected path graphs [3], grid graphs [27], triangular grid graphs [13], and supergrid graphs [17]. In the literature, there are many studies for the Hamiltonian connectivity of interconnection networks, including WK-recursive network [10], recursive dual-net [37], hypercomplete network [5], alternating group graph [29], arrangement graph [39], augmented hypercube [16], generalized base-b hypercube [23], hyercube-like network [41], twisted cube [25], crossed cube [24], Möbius cube [7], folded hypercube [15], and enhanced hypercube [38]. In this paper, we will verify the Hamiltonian connectivity of alphabet supergrid graphs.

The *two-dimensional integer grid graph* G^∞ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if the (Euclidean) distance between them is equal to 1. The *two-dimensional triangular grid graph* T^∞ is an infinite graph obtained from G^∞ by adding all edges on the lines traced from up-left to down-right. A *grid graph* is a finite, vertex-induced subgraph of G^∞. For a node v in the plane with integer coordinates, let v_x and v_y represent its x and y *coordinates*, respectively, denoted by $v = (v_x, v_y)$. If v is a vertex in a grid graph, then its possible adjacent vertices include (v_x, v_y-1), (v_x-1, v_y), (v_x+1, v_y), and (v_x, v_y+1). A *triangular grid graph* is a finite, vertex-induced subgraph of T^∞. If v is a vertex in a triangular grid graph, then its possible neighboring vertices include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, and $(v_x + 1, v_y + 1)$. Thus, triangular grid graphs contain grid graphs as subgraphs. For example, Fig. 1(a) and Fig. 1(b) depict a grid graph and a triangular graph, respectively. The triangular grid graphs defined above are isomorphic to the original triangular grid graphs in [13] but these graphs are different when considered as geometric graphs. By the same construction of triangular grid graphs obtained from grid graphs, we defined a new class of graphs, namely *supergrid graphs*, in [17]. The *two-dimensional supergrid graph* S^∞ is an infinite graph obtained from T^∞ by adding all edges on the lines traced from up-right to down-left. That is, S^∞ is the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if the difference of their x or y coordinates is not larger than 1. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞. We will color vertex v *white* if $v_x + v_y \equiv 0 \pmod 2$; otherwise, v is colored *black*. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph hence include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, $(v_x + 1, v_y + 1)$, $(v_x + 1, v_y - 1)$, and $(v_x - 1, v_y + 1)$. Then, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. For instance, Fig.

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[1]Ruo-Wei Hung is with the Department of Computer Science and Information Engineering, Chaoyang University of Technology, Wufeng, Taichung 41349, Taiwan.

[2]Fatemeh Keshavarz-Kohjerdi is with the Department of Computer Science, Shahed University, Tehran, Iran.

[3]Chuan-Bi Lin and Jong-Shin Chen are with the Department of Information and Communication Engineering, Chaoyang University of Technology, Wufeng, Taichung 41349, Taiwan.

*Corresponding author e-mail: rwhung@cyut.edu.tw.

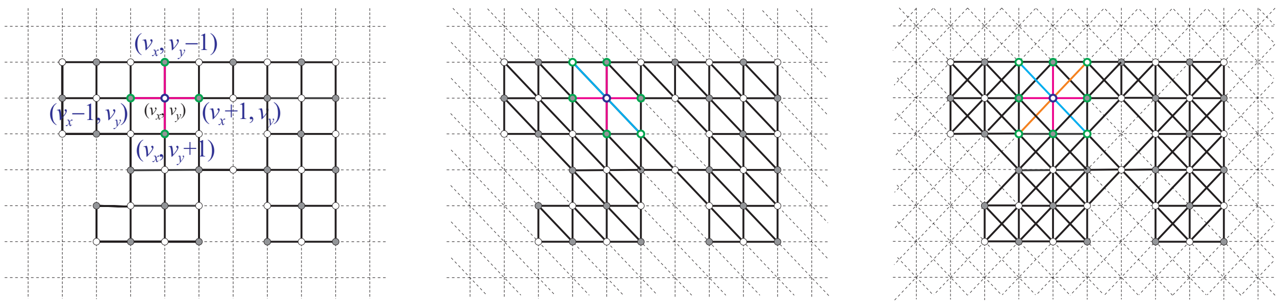
1(c) shows a supergrid graph. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (vertices) but in general they are distinct since the edge sets of these graphs are different. Obviously, all grid graphs are bipartite [27] but triangular grid graphs and supergrid graphs are not always bipartite. An edge (u, v) in a supergrid graph is said to be *horizontal* (resp., *vertical*) if $u_y = v_y$ and $u_x \neq v_x$ (resp., $u_x = v_x$ and $u_y \neq v_y$), and is called *skewed* if it is neither a horizontal nor a vertical edge. In the figures we will assume that $(1, 1)$ are coordinates of the up-left vertex, i.e. the leftmost vertex of the first row, in a supergrid graph.

The Hamiltonian cycle and path problems for grid and triangular grid graphs were known to be NP-complete [13], [27]. In [17], we have showed that the Hamiltonian path and cycle problems on supergrid graphs are also NP-complete. Thus, it is NP-complete for determining whether a supergrid graph is Hamiltonian connected. In the past, we have verified the Hamiltonian and Hamiltonian connected properties of some special supergrid graphs. The Hamiltonian cycle problem on linear-convex supergrid graphs can be solvable in linear time [18]. Recently, we verified the Hamiltonicity and Hamiltonian connectivity of some special supergrid graphs, including rectangular [19], triangular, parallelogram, trapezoid [20], and L-shaped [22].

Rectangular, parallelogram, and alphabet supergrid graphs first appeared in [17], in which they are proved to be Hamiltonian. An alphabet supergrid graph is a finite vertex-induced subgraph of the rectangular supergrid graph. There are 26 types of alphabet supergrid graphs in which the shape of each type of alphabet supergrid graphs forms a capital letter. In this paper, we first prove L-, C-, F-, and E-alphabet supergrid graphs to be Hamiltonian connected by decomposing them into disjoint rectangular supergrid subgraphs. However, many other alphabet supergrid graphs can not be decomposed into only rectangular supergrid subgraphs, e.g., N- and Y-alphabet supergrid graphs. We observe that these alphabet supergrid graphs can be decomposed into disjoint rectangular, triangular, parallelogram, and trapezoid supergrid subgraphs. In [20], we provided a constructive proof to show that triangular, parallelogram, and trapezoid supergrid graphs are Hamiltonian and Hamiltonian connected. Based on the Hamiltonian connectivity of triangular, parallelogram, trapezoid, and rectangular supergrid graphs, we will prove N- and Y-alphabet supergrid graphs to be Hamiltonian connected. The Hamiltonian connectivity of the other alphabet supergrid graphs can be verified similarly.

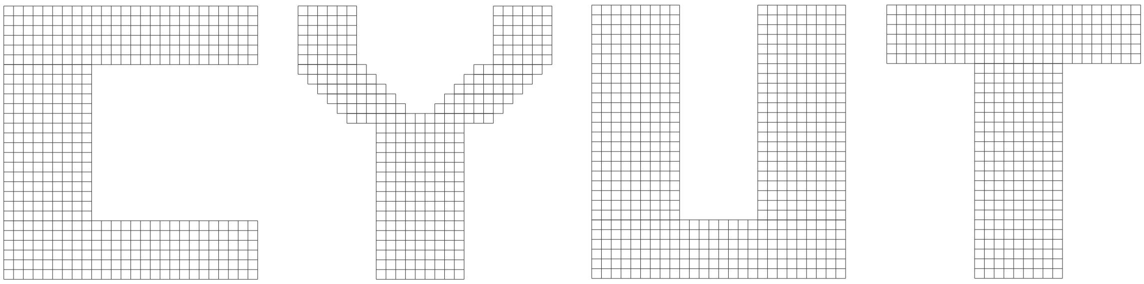
The possible application of the Hamiltonian connectivity of alphabet supergrid graphs is presented as follows. Consider a computerized embroidery machine which will sew a k-letters string into object. Its computerized embroidery software is used to compute the sewing track of the input string. First, the software produces k sets of lattices in which every set of lattices represents a letter in the string. It then computes a path to visit the lattices of the sets such that each lattice is visited exactly once and the sum of lengths between any two disconnected sets of lattices is minimum. Finally, the software transmits the stitching track of the computed path to the computerized embroidery machine, and the machine then performs the sewing work along the track on the object, e.g., clothes. For example, given a string "CYUT" the computerized embroidery software first produces a series of sets of lattices in which each set forms an alphabet supergrid graph, as depicted in Fig. 2(a). It then computes a path to visit the lattices of the sets such that each lattice is visited exactly once, as shown in Fig. 2(b). Since each stitch position of a embroidery machine can be moved to its eight neighboring positions (left, right, up, down, up-left, up-right, down-left, and down-right), one set of neighboring lattices forms a connected alphabet supergrid graph. Note that each lattice will be represented by a vertex of a supergrid graph. The desired sewing track of each set of adjacent lattices is a Hamiltonian path of the corresponding alphabet supergrid graph. Note that if the corresponding alphabet supergrid graph contains no Hamiltonian path, then the sewing track in it contains more than one path and these paths must be concatenated by jump lines. In this paper, we will show that alphabet supergrid graphs are always Hamiltonian connected and hence there exists no jump line on the inside of an alphabet supergrid graph. By the Hamiltonian connectivity of alphabet supergrid graphs, we can seek the end vertices of Hamiltonian paths in the corresponding alphabet supergrid graphs so that the total length of jump lines connecting two alphabet supergrid graphs is minimum. For an example, Fig. 2(b) shows such a minimum sewing track for the sets of lattices in Fig. 2(a).

Previous related works are summarized as follows. Itai *et al.* [27] showed that the Hamiltonian path and cycle problems for grid graphs are NP-complete. They also gave the necessary and sufficient conditions for a rectangular grid graph to be Hamiltonian connected. Thus, rectangular grid graphs are not always Hamiltonian connected. Zamfirescu *et al.* [46] gave the sufficient conditions for a grid graph having a Hamiltonian cycle, and proved that all grid graphs of positive width have Hamiltonian line graphs. Later, Chen *et al.* [6] improved the Hamiltonian path algorithm of [27] on rectangular grid graphs and presented a parallel algorithm for the Hamiltonian path problem with two given end vertices in rectangular grid graph. Also Lenhart and Umans [36] showed the Hamiltonian cycle problem on solid grid graphs, which are grid graphs without holes, is solvable in polynomial time. Recently, Keshavarz-Kohjerdi *et al.* [31] presented a linear-time algorithm to compute the longest path between two given vertices in rectangular grid graphs. Reay and Zamfirescu [43] proved that all 2-connected, linear-convex triangular grid graphs contain Hamiltonian cycles except one special case. The Hamiltonian cycle and path problems on triangular grid graphs were known to be NP-complete [13]. In addition, the Hamiltonian cycle problem on hexagonal grid graphs has been shown to be NP-complete [26]. Alphabet grid graphs first appeared in [44], in which Salman determined the classes of alphabet grid graphs containing Hamiltonian cycles. Keshavarz-Kohjerdi and Bagheri [30] gave the necessary and sufficient conditions for the existence of Hamiltonian paths in alphabet grid graphs, and presented a linear-time algorithm for finding Hamiltonian path with two given endpoints in these graphs. Recently, Keshavarz-Kohjerdi and Bagheri [32] verified the Hamiltonian connectivity of L-shaped grid graphs. Very recently, Keshavarz-Kohjerdi and Bagheri presented a linear-time algorithm to find Hamiltonian (s, t)-paths in rectangular

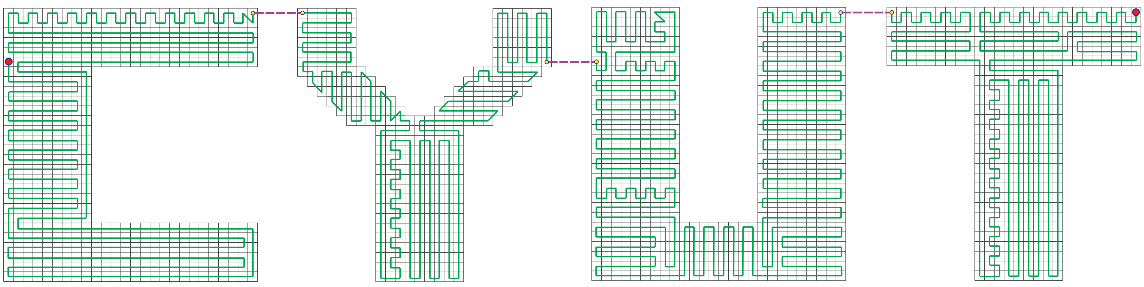


(a) (b) (c)

Fig. 1. (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, where circles represent the vertices and solid lines indicate the edges in the graphs.



(a)



(b)

Fig. 2. (a) Four sets of lattices for string "CYUT" in which each letter is represented as a set of connected lattices, and (b) a possible sewing trace for the sets of lattices in (a), where solid lines indicate the computed trace and dashed lines indicate the jump lines connecting two continuous letters.

grid graphs with a rectangular hole [33], [34]. The supergrid graphs were first introduced in [17], in which we proved that the Hamiltonian cycle and path problems on supergrid graphs are NP-complete, and every rectangular supergrid graph is Hamiltonian. Recently, we proved that linear-convex supergrid graphs, which form a subclass of supergrid graphs, always contain Hamiltonian cycles [18]. In [19], we have proved that rectangular supergrid graphs (with one trivial exception) are always Hamiltonian connected. Very recently, we verified the Hamiltonicity and Hamiltonian connectivity of some shaped supergrid graphs, including triangular, parallelogram, and trapezoid [20].

The rest of the paper is organized as follows. In Section II, some notations and background results are introduced. By using the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs in [19], Section III proves L-, C-, F-, and E-alphabet supergrid graphs to be Hamiltonian connected. Based on the Hamiltonicity and Hamiltonian connectivity of shaped supergrid graphs in [20], we verify the Hamiltonicity and the Hamiltonian connectivity of N- and Y-alphabet supergrid graphs in Section IV. The other types of alphabet supergrid graphs can be verified to be Hamiltonian and Hamiltonian connected by similar arguments as in Sections III–IV. Finally, we make some concluding remarks in Section V.

II. Notations and Background Results

In this section, we will introduce terminology and symbols. Some observations and previously established results for the Hamiltonicity and Hamiltonian connectivity of shaped supergrid graphs are also presented. For graph-theoretic terminology not defined in this paper, the reader is referred to [4].

A. Notations

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let S be a subset of vertices in G, and let

u and v be two distinct vertices in G. We write $G[S]$ for the subgraph of G *induced* by S, $G - S$ for the subgraph $G[V - S]$, i.e., the subgraph induced by $V - S$. In general, we write $G - v$ instead of $G - \{v\}$. If (u, v) is an edge in G, we say that u is *adjacent* to v. A *neighbor* of v in G is any vertex that is adjacent to v. We use $N_G(v)$ to denote the set of neighbors of v in G. The subscript 'G' of $N_G(v)$ can be removed from the notation if it has no ambiguity. The *degree* of vertex v, denoted by $deg(v)$, is the number of vertices adjacent to vertex v. The notation $u \sim v$ (resp., $u \nsim v$) means that vertices u and v are adjacent (resp., non-adjacent). A vertex w *adjoins* edge (u, v) if $w \sim u$ and $w \sim v$. Two nonincident edges e_1 and e_2 are *parallel* if each end vertex of e_1 is adjacent to some end vertex of e_2, denote this by $e_1 \approx e_2$. A path P of length $|P|$ in G, denoted by $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{|P|-1} \rightarrow v_{|P|}$, is a sequence $(v_1, v_2, \cdots, v_{|P|-1}, v_{|P|})$ of vertices such that $v_i \neq v_j$ for $i \neq j$, and $(v_i, v_{i+1}) \in E(G)$ for $1 \leqslant i < |P|$. The first and last vertices visited by P are denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote "P visits vertex v_i" and use $(v_i, v_{i+1}) \in P$ to denote "P visits edge (v_i, v_{i+1})". A path from v_1 to v_k is called a (v_1, v_k)-path. In addition, we use P to refer to the set of vertices visited by path P if it is understood without ambiguity. A path P is a cycle if $|V(P)| \geqslant 3$ and $end(P) \sim start(P)$. Two paths (or cycles) P_1 and P_2 of graph G are called *vertex-disjoint* if and only if $V(P_1) \cap V(P_2) = \emptyset$. Two vertex-disjoint paths P_1 and P_2 can be concatenated to a path, denoted by $P_1 \Rightarrow P_2$, if $end(P_1) \sim start(P_2)$.

Rectangular supergrid graphs first appeared in [17], in which the Hamiltonian cycle problem was solved. Let $R(m, n)$ be the supergrid graph whose vertex set $V(R(m, n))$ equals to $\{v = (v_x, v_y) \mid 1 \leqslant v_x \leqslant m$ and $1 \leqslant v_y \leqslant n\}$. That is, $R(m, n)$ contains m columns and n rows of vertices in S^∞. A *rectangular supergrid graph* is a supergrid graph which is isomorphic to $R(m, n)$. Then m and n, the *dimensions*, specify a rectangular supergrid graph up to isomorphism. The size of $R(m, n)$ is defined to be mn, and $R(m, n)$ is called an n-rectangle. Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. The vertex v is called the *up-left* (resp., *up-right*, *down-left*, *down-right*) *corner* of $R(m, n)$ if for any vertex $w = (w_x, w_y) \in R(m, n)$, $w_x \geqslant v_x$ and $w_y \geqslant v_y$ (resp., $w_x \leqslant v_x$ and $w_y \geqslant v_y$, $w_x \geqslant v_x$ and $w_y \leqslant v_y$, $w_x \leqslant v_x$ and $w_y \leqslant v_y$). There are four boundaries (borders) in a rectangular supergrid graph $R(m, n)$ with $m, n \geqslant 2$. An edge in any boundary of $R(m, n)$ is called *boundary edge*. For example, Fig. 3(a) shows a rectangular supergrid graph $R(10, 10)$ which is called a 10-rectangle and contains $2(9 + 9) = 36$ boundary edges. Fig. 3(a) also indicates the types of corners.

Next, we will introduce some shaped supergrid graphs, including triangular, parallelogram, and trapezoid, defined in [20]. The triangular supergrid graphs are subgraphs of rectangular supergrid graphs and are defined as follows.

Definition 1. Let ℓ be a diagonal line of $R(n, n)$ with $n \geqslant 2$ from the up-left corner to the down-right corner. Let $\Delta(n, n)$ be the supergrid graph obtained from $R(n, n)$ by removing all vertices under ℓ. A *triangular supergrid graph* is a supergrid graph that is isomorphic to $\Delta(n, n)$.

For instance, Fig. 3(b) depicts a triangular supergrid graph

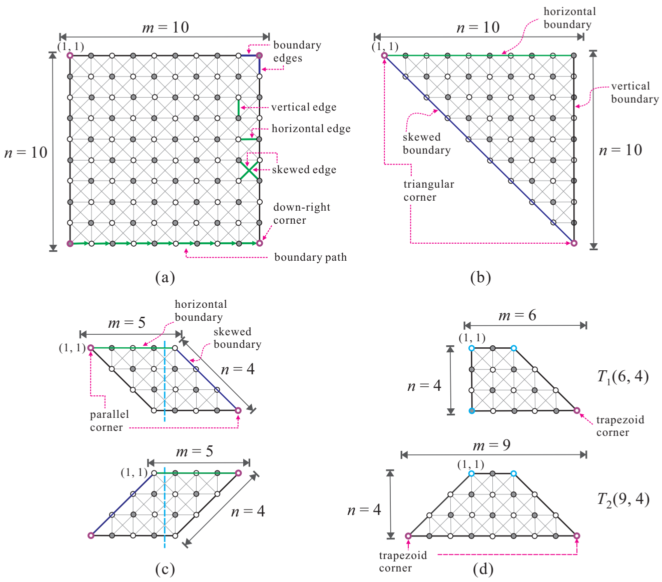


Fig. 3. (a) A rectangular supergrid graph $R(10, 10)$, (b) a triangular supergrid graph $\Delta(10, 10)$, (c) two types of parallelogram supergrid graph $P(5, 4)$, and (d) two types of trapezoid supergrid graphs $T_1(6, 4)$ and $T_2(9, 4)$, where solid arrow lines in (a) indicate a boundary path on $R(10, 10)$ and dashed line in (c) indicates a vertical separation.

$\Delta(10, 10)$. Each triangular supergrid graph contains three boundaries, namely *horizontal*, *vertical*, and *skewed*, and these boundaries form a triangle, as illustrated in Fig. 3(b). The triangular supergrid graph $\Delta(n, n)$ is called an n-triangle, and the vertex v in $\Delta(n, n)$ is called *triangular corner* if $deg(v) = 2$ and it is the intersection of horizontal (or vertical) and skewed boundaries.

Parallelogram supergrid graphs are defined similar to rectangular supergrid graphs as follows.

Definition 2. Let $P(m, n)$ be the supergrid graph with $m \geqslant n$ whose vertex set $V(P(m, n))$ equals to $\{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y + m - 1\}$ or $\{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $-v_y + 2 \leqslant v_x \leqslant m - (v_y - 1)\}$. A *parallelogram supergrid graph* is a supergrid graph which is isomorphic to $P(m, n)$.

In the above definition, there are two types of parallelogram supergrid graphs. We can see that they are isomorphic although they are different when considered as geometric graphs. In this paper, it suffices to consider the parallelogram supergrid graph $P(m, n)$ with $V(P(m, n)) = \{v = (v_x, v_y) \mid 1 \leqslant v_y \leqslant n$ and $v_y \leqslant v_x \leqslant v_y + m - 1\}$. Each parallelogram supergrid graph contains four boundaries, two *horizontal* boundaries and two *skewed* boundaries, and these boundaries form a parallelogram, as illustrated in Fig. 3(c). The size of $P(m, n)$ is defined to be mn, and $P(m, n)$ is called an n-parallelogram. The vertex w of $P(m, n)$ is called *parallelogram corner* if $deg(w) = 2$. We can see that a parallelogram supergrid graph contains two parallelogram corners and it can be decomposed into disjoint triangular or rectangular supergrid subgraphs. For instance, Fig. 3(c) depicts a parallelogram supergrid graph $P(5, 4)$ which can be partitioned into two triangular supergrid graphs $\Delta(4, 4)$.

We then introduce trapezoid supergrid graphs. Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. A trapezoid graph $T_1(m, n)$ or $T_2(m, n)$ is obtained from

$R(m,n)$ by removing one or two triangular supergrid graphs $\Delta(n-1,n-1)$ from its corners. The trapezoid supergrid graphs $T_1(m,n)$ and $T_2(m,n)$ are defined as follows.

Definition 3. Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. A trapezoid supergrid graph $T_1(m,n)$ with $m \geqslant n+1$ is obtained from $R(m,n)$ by removing a triangular supergrid graph $\Delta(n-1,n-1)$ from the corner of $R(m,n)$. A trapezoid supergrid graph $T_2(m,n)$ is constructed from $R(m,n)$ with $m \geqslant 2n$ by removing two triangular supergrid graphs $\Delta(n-1,n-1)$ from the up-left and up-right corners of $R(m,n)$. Fig. 3(d) illustrates these two types of trapezoid graphs. A *trapezoid supergrid graph* is a supergrid graph which is isomorphic to $T_1(m,n)$ or $T_2(m,n)$.

In a trapezoid supergrid graph, a vertex v is said to be *trapezoid corner* if $deg(v)=2$. We can see that $T_1(m,n)$ contains a trapezoid corner, $T_2(m,n)$ contains two trapezoid corners, $T_1(m,n)$ contains two horizontal boundaries, one vertical boundary and one skewed boundary, and $T_2(m,n)$ contains two horizontal boundaries and two skewed boundaries. By definition, each boundary of $T_1(m,n)$ and $T_2(m,n)$ contains at least two vertices. On the other hand, $T_1(m,n)$ and $T_2(m,n)$ are called an n_{T_1}-trapezoid and an n_{T_2}-trapezoid, respectively. For instance, Fig. 3(d) shows $T_1(6,4)$ and $T_2(9,4)$ that are a 4_{T_1}- and a 4_{T_2}-trapezoid, respectively.

In [30], the authors studied the Hamiltonian path problem on alphabet grid graphs. We extend their definition of alphabet grid graphs to alphabet supergrid graphs and prove that alphabet supergrid graphs are Hamiltonian connected. An *alphabet supergrid graph* is a finite vertex-induced subgraph of the rectangular supergrid graph of a certain type, as follows. Let $R(3m-2,5n-4)$ be a rectangular supergrid graph such that $m \geqslant n+1$ and $5n-4 \geqslant 3m-2$. For each letter of the alphabet, a corresponding alphabet supergrid graph is an induced subgraph of $R(3m-2,5n-4)$. The L-, C-, F-, E-, N-, and Y-alphabet supergrid graphs studied in the paper are denoted by $L(m,n)$, $C(m,n)$, $F(m,n)$, $E(m,n)$, $N(m,n)$, and $Y(m,n)$, respectively. These studied alphabet supergrid graphs are shown in Fig. 4, where $m=4$ and $n=3$.

Every alphabet supergrid graph $A(m,n)$ satisfies that $5n-4 \geqslant 3m-2 \geqslant 3(n+1)-2 = 3n+1$, and hence $n \geqslant 3$ and $m \geqslant 4$. There are 26 types of alphabet supergrid graphs in which each type of alphabet supergrid graph forms a capital letter. In this paper, it suffices to consider the alphabet supergrid graphs shown in Fig. 4. The other types of alphabet supergrid graphs can be verified to be Hamiltonian connected similarly. Let $A(m,n)$ be an alphabet supergrid graph. Then, it can be embedded into a rectangular supergrid graph $R(3m-2,5n-4)$, where $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$. The parameters m and n are used to adjust the width and height of the alphabet supergrid graph. By the structure of alphabet supergrid graphs, we can see that they can be decomposed into disjoint rectangular, triangular, parallelogram, or trapezoid supergrid subgraphs. In proving our results, we need to partition a supergrid graph into two disjoint parts. The partition is defined as follows.

Definition 4. Let $S(m,n)$ be a triangular, parallelogram, trapezoid, or alphabet supergrid graph, and let Z be a subset of the edge set $E(S(m,n))$. Z is called an *edge separator* of $S(m,n)$ if the removal of Z from $S(m,n)$ results in two disjoint supergrid subgraphs S_1 and S_2. An edge separator Z is called *vertical* (resp., *horizontal*) if Z is a set of horizontal (resp., vertical) edges and it separates $S(m,n)$ into S_1 and S_2 so that S_1 is to the left (resp., upper) of S_2. The *vertical* (resp., *horizontal*) separation operation on $S(m,n)$ is to compute a *vertical* (resp., *horizontal*) edge separator of $S(m,n)$.

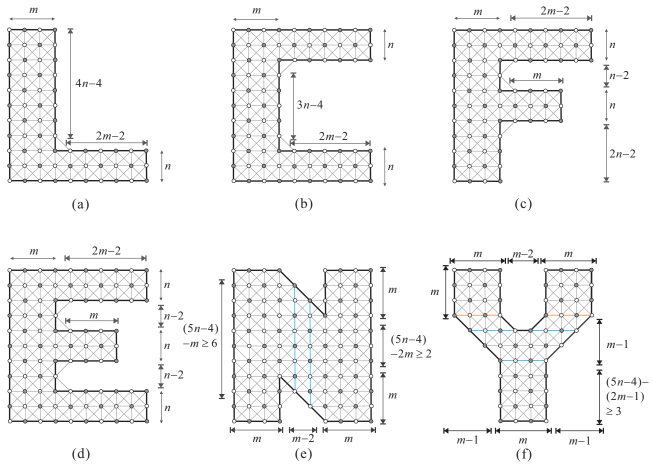


Fig. 4. Alphabet supergrid graphs studied in the paper, where (a) an L-alphabet supergrid graph $L(4,3)$, (b) an C-alphabet supergrid graph $C(4,3)$, (c) an F-alphabet supergrid graph $F(4,3)$, (d) an E-alphabet supergrid graph $E(4,3)$, (e) an N-alphabet supergrid graph $N(4,3)$, and (f) an Y-alphabet supergrid graph $Y(4,3)$.

For instance, the bold dashed line in Fig. 3(c) shows a vertical separation on $P(5,4)$ that is to partition it into two disjoint triangular supergrid subgraphs $\Delta(4,4)$.

Let $S(m,n)$ be a triangular, parallelogram, trapezoid, or alphabet supergrid graph. Let $\mathcal{C}$ be a Hamiltonian cycle or path of $S(m,n)$ and let H be a boundary of $S(m,n)$, where H is a subgraph of $S(m,n)$. The restriction of $\mathcal{C}$ to H is denoted by $\mathcal{C}_{|H}$. If $|\mathcal{C}_{|H}|=1$, i.e. the number of paths in $\mathcal{C}_{|H}$ equals to one, then $\mathcal{C}_{|H}$ is called *flat face* on H. If $|\mathcal{C}_{|H}|>1$ and $\mathcal{C}_{|H}$ contains at least one boundary edge of H, then $\mathcal{C}_{|H}$ is called *concave face* on H. In proving our result, we will construct a Hamiltonian cycle (path) of a triangular, parallelogram, trapezoid, or alphabet supergrid graph. The constructed Hamiltonian cycle (path) is called *canonical* defined below.

Definition 5. Let $S(m,n)$ be a triangular, parallelogram, trapezoid, or alphabet supergrid graph with κ boundaries, and let s and t be two distinct vertices of $S(m,n)$. A Hamiltonian cycle of $S(m,n)$ is called *canonical* if it contains $\kappa-1$ flat faces on $\kappa-1$ boundaries, and it contains at least one boundary edge in the other boundary. A Hamiltonian (s,t)-path of $S(m,n)$ is called *canonical* if it contains at least one boundary edge of each boundary in $S(m,n)$.

B. Background results

In [17], we have shown that rectangular supergrid graphs always contain canonical Hamiltonian cycles except 1-rectangles.

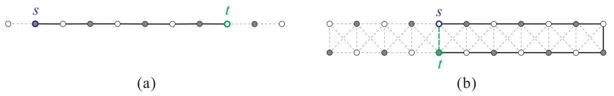


Fig. 5. Rectangular supergrid graph in which there exists no Hamiltonian (s,t)-path for (a) $R(m,1)$, and (b) $R(m,2)$, where solid lines indicate the longest path between s and t.

Lemma 1. *(See [17].) Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. Then, the following statements hold true:*
(1) if $n=3$, then $R(m,3)$ contains a canonical Hamiltonian cycle;
(2) if $n=2$ or $n \geqslant 4$, then $R(m,n)$ contains four canonical Hamiltonian cycles with concave faces being located on different boundaries.

Let (G,s,t) denote the supergrid graph G with two given distinct vertices s and t. Without loss of generality, we will assume that $s_x \leqslant t_x$, i.e., s is to the left of t, in the rest of the paper. We denote a Hamiltonian path between s and t in G by $HP(G,s,t)$. We say that $HP(G,s,t)$ does exist if there is a Hamiltonian (s,t)-path of G. It is clear that there exists a Hamiltonian (s,t)-path of graph G if edge (s,t) is in a Hamiltonian cycle of G. In [19], we proved that $HP(R(m,n),s,t)$ always exists for $m,n \geqslant 3$ as follows.

Lemma 2. *(See [19].) For $(R(m,n),\ s,\ t)$ with $m \geqslant n \geqslant 3$, $R(m,n)$ contains a canonical Hamiltonian (s,t)-path, and hence $HP(R(m,n),s,t)$ does exist.*

Recently, we verified the Hamiltonian connectivity of rectangular supergrid graphs except one condition [19]. The exception for $HP(R(m,n),s,t)$ holds only for 1-rectangles or 2-rectangles. To describe the exception condition, we define the vertex cut and cut vertex of a graph as follows.

Definition 6. Let G be a connected graph and let V_1 be a subset of the vertex set $V(G)$. The set V_1 is called a *vertex cut* of G if $G-V_1$ is disconnected. A vertex v of G is said to be a *cut vertex* of G if $\{v\}$ is a vertex cut of G. For example, in Fig. 5(b) $\{s,t\}$ is a vertex cut and in Fig. 5(a) t is a cut vertex.

The following condition implies that $HP(R(m,1),s,t)$ and $HP(R(m,2),s,t)$ do not exist.

- **(F1)** s or t is a cut vertex of $R(m,1)$, or $\{s,t\}$ is a vertex cut of $R(m,2)$ (see Fig. 5(a) and Fig. 5(b)). Notice that, here, s or t is a cut vertex of $R(m,1)$ if either s or t is not a corner vertex, and $\{s,t\}$ is a vertex cut of $R(m,2)$ if $2 \leqslant s_x = t_x \leqslant m-1$.

Obviously, the following lemma, showing that $HP(R(m,n),s,t)$ does not exist if $(R(m,n),s,t)$ satisfies condition (F1), holds true.

Lemma 3. *Let $R(m,n)$ be a rectangular supergrid graph with two vertices s and t. If $(R(m,n),s,t)$ satisfies condition* (F1)*, then $R(m,n)$ contains no Hamiltonian (s,t)-path.*

In [20], we have verified the Hamiltonicity of triangular, parallelogram, and trapezoid supergrid graphs as follows.

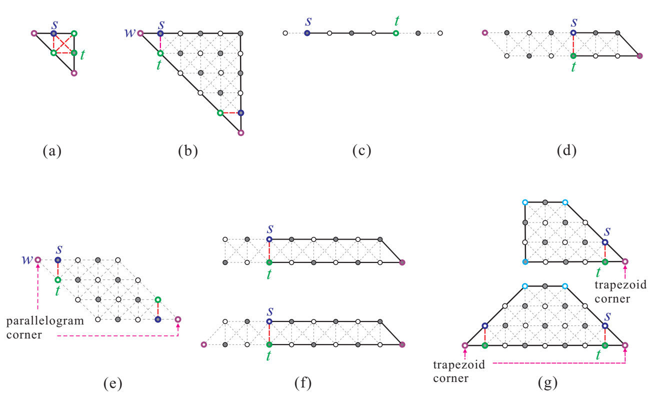


Fig. 6. The conditions for that $HP(S(m,n),s,t)$ does not exist, where (a)–(b) $S(m,n) = \Delta(n,n)$, (c)–(e) $S(m,n) = P(m,n)$, and (f)–(g) $S(m,n) = T(m,n)$, where dashed lines indicate the forbidden edges (s,t) and $T(m,n) = T_1(m,n)$ or $T_2(m,n)$.

Lemma 4. *(See [20].) Let $S(m,n)$ be a triangular, parallelogram, or trapezoid supergrid graph with $m \geqslant n \geqslant 2$. Then, $S(m,n)$ contains a canonical Hamiltonian cycle.*

For a triangular, parallelogram, or trapezoid supergrid graph $S(m,n)$, $HP(S(m,n),s,t)$ does exist except some trivial conditions [20]. These conditions for that $HP(S(m,n),s,t)$ does not exist are stated as the following seven conditions:

- **(F2)** $\Delta(n,n)$ is a 3-triangle, and (s,t) is a nonboundary edge of $\Delta(n,n)$ (see Fig. 6(a)).
- **(F3)** $\Delta(n,n)$ satisfies $n \geqslant 3$, and (s,t) is an edge of $\Delta(n,n)$ such that s and t are adjacent to a triangular corner w of $\Delta(n,n)$, i.e., $\{s,t\}$ is a vertex cut of $\Delta(n,n)$ (see Fig. 6(b)).
- **(F4)** $P(m,n)$ is a 1-parallelogram, and s or t is a cut vertex of $P(m,n)$ (see Fig. 6(c)).
- **(F5)** $P(m,n)$ is a 2-parallelogram with $m \geqslant 2$, and $\{s,t\}$ is a vertex cut of $P(m,n)$ (see Fig. 6(d)).
- **(F6)** $P(m,n)$ satisfies $m \geqslant n \geqslant 2$, and (s,t) is an edge of $P(m,n)$ such that $s \sim w$ and $t \sim w$ for any parallelogram corner w of $P(m,n)$, where $s \neq w$, $t \neq w$, and $deg(w)=2$, i.e., $\{s,t\}$ is a vertex cut of $P(m,n)$ (see Fig. 6(e)).
- **(F7)** $T(m,n)$ is a 2_{T_1}-trapezoid or 2_{T_2}-trapezoid, and (s,t) is a vertical and nonboundary edge of $T(m,n)$, i.e., $\{s,t\}$ is a vertex cut of $T(m,n)$ (see Fig. 6(f)).
- **(F8)** $T(m,n)$ is a trapezoid supergrid graph for $n \geqslant 2$, w is a trapezoid corner of $T(m,n)$, $s,t \neq w$, $s \sim w$, and $t \sim w$, i.e., $\{s,t\}$ is a vertex cut of $T(m,n)$ (see Fig. 6(g)).

In [20], we verified the Hamiltonian connectivity of triangular, parallelogram, and trapezoid supergrid graphs as follows.

Lemma 5. *(See [20].) Let $S(m,n)$ be a triangular, parallelogram, or trapezoid supergrid graph, and let s and t be two distinct vertices of $S(m,n)$. If $(S(m,n),s,t)$ does not satisfy conditions* (F2)–(F8)*, then $S(m,n)$ contains a canonical Hamiltonian (s,t)-path, and hence $HP(S(m,n),s,t)$ does exist.*

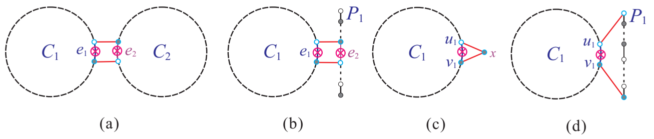


Fig. 7. A schematic diagram for (a) Statement (1), (b) Statement (2), (c) Statement (3), and (d) Statement (4) of Proposition 6, where bold dashed lines indicate the cycles (paths) and ⊗ represents the destruction of an edge while constructing a cycle or path.

In the past, we have obtained some observations on the relations among cycle, path, and vertex [18], [19]. They will be used in verifying the Hamiltonian connectivity of alphabet supergrid graphs.

Proposition 6. *(See [18], [19].) Let C_1 and C_2 be two vertex-disjoint cycles of a graph G, let C_1 and P_1 be a cycle and a path, respectively, of G with $V(C_1) \cap V(P_1) = \emptyset$, and let x be a vertex in $G - V(C_1)$ or $G - V(P_1)$. Then, the following statements hold true:*
(1) If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be combined into a cycle of G (see Fig. 7(a)).
(2) If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path of G (see Fig. 7(b)).
(3) If vertex x adjoins one edge (u_1, v_1) of C_1 (resp., P_1), then C_1 (resp., P_1) and x can be combined into a cycle (resp., path) of G (see Fig. 7(c)).
(4) If there exists one edge $(u_1, v_1) \in C_1$ such that $u_1 \sim start(P_1)$ and $v_1 \sim end(P_1)$, then C_1 and P_1 can be combined into a cycle C of G (see Fig. 7(d)).

III. The Hamiltonian Connectivity of L-, C-, F-, and E-alphabet Supergrid Graphs

In this section, we will show that L-, C-, F-, and E-alphabet supergrid graphs are Hamiltonian connected. Let $A(m, n)$ be an L-, C-, F-, or E-alphabet supergrid graph, and let s and t be two distinct vertices in $A(m, n)$. We will provide a constructive proof to show that $HP(A(m, n), s, t)$ does exist. Our basic idea is described as follows. First, we perform a series of separation operations on $A(m, n)$ to obtain k disjoint rectangular supergrid subgraphs A_1–A_k. Consider the relative positions of s and t. We then use Lemmas 1 and 2 to construct canonical Hamiltonian cycles or paths of A_1–A_k. By using Proposition 6, we finally combine these cycles and paths into a Hamiltonian (s, t)-path of $A(m, n)$. Our constructed Hamiltonian (s, t)-path contains at least one boundary edge in each boundary of $A(m, n)$ and hence is canonical.

Lemma 7. *Let $L(m, n)$ be an L-alphabet supergrid graph with $m \geqslant n + 1 \geqslant 4$, and let s and t be two distinct vertices of $L(m, n)$. Then, $L(m, n)$ contains a canonical Hamiltonian (s, t)-path, and hence $HP(L(m, n), s, t)$ does exist.*

Proof: We prove this lemma by constructing a canonical Hamiltonian (s, t)-path of $L(m, n)$. Note that $s_x \leqslant t_x$, i.e., s is to the left of t. We first make a series of separation operations on $L(m, n)$ as follows:
① a vertical separation on $L(m, n)$ to partition it into two disjoint rectangular supergrid subgraphs $L_1 = R(2m - 2, n)$ and $L^* = R(m, 5n - 4)$;
② a horizontal separation on L^* to partition it into two disjoint rectangular supergrid subgraphs $L_2 = R(m, n)$ and $L_3 = R(m, 4n - 4)$.

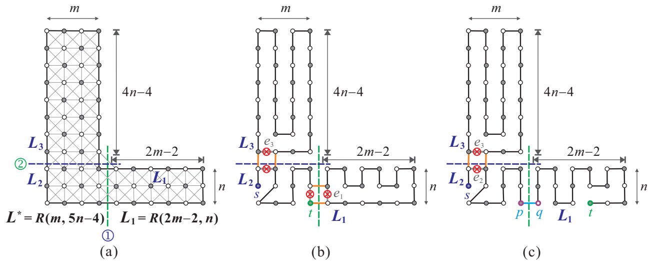


Fig. 8. (a) The separations on $L(m, n)$, where bold dashed lines indicate the separation operations, and (b)–(c) the construction of $HP(L(m, n), s, t)$, where ⊗ represents the destruction of an edge while constructing a Hamiltonian (s, t)-path.

Fig. 8(a) depicts the above separation operations. Since $m \geqslant n + 1 \geqslant 4$, $L_1 = R(2m - 2, n)$, $L_2 = R(m, n)$, and $L_3 = R(m, 4n - 4)$ satisfy that $2m - 2, n \geqslant 3$, $m, n \geqslant 3$, and $m, 4n - 4 \geqslant 3$. Depending on the locations of s and t, there are the following two cases:

Case 1: $s, t \in L_i$ for $1 \leqslant i \leqslant 3$. In this case, s and t are located in the same partitioned rectangular supergrid subgraph. There are two subcases:

Case 1.1: $s, t \in L_2$. Since $L_2 = R(m, n)$ satisfies $m, n \geqslant 3$, by Lemma 2 there exists a canonical Hamiltonian (s, t)-path P_2 of L_2. Then, P_2 visits at least one boundary edge in each boundary of L_2. On the other hand, $L_1 = R(2m - 2, n)$ and $L_3 = R(m, 4n - 4)$ satisfy that $2m - 2, n \geqslant 3$ and $m, 4n - 4 \geqslant 3$. By Lemma 1, there exist two canonical Hamiltonian cycles HC_1 and HC_3 of L_1 and L_3, respectively. We can place one flat face of HC_i, $i = 1$ or 3, to face its neighboring rectangular supergrid subgraph L_2. Thus, there exist four edges $e_1^*, e_3^* \in P_2$, $e_1 \in HC_1$, and $e_3 \in HC_3$ such that $e_1^* \approx e_1$ and $e_3^* \approx e_3$. By Statement (2) of Proposition 6, P_2, HC_1, and HC_3 can be combined into a (s, t)-path P. Clearly, P is a canonical Hamiltonian (s, t)-path of $L(m, n)$. The construction of such a canonical Hamiltonian (s, t)-path is depicted in Fig. 8(b).

Case 1.2: $s, t \in L_1$ or L_3. Suppose that $s, t \in L_1$. By Lemma 2, there exists a canonical Hamiltonian (s, t)-path P_1 of L_1. Then, P_1 visits at least one boundary edge of each boundary in L_1. Let $L^* = L_2 \cup L_3$. Then, $L^* = R(m, 5n - 4)$ satisfies that $m, 5n - 4 \geqslant 3$. By Lemma 1, there exists a canonical Hamiltonian cycle HC^* of L^*. Then, HC^* contains a flat face that is placed to face L_1. Thus, there exist two edges $e_1 \in P_1$ and $e^* \in HC^*$ such that $e_1 \approx e^*$. By Statement (2) of Proposition 6, P_1 and HC^* can be combined into a canonical Hamiltonian (s, t)-path P of $L(m, n)$. The case of $s, t \in L_3$ can be proved by the same construction.

Case 2: $s \in L_i$ and $t \in L_j$ for $i \neq j$. In this case, s and t are located in the different partitioned rectangles. There are two subcases:

Case 2.1: exactly one of s and t is in L_2. By symmetry, it suffices to consider that $s \in L_2$ and $t \in L_1$. Let $p \in L_2$ and $q \in L_1$ such that $p \neq s$, $q \neq t$, and $p \sim q$. By Lemma 2, L_2 contains a canonical Hamiltonian (s, p)-path P_2, and L_1

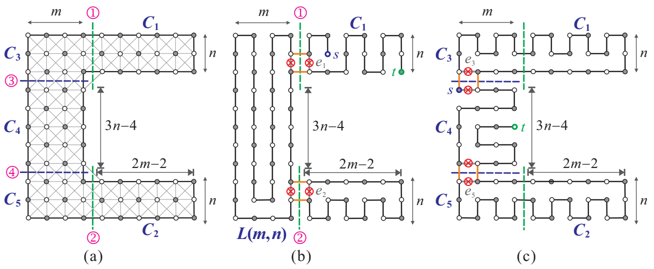


Fig. 9. (a) A series of separation operations on $C(m,n)$, where bold dashed lines indicate the separation operations, (b) the construction of $HP(C(m,n),s,t)$ for $s,t \in C_1$, and (c) the construction of $HP(C(m,n),s,t)$ for $s,t \in C_4$, where ⊗ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path.

contains a canonical Hamiltonian (q,t)-path P_1. By Lemma 1, there exists a canonical Hamiltonian cycle HC_3 of L_3 in which one flat face of HC_3 is placed to face L_2. Then, there exist two edges $e_2 \in P_2$ and $e_3 \in HC_3$ with $e_2 \approx e_3$. By Statement (2) of Proposition 6, P_2 and HC_3 can be combined into a canonical Hamiltonian (s,p)-path P_2^* of $L_2 \cup L_3$. Therefore, $P_2^* \Rightarrow P_1$ forms a canonical Hamiltonian (s,t)-path of $L(m,n)$. The construction of such a canonical Hamiltonian (s,t)-path is shwon in Fig. 8(c).

Case 2.2: s and t are not in L_2. In this subcase, $s \in L_3$ and $t \in L_1$. Let $p \in L_3$, $q \in L_1$, and $r_3^*, r_1^* \in L_2$ such that $p \neq s$, $q \neq t$, $r_3^* \sim p$, and $r_1^* \sim q$. By Lemma 2, L_3, L_2, and L_1 contain canonical Hamiltonian (s,p)-path P_3, (r_3^*, r_1^*)-path P_2, and (q,t)-path P_1, respectively. Then, $P_3 \Rightarrow P_2 \Rightarrow P_1$ forms a canonical Hamiltonian (s,t)-path of $L(m,n)$.

We have considered any case to construct a canonical Hamiltonian (s,t)-path of $L(m,n)$. Thus, $HP(L(m,n),s,t)$ does exist. ■

We next consider C- and F-alphabet supergrid graphs. By the structures of considered alphabet supergrid graphs in Fig. 4, $L(m,n)$ forms a subgraph of $C(m,n)$ or $F(m,n)$. We first prove the Hamiltonian connectivity of $C(m,n)$ as follows.

Lemma 8. *Let $C(m,n)$ be an C-alphabet supergrid graph with $m \geqslant n+1 \geqslant 4$, and let s and t be two distinct vertices of $C(m,n)$. Then, $C(m,n)$ contains a canonical Hamiltonian (s,t)-path, and hence $HP(C(m,n),s,t)$ does exist.*

Proof: We prove this lemma by constructing a canonical Hamiltonian (s,t)-path of $C(m,n)$. We first partition $C(m,n)$ into two disjoint supergrid subgraphs C_1 and $L(m,n)$ by a vertical separation, as depicted in Fig. 9(a), where the circled number ① indicates the separation operation and $C_1 = R(2m-2,n)$ satisfies $2m-2, n \geqslant 3$. Depending on the positions of s and t, there are the following three cases:

Case 1: $s,t \in C_1$. By Lemma 2, there exists a canonical Hamiltonian (s,t)-path P_1 of C_1. We next make a vertical separation on $L(m,n)$ to obtain two disjoint rectangular supergrid subgraphs $C_2 = R(2m-2,n)$ and $C^* = R(m,5n-4)$, where $2m-2, n \geqslant 3$ and $m, 5n-4 \geqslant 3$, as depicted in the separation ② of Fig. 9(a). By Lemma 1, C_2 and C^* contain canonical Hamiltonian cycles HC_2 and HC^*, respectively. Since HC^* is a canonical Hamiltonian cycle of C^*, we can place one flat face of HC^* to face its neighboring rectangular supergrid subgraphs C_1 and C_2. Then, there exist four edges $e_1 \in P_1$, $e_2 \in HC_2$, and $e_1^*, e_2^* \in HC^*$ such that $e_1 \approx e_1^*$ and $e_2 \approx e_2^*$. By Statements (1) and (2) of Proposition 6, P_1, HC_2, and HC^* can be combined into a canonical Hamiltonian (s,t)-path of $C(m,n)$. The construction of such a Hamiltonian (s,t)-path is depicted in Fig. 9(b).

Case 2: exactly one of s and t is in C_1. Without loss of generality, assume that $s \in C_1$ and $t \in L(m,n)$. Let $p \in C_1$ and $q \in L(m,n)$ such that $p \neq s$, $q \neq t$, and $p \sim q$. By Lemma 2, there exists a canonical Hamiltonian (s,p)-path P_1 of C_1. By Lemma 7, there exists a canonical Hamiltonian (q,t)-path P_L of $L(m,n)$. Then, $P_1 \Rightarrow P_L$ forms a canonical Hamiltonian (s,t)-path of $C(m,n)$.

Case 3: $s,t \notin C_1$. In this case, $s,t \in L(m,n)$. We then perform a vertical separation on $L(m,n)$ to obtain two disjoint rectangular supergrid subgraphs $C_2 = R(2m-2,n)$ and $C^* = R(m,5n-4)$ such that $2m-2, n \geqslant 3$ and $m, 5n-4 \geqslant 3$, as shown in the separation ② of Fig. 9(a). If $\{s,t\} \cap C_2 \neq \emptyset$, then, by symmetry, a canonical Hamiltonian (s,t)-path of $C(m,n)$ can be constructed by using the same construction of Case 1 or Case 2. In the following, suppose that $s,t \notin C_2$. We then make two horizontal separations on C^* to partition it into three disjoint rectangular supergrid subgraphs $C_3 = R(m,n)$, $C_4 = R(m,3n-4)$, and $C_5 = R(m,n)$, where $m,n \geqslant 3$ and $m, 3n-4 \geqslant 3$, as depicted in the separations ③–④ of Fig. 9(a). There are three subcases:

Case 3.1: $s,t \in C_3$ or C_5. By symmetry, it suffices to consider that $s,t \in C_3$. By Lemma 2, there exists a canonical Hamiltonian (s,t)-path P_3 of C_3. Then, P_3 contains two boundary edges e_1^* and e_4^* that are to face C_1 and C_4, respectively. Let $C_{45} = C_4 \cup C_5$. Then, $C_{45} = R(m, 4n-4)$. By Lemma 1, C_{45} contains a canonical Hamiltonian cycle HC_{45}. We can place two flat faces of HC_{45} to face its two neighboring rectangular supergrid subgraphs C_2 and C_3. By Lemma 1, there exist canonical Hamiltonian cycles HC_1 and HC_2 of C_1 and C_2, respectively. Then, there exist four edges $e_1 \in HC_1$, $e_2 \in HC_2$, and $e_4, e_5 \in HC_{45}$ such that $e_1 \approx e_1^*$, $e_4 \approx e_4^*$, and $e_2 \approx e_5$. By Statements (1) and (2) of Proposition 6, P_3, HC_1, HC_2, and HC_{45} can be combined into a canonical Hamiltonian (s,t)-path of $C(m,n)$.

Case 3.2: exactly one of s and t is in C_3 or C_5. Without loss of generality, assume that $s \in C_3$ and $t \notin C_3$. Consider that $t \in C_4$. Let $p \in C_3$ and $q \in C_4$ such that $p \neq s$, $q \neq t$, and $p \sim q$. By Lemma 2, C_3 contains a canonical Hamiltonian (s,p)-path P_3. By Lemma 1, C_1 contains a canonical Hamiltonian cycle HC_1 whose one flat face is placed to face C_3. Then, there exist two edges $e_3^* \in P_3$ and $e_1 \in HC_1$ such that $e_3^* \approx e_1$. By Statement (2) of Proposition 6, P_3 and HC_1 can be combined into a canonical Hamiltonian (s,p)-path P_3^* of $C_1 \cup C_3$. Let $C_{25} = C_2 \cup C_5$. Then, $C_{25} = R(3m-2,n)$ satisfies $3m-2, n \geqslant 3$. By Lemma 1, there exists a canonical Hamiltonian cycle HC_{25} of C_{25} such that one flat face of HC_{25} is placed to face C_4. By Lemma 2, there exists a canonical Hamiltonian (q,t)-path P_4 of C_4. Then, there exist two edges $e_5 \in HC_{25}$ and $e_5^* \in P_4$ such that $e_5 \approx e_5^*$. By Statement (2) of Proposition 6, P_4 and C_{25} can be combined into a canonical Hamiltonian (q,t)-path P_4^* of $C_{25} \cup C_4$. Then, $P_3^* \Rightarrow P_4^*$ forms a canonical Hamiltonian (s,t)-path of $C(m,n)$. On the other hand, consider that $t \in C_5$. Let $p \in C_3$, $q \in C_5$,

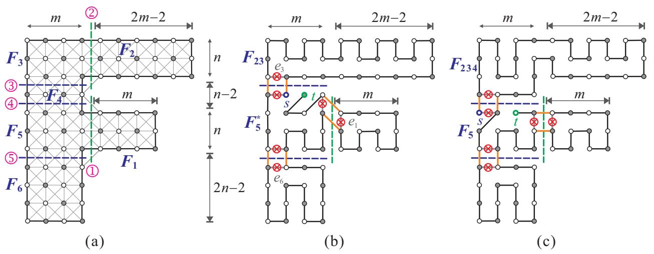


Fig. 10. (a) A series of separations on $F(m,n)$, (b) the construction of $HP(F(m,n),s,t)$ under that $s,t \in F_4$ and F_4 is a 1-rectangle, and (c) the construction of $HP(F(m,n),s,t)$ under that $s,t \in F_5$, where bold dashed lines indicate the separation operations and ⊗ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path.

and $r_3^*, r_5^* \in C_4$ such that $p \neq s$, $q \neq t$, $p \sim r_3^*$, and $q \sim r_5^*$. By the same construction in proving the case of $t \in C_4$, $C_1 \cup C_3$ contains a canonical Hamiltonian (s,p)-path P_3^*, and $C_2 \cup C_5$ contains a canonical Hamiltonian (q,t)-path P_5^*. By Lemma 2, there exists a canonical Hamiltonian (r_3^*, r_5^*)-path P_4^* of C_4. Then, $P_3^* \Rightarrow P_4^* \Rightarrow P_5^*$ forms a canonical Hamiltonian (s,t)-path of $C(m,n)$.

Case 3.3: $s,t \notin C_3 \cup C_5$. In this subcase, $s,t \in C_4$. Let $C_{13} = C_1 \cup C_3$ and $C_{25} = C_2 \cup C_5$. Then, $C_{13} = R(3m-2,n)$ and $C_{25} = R(3m-2,n)$ satisfy that $3m-2, n \geqslant 3$. By Lemma 1, C_{13} and C_{25} respectively contain canonical Hamiltonian cycles HC_{13} and HC_{25} in which their one flat face is placed to face C_4. By Lemma 2, there exists a canonical Hamiltonian (s,t)-path P_4 of C_4. Then, there exist four edges $e_3 \in HC_{13}$, $e_5 \in HC_{25}$, and $e_3^*, e_5^* \in P_4$ such that $e_3 \approx e_3^*$ and $e_5 \approx e_5^*$. By Statement (2) of Proposition 6, P_4, HC_{13}, and HC_{25} can be combined into a canonical Hamiltonian (s,t)-path of $C(m,n)$. The construction of such a Hamiltonian path is depicted in Fig. 9(c).

We have considered any case to construct a canonical Hamiltonian (s,t)-path of $C(m,n)$. Thus, the lemma holds true. ■

By similar arguments in proving the Hamiltonian connectivity of $C(m,n)$, we prove $F(m,n)$ to be Hamiltonian connected as follows.

Lemma 9. *Let $F(m,n)$ be an F-alphabet supergrid graph with $m \geqslant n+1 \geqslant 4$, and let s and t be two distinct vertices of $F(m,n)$. Then, $F(m,n)$ contains a canonical Hamiltonian (s,t)-path, and hence $HP(F(m,n),s,t)$ does exist.*

Proof: We first make a vertical separation on $F(m,n)$ to partition it into $L(m,n)$ and $F_1 = R(m,n)$, as depicted in separation ① of Fig. 10(a). Then, $F_1 = R(m,n)$ satisfies $m,n \geqslant 3$. If $\{s,t\} \cap F_1 \neq \emptyset$, then a canonical Hamiltonian (s,t)-path of $F(m,n)$ can be constructed by the same construction in Case 1 or Case 2 of Lemma 8. In the following, assume that $s,t \notin F_1$. Then, $s,t \in L(m,n)$. We next perform a series of separation operations, including one vertical separation and three horizontal separations, on $L(m,n)$ to obtain five disjoint rectangular supergrid subgraphs $F_2 = R(2m-2,n)$, $F_3 = R(m,n)$, $F_4 = R(m,n-2)$, $F_5 = R(m,n)$, and $F_6 = R(m,2n-2)$, where $2m-2, n \geqslant 3$, $m,n \geqslant 3$, and $m, 2n-2 \geqslant 3$, as shown in separations ②–⑤ of Fig. 10(a). Consider the following two cases:

Case 1: $s,t \in F_i$ for $2 \leqslant i \leqslant 6$. In this subcase, s and t are in the same partitioned rectangular supergrid subgraph. There are four subcases:

Case 1.1: $s,t \in F_2$. By Lemma 2, F_2 contains a canonical Hamiltonian (s,t)-path P_2. Then, P_2 contains a boundary edge e_2 that is placed to face its neighboring subgraph F_3. Let $F^* = F_3 \cup F_4 \cup F_5 \cup F_6$. Then, $F^* = R(m,5n-4)$ and $F_1 = R(m,n)$ satisfy that $m, 5n-4 \geqslant 3$ and $m,n \geqslant 3$. By Lemma 1, F^* and F_1 contain canonical Hamiltonian cycles HC^* and HC_1, respectively. We can place one flat face of HC^* to face F_1 and F_2. Thus, there exist four edges $e_1^*, e_2^* \in HC^*$, $e_1 \in HC_1$, and $e_2 \in P_2$ such that $e_1^* \approx e_1$ and $e_2^* \approx e_2$. By Statements (1) and (2) of Proposition 6, P_2, HC^*, and HC_1 can be combined into a canonical Hamiltonian (s,t)-path of $F(m,n)$.

Case 1.2: $s,t \in F_3$. Let $F^* = F_4 \cup F_5 \cup F_6$. Then, $F^* = R(m,4n-4)$, $F_1 = R(m,n)$, and $F_2 = R(2m-2,n)$ satisfy that $m, 4n-4 \geqslant 3$, $m,n \geqslant 3$, and $2m-2, n \geqslant 3$. By Lemma 1, F^*, F_1, and F_2 contain canonical Hamiltonian cycles HC^*, HC_1, and HC_2, respectively. We can place two flat faces of HC^* to respectively face F_1 and F_3. Thus, HC^* and HC_1 can be combined into a Hamiltonian cycle HC of $F^* \cup F_1$ such that HC contains a flat face of F_4 to face F_3. By Lemma 2, F_3 contains a canonical Hamiltonian (s,t)-path P_3. Then, P_3 contains two boundary edge e_4^* and e_2^* that are placed to face its neighboring subgraphs F_2 and F_4, respectively. Thus, there exist two edges $e_4 \in HC$ and $e_2 \in HC_2$ such that $e_4 \approx e_4^*$ and $e_2 \approx e_2^*$. By Statement (2) of Proposition 6, P_3, HC_2, and HC can be combined into a canonical Hamiltonian (s,t)-path of $F(m,n)$.

Case 1.3: $s,t \in F_4$. If $HP(F_4,s,t)$ does exist, then a canonical Hamiltonian (s,t)-path can be constructed by similar construction in Case 1.2. Suppose that $HP(F_4,s,t)$ does not exist. Then, F_4 is either a 1-rectangle or 2-rectangle (see [19]). Consider that F_4 is a 1-rectangle. Let $F_5^* = F_5 \cup F_4$ and $F_{23} = F_2 \cup F_3$. Then, $F_5^* = R(m,2n-2)$ and $F_{23} = R(3m-2,n)$ satisfy that $m, 2n-2 \geqslant 3$ and $3m-2, n \geqslant 3$. By Lemma 2, there exists a canonical Hamiltonian (s,t)-path P_5^* of F_5^*. By Lemma 1, there exist canonical Hamiltonian cycles HC_{23}, HC_1, and HC_6 of F_{23}, F_1, and F_6, respectively. Then, there exists edges $e_1^*, e_6^*, e_3^* \in P_5^*$, $e_3 \in HC_{23}$, $e_1 \in HC_1$, and $e_6 \in HC_6$ such that $e_1^* \approx e_1$, $e_6^* \approx e_6$, and $e_3^* \approx e_3$. By Statement (2) of Proposition 6, P_5^*, HC_{23}, HC_1, and HC_6 can be combined into a canonical Hamiltonian (s,t)-path of $F(m,n)$. The construction of such a canonical Hamiltonian (s,t)-path is depicted in Fig. 10(b). On the other hand, consider that F_4 is a 2-rectangle. Then, (s,t) is a vertical and nonboundary edge in F_4 (see [19]) and hence $s_x = t_x$. We then make a horizontal separation on F_4 to obtain two 1-rectangles F_{41} and F_{42} such that F_{41} is to the upper of F_{42}. Without loss of generality, assume that $s_y \leqslant t_y$. Then, $s \in F_{41}$ and $t \in F_{42}$. Let $F_3^* = F_3 \cup F_{41}$ and $F_5^* = F_5 \cup F_{42}$. Then, $F_3^* = R(m,n+1)$ and $F_5^* = R(m,n+1)$. Let $p \in F_3^*$ and $q \in F_5^*$ such that $p \neq s$, $q \neq t$, and $p \sim q$. By Lemma 2, F_3^* and F_5^* contain canonical Hamiltonian (s,p)-path P_3^* and (q,t)-path P_5^*, respectively. By Lemma 1, F_1, F_2, and F_6 contain canonical Hamiltonian cycles HC_1, HC_2, and HC_6, respectively. Then, there exist edges $e_2^* \in P_3^*$, $e_1^*, e_6^* \in P_5^*$, $e_2 \in HC_2$, $e_1 \in HC_1$, and $e_6 \in HC_6$ such that $e_2^* \approx e_2$, $e_1^* \approx e_1$, and $e_6^* \approx e_6$. By Statement (2) of Proposition 6, P_3^* and HC_2 can be combined into a canonical

Hamiltonian (s,p)-path P_3 of $F_3^* \cup F_2$, and P_5^*, HC_1, and HC_6 can be combined into a canonical Hamiltonian (q,t)-path P_5 of $F_5^* \cup F_1 \cup F_6$. Then, $P_3 \Rightarrow P_5$ forms a canonical Hamiltonian (s,t)-path of $F(m,n)$.

Case 1.4: $s,t \in F_5$ or F_6. This subcase can be proved by similar arguments in proving Case 1.2. For example, Fig. 10(c) shows a canonical Hamiltonian (s,t)-path of $F(m,n)$ under that $s,t \in F_5$.

Case 2: $s \in F_\imath$ and $t \in F_\jmath$ for $2 \leqslant \imath, \jmath \leqslant 6$ and $\imath \neq \jmath$. Without loss of generality, assume that $\imath < \jmath$. There are the following two subcases:

Case 2.1: $F_4 = R(m,n-2)$ satisfies $n-2 \geqslant 3$, i.e, $n \geqslant 5$. Let $s_\imath = s$ and $t_\jmath = t$. For $\imath \leqslant \tau \leqslant \jmath$, let s_τ and t_τ be two distinct vertices of F_τ so that $t_k \sim s_{k+1}$ for $\imath \leqslant k \leqslant \jmath - 1$. Since every $F_\tau = R(m_\tau, n_\tau)$, $\imath \leqslant \tau \leqslant \jmath$, satisfies that $m_\tau, n_\tau \geqslant 3$, F_τ contains a canonical Hamiltonian (s_τ, t_τ)-path P_τ by Lemma 2. Then, $P_\imath \Rightarrow P_{\imath+1} \Rightarrow \cdots \Rightarrow P_\jmath$ forms a canonical Hamiltonian (s,t)-path P of $\cup_{\imath \leqslant \tau \leqslant \jmath} F_\tau$. For $2 \leqslant k \leqslant \imath - 1$, there exists a canonical Hamiltonian cycle HC_k of F_k by Lemma 1, and one boundary of F_k is adjacent to one boundary of F_{k+1}. By Statements (1) and (2) of Proposition 6, P and $\cup_{2 \leqslant k \leqslant \imath-1} HC_k$ can be combined into a canonical Hamiltonian (s,t)-path P^* of $\cup_{2 \leqslant k \leqslant \jmath} F_k$. For the subgraphs F_k's, $\jmath + 1 \leqslant k \leqslant 6$, their canonical Hamiltonian cycles can be also merged into P^* by the same construction. In addition, a canonical Hamiltonian cycle HC_1 of F_1 contains a flat face that is placed to face F_5 and hence it can be combined into the canonical Hamiltonian (s,t)-path P^* of $\cup_{2 \leqslant k \leqslant 6} F_k$ by Statement (2) of Proposition 6. Thus, $F(m,n)$ contains a canonical Hamiltonian (s,t)-path.

Case 2.2: $F_4 = R(m,n-2)$ does not satisfy $n-2 \geqslant 3$, i.e, $n \leqslant 4$. In this subcase, F_4 is either a 1-rectangle or 2-rectangle. By the same arguments in proving Case 1.3, let $F_5^* = F_5 \cup F_4$ if $F_4 = R(m,1)$, and let $F_3^* = F_3 \cup F_{41}$ and $F_5^* = F_5 \cup F_{42}$ if $F_4 = R(m,2)$ in which $F_4 = R(m,2)$ is partitioned into two 1-rectangles F_{41} and F_{42} such that F_{41} is to the upper of F_{42}. Then, each partitioned subgraph $F_\tau = R(m', n')$ satisfies that $m', n' \geqslant 3$. By the same construction in Case 2.1, a canonical Hamiltonian (s,t)-path of $F(m,n)$ can be constructed.

It follows from the above cases that a canonical Hamiltonian (s,t)-path of $F(m,n)$ can be constructed, and hence $HP(F(m,n),s,t)$ does exist. ■

Based on Lemmas 8 and 9, we verify the Hamiltonian connectivity of $E(m,n)$ as follows.

Lemma 10. *Let $E(m,n)$ be an E-alphabet supergrid graph with $m \geqslant n+1 \geqslant 4$, and let s and t be two distinct vertices of $E(m,n)$. Then, $E(m,n)$ contains a canonical Hamiltonian (s,t)-path, and hence $HP(E(m,n),s,t)$ does exist.*

Proof: We first make three vertical separations on $E(m,n)$ to obtain four disjoint rectangles $E_1 = R(2m-2,n)$, $E_2 = R(m,n)$, $E_3 = R(2m-2,n)$, and $E^* = R(m,5n-4)$, as shown in Fig. 11(a). Since $m,n \geqslant 3$, we get that $E_i = R(m_i,n_i)$, $1 \leqslant i \leqslant 3$, and $E^* = R(m^*,n^*)$ satisfy $m_i, n_i \geqslant 3$ and $m^*, n^* \geqslant 3$. Consider the following three cases:

Case 1: $s,t \in E_\imath$ for some $1 \leqslant \imath \leqslant 3$. In this case, s and t are in the same rectangle $E_\imath$. Suppose that $\imath = 1$. Then, $s,t \in E_1$. By Lemma 2, there exists a canonical Hamiltonian (s,t)-path P_1 of E_1. By Lemma 1,

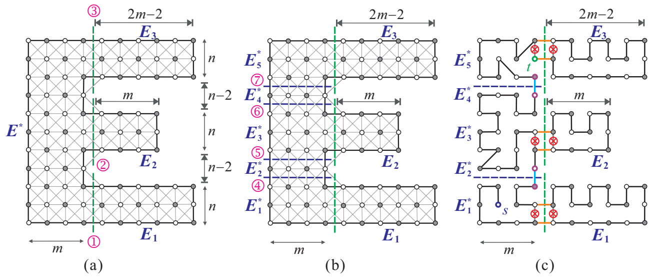


Fig. 11. (a) Three vertical separations on $E(m,n)$ to obtain E_1, E_2, E_3, and E^*, (b) four horizontal separations on E^*, and (c) the construction of $HP(E(m,n),s,t)$ under that $s \in E_1^*$, $t \in E_5^*$ and E_2^*, E_4^* are 1-rectangles, where bold dashed lines indicate the separation operations and ⊗ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path.

E^*, E_2, and E_3 contain canonical Hamiltonian cycles HC^*, HC_2, and HC_3, respectively. We can place one flat face of HC^* to face E_1, E_2, and E_3. Then, there exist six edges $e_1^*, e_2^*, e_3^* \in HC^*$, $e_1 \in P_1$, $e_2 \in HC_2$, and $e_3 \in HC_3$ such that $e_1^* \approx e_1$, $e_2^* \approx e_2$, and $e_3^* \approx e_3$. By Statements (1) and (2) of Proposition 6, P_1, HC^*, HC_2, and HC_3 can be combined into a canonical Hamiltonian (s,t)-path of $E(m,n)$. The other cases of $\imath = 2$ and $\imath = 3$ can be proved by the same arguments.

Case 2: exactly one of s and t is in $E_\imath$ for some $1 \leqslant \imath \leqslant 3$. Without loss of generality, assume that $s \in E_\imath$. Let $E' = E(m,n) - E_\imath$. Then, E' is either $C(m,n)$ or $F(m,n)$, and $t \in E'$. Let $p \in E_\imath$ and $q \in E'$ such that $p \neq s$, $q \neq t$, and $p \sim q$. By Lemma 2, there exists a canonical Hamiltonian (s,p)-path $P_\imath$ of $E_\imath$. By Lemma 8 or Lemma 9, E' contains a canonical Hamiltonian (q,t)-path P'. Then, $P_\imath \Rightarrow P'$ forms a canonical Hamiltonian (s,t)-path of $E(m,n)$.

Case 3: $s,t \in E^*$. In this case, we make four horizontal separations on E^* to partition it into five disjoint rectangles $E_\jmath^*$ for $1 \leqslant \jmath \leqslant 5$, as depicted in Fig. 11(b). By similar constructions in Cases 1–2 of Lemma 9, a canonical Hamiltonian (s,t)-path of $E(m,n)$ can be constructed. For example, when $s \in E_1^*$, $t \in E_5^*$, and $n = 3$, the constructed canonical Hamiltonian (s,t)-path of $E(m,n)$ is shown in Fig. 11(c).

In any case, a canonical Hamiltonian (s,t)-path of $E(m,n)$ is constructed. Thus, $HP(E(m,n),s,t)$ does exist. ■

It immediately follows from Lemmas 7–10 that we conclude the following theorem.

Theorem 11. *Let $A(m,n)$ be an L-alphabet, C-alphabet, F-alphabet, or E-alphabet supergrid graph with $m \geqslant n+1 \geqslant 4$, and let s and t be two distinct vertices of $A(m,n)$. Then, $A(m,n)$ contains a canonical Hamiltonian (s,t)-path, and hence $HP(A(m,n),s,t)$ does exist.*

By the proofs of Lemmas 7–10, we can see that if an alphabet supergrid graph can be decomposed into disjoint rectangular supergrid subgraphs by a series of vertical and horizontal separations, then it contains a canonical Hamiltonian path between any two vertices. Thus, we conclude the following corollary.

Corollary 12. *Let $A(m,n)$ be an alphabet supergrid graph with $m \geqslant n+1 \geqslant 4$ such that it can be partitioned into disjoint rectangular supergrid subgraphs by a series of vertical* *and horizontal separations, and let s and t be two distinct vertices of $A(m, n)$. Then, $A(m, n)$ contains a canonical Hamiltonian (s, t)-path, and hence $HP(A(m, n), s, t)$ does exist.*

We can construct G-, H-, I-, J-, O-, P-, S-, T-, and U-alphabet supergrid graphs to satisfy the above corollary. However, many other alphabet supergrid graphs can not be partitioned into only disjoint rectangular supergrid subgraphs. For example, N- and Y-alphabet supergrid graphs are such alphabet supergrid graphs. However, they can be separated into some disjoint shaped supergrid subgraphs, including rectangle, triangle, parallelogram, and trapezoid. In the following section, we will verify the Hamiltonian connectivity of N- and Y-alphabet supergrid graphs.

IV. The Hamiltonian Connectivity of N- and Y-alphabet Supergrid Graphs

In this section, we will verify the Hamiltonian connectivity of N- and Y-alphabet supergrid graphs. Let $B(m, n)$ be either an N- or Y-alphabet supergrid graph. We can see from the structures of these two types of alphabet supergrid graphs that they can be decomposed into disjoint shaped supergrid subgraphs, including rectangles, triangles, parallelograms, and trapezoids. By the Hamiltonicity and Hamiltonian connectivity of shaped supergrid graphs (see Lemmas 1–2 and 4–5), we will construct a Hamiltonian (s, t)-path of $B(m, n)$ through Statements (1)–(4) of Proposition 6.

We first verify the Hamiltonian connectivity of N-alphabet supergrid graphs. Let $N(m, n)$ be an N-alphabet supergrid graph with $m \geqslant n+1$ and $5n-4 \geqslant 3m-2$, as shown in Fig. 4(e). Then, $m \geqslant 4$ and $n \geqslant 3$. We first make two vertical separations on $N(m, n)$ to partition it into three disjoint subgraphs $N_1 = R(m, 5n-4)$, $N_2 = P(5n-4-m, m-2)$, and $N_3 = R(m, 5n-4)$, as illustrated in Fig. 12(a). Since $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$, we get that N_1 and N_3 satisfy that $m \geqslant 4$ and $5n-4 \geqslant 11$, and $N_2 = P(5n-4-m, m-2)$ satisfies that $5n-4-m \geqslant 6$ and $m-2 \geqslant 2$. The following lemma shows the Hamiltonian connectivity of $N(m, n)$, where $m \geqslant 4$ and $n \geqslant 3$.

Lemma 13. *Let $N(m, n)$ be an N-alphabet supergrid graph with $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$, and let s and t be two distinct vertices of $N(m, n)$. Then, $N(m, n)$ contains a Hamiltonian (s, t)-path, and hence $HP(N(m, n), s, t)$ does exist.*

Proof: In this lemma, $N_2 = P(5n-4-m, m-2)$ satisfies $5n-4-m \geqslant 6$ and $m-2 \geqslant 2$, and $N_1 = R(m, 5n-4)$ and $N_3 = R(m, 5n-4)$ satisfy $m \geqslant 4$ and $5n-4 \geqslant 11$. Without loss of generality, assume that $s_x \leqslant t_x$. Depending on the relative locations of s and t, there are four cases:

Case 1: $s, t \in N_1$ or N_3. By symmetry, it suffices to consider that $s, t \in N_1$. By Lemma 4, N_2 contains a canonical Hamiltonian cycle HC_2 whose two flat faces are respectively placed to face N_1 and N_3. By Lemma 1, N_3 contains a canonical Hamiltonian cycle HC_3 such that its one flat face is placed to face N_2. By Lemma 2, N_1 contains a canonical Hamiltonian (s, t)-path P_1. Then, there exist four edges $e_1 \in P_1$, $e_1^*, e_3^* \in HC_2$, and $e_3 \in HC_3$ such that $e_1 \approx e_1^*$ and $e_3 \approx e_3^*$. By Statements (1) and (2) of Proposition 6, P_1, HC_2, and HC_3 can be combined into a Hamiltonian (s, t)-path of $N(m, n)$. Fig. 12(b) depicts a such constructed Hamiltonian (s, t)-path.

Case 2: $s, t \in N_2$. By Lemma 1, there exist canonical Hamiltonian cycles HC_1 and HC_3 of N_1 and N_3, respectively. We can place two flat faces of HC_1 and HC_3 to face N_2. Consider the following two subcases:

Case 2.1: (N_2, s, t) does not satisfy conditions (F5) and (F6). By Lemma 5, there exists a Hamiltonian (s, t)-path P_2 of N_2 such that P_2 contains at least one boundary edge of each horizontal or skewed boundary in N_2. Then, there exist four edges $e_1 \in HC_1$, $e_1^*, e_3^* \in P_2$, and $e_3 \in HC_3$ such that $e_1 \approx e_1^*$ and $e_3 \approx e_3^*$. By Statement (2) of Proposition 6, P_2, HC_1, and HC_3 can be combined into a Hamiltonian (s, t)-path of $N(m, n)$.

Case 2.2: (N_2, s, t) satisfies condition (F5) or (F6). Suppose that (N_2, s, t) satisfies condition (F5). Then, N_2 is a 2-parallelogram, (s, t) is a horizontal edge of N_2, and $HP(N_2, s, t)$ does not exist. Let P_2 be the longest (s, t)-path of N_2 computed in [20], and let $N_2' = N_2 - P_2$. Then, for every vertex $v \in N_2'$, v adjoins one edge in HC_1 or HC_3. By Statement (3) Proposition 6, all of vertices of N_2' can be merged into HC_1 or HC_3. Let the above combined cycles of HC_1 and HC_3 be HC_1' and HC_3', respectively. Then, there exist four edges $e_1 \in HC_1'$, $e_1^*, e_3^* \in P_2$, and $e_3 \in HC_3'$ such that $e_1 \approx e_1^*$ and $e_3 \approx e_3^*$. By Statement (2) of Proposition 6, P_2, HC_1', and HC_3' can be combined into a Hamiltonian (s, t)-path of $N(m, n)$. Fig. 12(c) shows a such constructed Hamiltonian (s, t)-path. On the other hand, suppose that (N_2, s, t) satisfies condition (F6). Let P_2 be the longest (s, t)-path of N_2 computed in [20], and let $N_2' = N_2 - P_2$. Then, $V(N_2') = \{w\}$, where w is a parallelogram corner of N_2 such that $s \sim w$ and $t \sim w$. Since w adjoins an edge of HC_1 or HC_3, w can be merged into HC_1 or HC_3. Then, P_2 and these two above combined Hamiltonian cycles can be merged into a Hamiltonian (s, t)-path of $N(m, n)$ by the same arguments.

Case 3: ($s \in N_1$ and $t \in N_2$) or ($s \in N_2$ and $t \in N_3$). By symmetry, we can only consider that $s \in N_1$ and $t \in N_2$. Let $p \in N_1$ and $q \in N_2$ such that $p \neq s$, $q \neq t$, (N_2, q, t) does not satisfy conditions (F5) and (F6), and $p \sim q$. By Lemma 2, there exists a canonical Hamiltonian (s, p)-path P_1 of N_1. By Lemma 1, there exists a canonical Hamiltonian cycle HC_3 of N_3 such that one flat face of HC_3 is placed to face N_2. By Lemma 5, there exists a Hamiltonian (q, t)-path P_2 of N_2 such that P_2 contains at least one boundary edge in each horizontal or skewed boundary of N_2. Then, there exist two edges $e_3 \in HC_3$ and $e_2 \in P_2$ such that $e_3 \approx e_2$. By Statement (2) of Proposition 6, P_2 and HC_3 can be combined into a Hamiltonian (q, t)-path P_2' of $N_2 \cup N_3$. Then, $P_1 \Rightarrow P_2'$ forms a Hamiltonian (s, t)-path of $N(m, n)$.

Case 4: $s \in N_1$ and $t \in N_3$. Let $p \in N_1$, $q \in N_3$, and $r_1, r_2 \in N_2$ such that $p \neq s$, $q \neq t$, $p \sim r_1$, $q \sim r_2$, and (N_2, r_1, r_2) does not satisfy conditions (F5) and (F6). By Lemma 5, N_2 contains a Hamiltonian (r_1, r_2)-path P_2. By Lemma 2, there exist canonical Hamiltonian (s, p)-path P_1 and (q, t)-path P_3 of N_1 and N_3, respectively. Then, $P_1 \Rightarrow P_2 \Rightarrow P_3$ forms a Hamiltonian (s, t)-path of $N(m, n)$. Fig. 12(d) depicts such a constructed Hamiltonian (s, t)-path.

We have considered any case to construct a Hamiltonian (s, t)-path of $N(m, n)$. This completes the proof of the lemma. ■

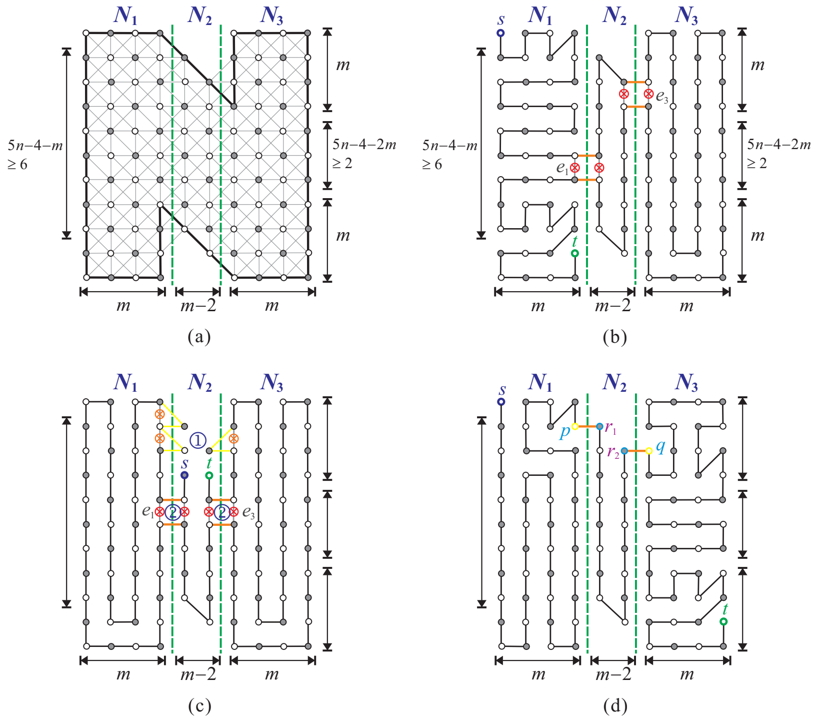


Fig. 12. (a) Two vertical separations on $N(m, n)$ to obtain two rectangular supergrid subgraphs, N_1 and N_3, and one parallelogram supergrid subgraph N_2, (b) the construction of $HP(N(m, n), s, t)$ under that $s, t \in N_1$, (c) the construction of $HP(N(m, n), s, t)$ under that $s, t \in N_2$ and (N_2, s, t) satisfies condition (F5), and (d) the construction of $HP(N(m, n), s, t)$ under that $s \in N_1$ and $t \in N_3$, where bold dashed lines indicate the separation operations and ⊗ represents the destruction of an edge while constructing a Hamiltonian (s, t)-path.

Finally, we will consider Y-alphabet supergrid graphs. Let $Y(m, n)$ be an Y-alphabet supergrid graph with $m \geqslant n+1 \geqslant 4$ and $5n - 4 \geqslant 3m - 2$, as illustrated in Fig. 4(f). Then, $m \geqslant 4$ and $n \geqslant 3$. We first make three horizontal separations on $Y(m, n)$ to obtain three rectangles $Y_1 = Y_2 = R(m, m)$, $Y_3 = R(m, (5n - 4) - (2m - 1))$, two parallelograms $Y_4 = Y_5 = P(m, \lceil \frac{m}{2} \rceil - 2)$, and one trapezoid $Y_6 = T_2(2m, \lfloor \frac{m}{2} \rfloor + 1)$ or $T_2(2m - 1, \lfloor \frac{m}{2} \rfloor + 1)$ depending on whether m is even, as depicted in Fig. 13(a). Since $5n - 4 \geqslant 3m - 2$, $m \geqslant 4$, and $n \geqslant 3$, we get that $(5n - 4) - (2m - 1) \geqslant m - 1 \geqslant 3$, $\lceil \frac{m}{2} \rceil - 2 \geqslant 0$, $2m - 1 \geqslant 7$, and $\lceil \frac{m}{2} \rceil + 1 \geqslant 3$. Let $Y_\iota = R(m', n')$ or $T_2(m', n')$ for $1 \leqslant \iota \leqslant 6$ and $\iota \neq 4, 5$. Then, Y_ι satisfies that $m' \geqslant 3$ and $n' \geqslant 3$. On the other hand, $Y_4 = Y_5 = P(m, \lceil \frac{m}{2} \rceil - 2)$ satisfies that $\lceil \frac{m}{2} \rceil - 2 \geqslant 0$, and hence, Y_4 and Y_5 may be empty. We first consider that $Y_4 = Y_5 = \emptyset$, i.e., $\lceil \frac{m}{2} \rceil - 2 = 0$, in Lemma 14.

Lemma 14. *Let $Y(m, n)$ be an Y-alphabet supergrid graph such that $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$, and let s and t be two distinct vertices of $Y(m, n)$. Let Y_i's, $1 \leqslant i \leqslant 6$, be partitioned subgraphs of $Y(m, n)$ as defined in Fig. 13(a). If $Y_4 = Y_5 = \emptyset$, then $Y(m, n)$ contains a Hamiltonian (s, t)-path, and hence $HP(Y(m, n), s, t)$ does exist.*

Proof: By the separation operations on $Y(m, n)$, as depicted in Fig. 13(a), $Y_\iota = R(m', n')$ or $T_2(m', n')$ for $1 \leqslant \iota \leqslant 6$ and $\iota \neq 4, 5$ satisfies $m' \geqslant 3$ and $n' \geqslant 3$. Since $Y_4 = Y_5 = \emptyset$ and $m \geqslant 4$, we get that $\lceil \frac{m}{2} \rceil - 2 = 0$ and hence $m = 4$. Since $m \geqslant n + 1 \geqslant 4$ and $5n - 4 \geqslant 3m - 2$, we obtain that $n = 3$. Thus, $Y(m, n) = Y(4, 3)$. Then, $Y_1 = Y_2 = R(4, 4)$, $Y_3 = R(4, 4)$, $Y_4 = Y_5 = \emptyset$, and $Y_6 = T_2(8, 3)$, as shown in Fig. 13(b). Depending on the relative locations of s and t, there are the following two cases:

Case 1: $s, t \in Y_i$ for $1 \leqslant i \leqslant 6$. In this case, s and t are in the same partitioned subgraph. Note that $Y_4 = Y_5 = \emptyset$. There are two subcases:

Case 1.1: $s, t \in Y_1, Y_2$, or Y_3. Suppose that $s, t \in Y_1$. By Lemma 2, Y_1 contains a canonical Hamiltonian (s, t)-path P_1. Then, P_1 visits at least one boundary edge of each boundary in $Y_1 = R(4, 4)$. By Lemma 1, Y_2 and Y_3 contain canonical Hamiltonian cycles HC_2 and HC_3, respectively. We can place the flat faces of HC_2 and HC_3 to face to their neighboring separated supergrid subgraphs. By Lemma 4, $Y_6 = T_2(2m, \lfloor \frac{m}{2} \rfloor + 1) = T_2(8, 3)$ contains a canonical Hamiltonian cycle HC_6 whose two flat faces are placed to face $Y_1 \cup Y_2$ and Y_3. Then, there exist edges $e_1 \in P_1$, $e_2 \in HC_2$, $e_3 \in HC_3$, and $e_{61}, e_{62}, e_{63} \in HC_6$ such that $e_1 \approx e_{61}$, $e_2 \approx e_{62}$, and $e_3 \approx e_{63}$. By Statements (1) and (2) of Proposition 6, P_1, HC_2, HC_3, and HC_6 can be combined into a Hamiltonian (s, t)-path of $Y(m, n) = Y(4, 3)$. The case of $s, t \in Y_2$ or Y_3 can be proved by the same construction.

Case 1.2: $s, t \in Y_6$. In this subcase, we first perform a vertical separation on Y_6 to obtain two trapezoid subgraphs Y_{61} and Y_{62}, as depicted in Fig. 13(b). Then, $Y_{61} = Y_{62} = T_1(m, \lfloor \frac{m}{2} \rfloor + 1) = T_1(4, 3)$. By Lemma 1, Y_1, Y_2, and Y_3 contain canonical Hamiltonian cycles HC_1, HC_2, and HC_3, respectively. We can place the flat faces of HC_1, HC_2 and HC_3 to face to their neighboring trapezoid Y_6. Consider the following subcases:

Case 1.2.1: $s, t \in Y_{61}$ or Y_{62}. Without loss of generality, assume that $s, t \in Y_{61}$. By Lemma 4, Y_{62} contains a canonical Hamiltonian cycle HC_{62}. We can place the flat faces of HC_{62} to face to their neighboring separated subgraphs Y_2, Y_3, Y_{61}. Suppose that (Y_{61}, s, t) does not satisfy condition (F8). By Lemma 5, Y_{61} contains a canonical Hamiltonian (s, t)-path P_{61}. Then, there exist edges $e_1 \in HC_1$, $e_2 \in HC_2$, $e_3 \in HC_3$, $e_{62}, e'_{66} \in HC_{62}$, and $e_{61}, e_{63}, e_{66} \in P_{61}$ such that $e_1 \approx e_{61}$, $e_2 \approx e_{62}$, $e_3 \approx e_{63}$, and $e'_{66} \approx e_{66}$. By Statements (1) and (2) of Proposition 6, HC_1, HC_2, HC_3, HC_{62}, and P_{61} can be combined into a Hamiltonian (s, t)-path of $Y(m, n) = Y(4, 3)$. On the other hand, suppose that (Y_{61}, s, t) satisfies condition (F8). Let w be a trapezoid corner of Y_{61} such that $w \sim s$ and $w \sim t$. Then, there exists one edge $e = (u, v)$ of HC_1 such that $w \sim u$ and $w \sim v$. By Statement (3) Proposition 6, HC_1 and w can be merged into a Hamiltonian cycle HC'_1 of $Y_1 \cup \{w\}$. By the construction in [20], we can construct a canonical Hamiltonian (s, t)-path P'_{61} of $Y_{61} - \{w\}$. Then, $P'_{61}, HC'_1, HC_2, HC_3$, and HC_{62} can be combined into a Hamiltonian (s, t)-path of $Y(m, n) = Y(4, 3)$ by Statements (1) and (2) of Proposition 6. The construction of a such Hamiltonian (s, t)-path is shown in Fig. 13(b).

Case 1.2.2: $s \in Y_{61}$ and $t \in Y_{62}$. Let $p \in Y_{61}$ and $q \in Y_{62}$ such that $p \sim q$, (Y_{61}, s, p) and (Y_{62}, q, t) do not satisfy condition (F8). The vertices p and q can be easily computed. By Lemma 5, Y_{61} and Y_{62} contain canonical Hamiltonian (s, p)-path P_{61} and (q, t)-path P_{62}, respectively. Then, $P_6 = P_{61} \Rightarrow P_{62}$ forms a canonical Hamiltonian (s, t)-path of Y_6. By Statements (1) and (2) of Proposition 6, P_6, HC_1, HC_2, and HC_3 can be combined into a Hamiltonian (s, t)-path of $Y(m, n) = Y(4, 3)$.

Case 2: $s \in Y_i$ and $t \in Y_j$ for $1 \leqslant i, j \leqslant 6$ and $i \neq j$. Depending on whether Y_i and Y_j are adjacent subgraphs, we consider the following subcases:

Case 2.1: Y_i and Y_j are adjacent, i.e. they are neighboring partitioned subgraphs. In this subcase, Y_i or Y_j is Y_6. Without loss of generality, assume that $Y_j = Y_6$. Then, $s \in Y_i$ for $1 \leqslant i \leqslant 3$, and $t \in Y_6$. Let $p \in Y_i - \{s\}$ and $q \in Y_6 - \{t\}$ such that $p \sim q$ and (Y_6, q, t) does not satisfy condition (F8). By Lemma 2, Y_i contains a canonical Hamiltonian (s, p)-path P_i. By similar construction in Case 1.2, Y_6 contains a canonical Hamiltonian (q, t)-path P_6. Then, $P = P_i \Rightarrow P_6$ forms a canonical Hamiltonian (s, t)-path of $Y_i \cup Y_6$. Let Y_α, Y_β be the partitioned subgraphs different from Y_i and Y_j. Then, Y_α and Y_β are rectangles. By Lemma 1, Y_α and Y_β contain canonical Hamiltonian cycles HC_α and HC_β, respectively, such that their flat faces are placed to face Y_6. By Statements (1) and (2) of Proposition 6, P, HC_α, and HC_β can be combined into a Hamiltonian (s, t)-path of $Y(m, n) = Y(4, 3)$.

Case 2.2: Y_i and Y_j are not adjacent. In this subcase, $Y_i, Y_j \neq Y_6$. Let $p \in Y_i - \{s\}$, $q \in Y_j - \{t\}$, and $r_1, r_2 \in Y_6$ such that $p \sim r_1$, $q \sim r_2$, and (Y_6, r_1, r_2) does not satisfy condition (F8). By Lemma 2, Y_i and Y_j contain Hamiltonian (s, p)-path P_i and (q, t)-path P_j, respectively. By similar construction in Case 1.2, Y_6 contains a canonical Hamiltonian (r_1, r_2)-path P_6. Then, $P = P_i \Rightarrow P_6 \Rightarrow P_j$ is a Hamiltonian (s, t)-path of $Y_i \cup Y_j \cup Y_6$. Let Y_α be the partitioned subgraph different from Y_i, Y_j, and Y_6. Then, Y_α is a rectangle. By Lemma 1, Y_α contains a canonical Hamiltonian cycle HC_α whose one flat face is placed to face Y_6. Then, there exist two edges $e \in P$ and $e_\alpha \in HC_\alpha$ such that $e \approx e_\alpha$. By Statement (2) of Proposition 6, P and HC_α can be combined into a Hamiltonian (s, t)-path of $Y(m, n) = Y(4, 3)$. Fig. 13(c) depicts the construction of a such Hamiltonian (s, t)-path.

It immediately follows from the above cases that the lemma holds true. ■

We have considered the case of $Y_4 = Y_5 = \emptyset$. Next, we will consider $Y_4 (= Y_5) \neq \emptyset$. Then, $Y_4 = P(m, \lceil \frac{m}{2} \rceil - 2) \neq \emptyset$, and hence $\lceil \frac{m}{2} \rceil - 2 > 0$. Thus, $m \geqslant 5$. When $m = 5$ or 6, $\lceil \frac{m}{2} \rceil - 2 = 1$ and hence Y_4 and Y_5 are 1-parallelograms. The following lemma shows the Hamiltonian connectivity of $Y(m, n)$ under that $6 \geqslant m \geqslant 5$.

Lemma 15. *Let $Y(m, n)$ be an Y-alphabet supergrid graph such that $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$, and let s and t be two distinct vertices of $Y(m, n)$. Let Y_i's, $1 \leqslant i \leqslant 6$, be partitioned subgraphs of $Y(m, n)$ as defined in Fig.* 13(a). *If Y_4 and Y_5 are 1-parallelograms, then $Y(m, n)$ contains a Hamiltonian (s, t)-path, and hence $HP(Y(m, n), s, t)$ does exist.*

Proof: Since Y_4 is a 1-parallelogram, we get that $m = 5$ or 6. The partitioned subgraphs of $Y(5, n)$ is depicted in Fig. 14(a). By Lemmas 1 and 4, Y_1, Y_2, Y_3, and Y_6 contain canonical Hamiltonian cycles HC_1, HC_2, HC_3, and HC_6, respectively, such that their flat faces are placed to face their neighboring partitioned subgraphs. Consider the following cases:

Case 1: $s, t \in Y_i$ for $1 \leqslant i \leqslant 6$. In this case, s and t are in the same partitioned subgraph. If $s, t \notin Y_4$ and Y_5, then the case can be verified by similar arguments in proving Case 1 of Lemma 14. Suppose that $s, t \in Y_4$ or Y_5. Without loss of generality, assume that $s, t \in Y_4$. By visiting every vertex of Y_5, we can obtain a Hamiltonian path P_5 of Y_5. By Statements (1), (2), and (4) of Proposition 6, HC_2, P_5, HC_6, and HC_3 can be combined into a Hamiltonian cycle HC' of $Y_2 \cup Y_5 \cup Y_6 \cup Y_3$. Consider the following subcases:

Case 1.1: (Y_4, s, t) satisfies condition (F4). In this subcase, Y_4 contains no Hamiltonian (s, t)-path. Let P_4 be the longest (s, t)-path of Y_4. Let e_{41} and e_{46} be two edges in Y_4 such that $e_{41} \in P_4$ and, $e_{46} \in P_4$ if $|V(P_4)| \geqslant 3$, and $e_{46} \in Y_4 - P_4$ otherwise. Since $m \geqslant 5$, e_{46} does exist. Let $W = Y_4 - P_4 - e_{46}$. By Statement (3) of Proposition 6, each vertex of W can be embedded into cycle HC_1 or HC'. Let the combined cycles to be HC'_1 and HC^*. Let $e_1 \in HC'_1$ and $e^* \in HC^*$ such that $e_{41} \approx e_1$ and $e_{46} \approx e^*$. By Statement (2) of Proposition 6, P_4, e_{46}, and HC'_1 can be merged into a path P^*. Then, there exist two edges $e_{46} \in P^*$ and $e^* \in HC^*$ such that $e_{46} \approx e^*$. By Statement (2) of Proposition 6, P^* and HC^* can be combined into a Hamiltonian (s, t)-path of $Y(m, n)$ with $m = 5$ or 6. The construction of such a Hamiltonian (s, t)-path is shown in Fig. 14(b).

Case 1.2: (Y_4, s, t) does not satisfy condition (F4). In this subcase, Y_4 contains a Hamiltonian (s, t)-path P_4. By Statement (2) of Proposition 6, P_4, HC_1, and HC' can be easily combined into a Hamiltonian (s, t)-path of $Y(m, n)$ with $m = 5$ or 6.

Case 2: $s \in Y_i$ and $t \in Y_j$ for $1 \leqslant i, j \leqslant 6$ and $i \neq j$. In this case, s and t are in the different partitioned subgraphs. We only consider the case of $s \in Y_1$ and $t \in Y_4$. The other case can be verified similarly. Let $p \in Y_1$ and $q \in Y_4$ such that $p \sim q$ and q is a corner of Y_4. Let P_4 be the longest (q, t)-path of Y_4 and let $W = Y_4 - P_4$. Then, every vertex

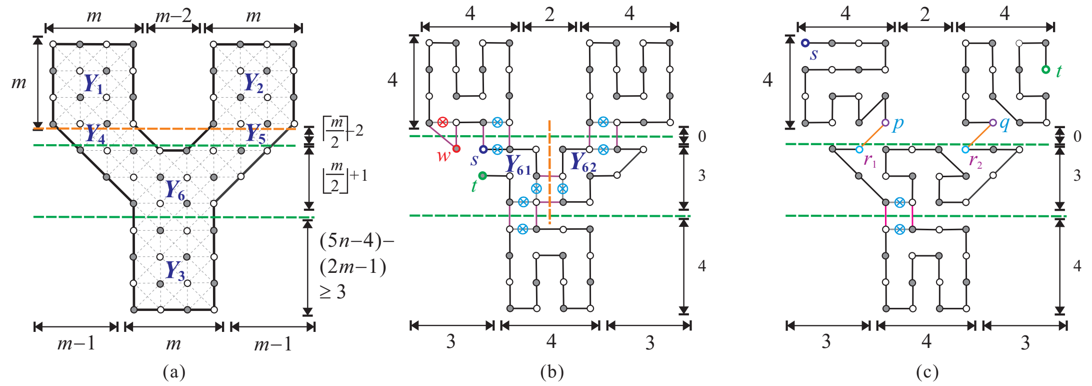


Fig. 13. (a) Three horizontal separations on $Y(m,n)$ to obtain three rectangles Y_1–Y_3, two parallelograms Y_4–Y_5, and one trapezoid Y_6, (b) the construction of $HP(Y(4,3),s,t)$ under that $s,t \in Y_{61}$ and (Y_{61},s,t) satisfies condition (F8), and (c) the construction of $HP(Y(4,3),s,t)$ under that $s \in Y_1$ and $t \in Y_2$, where bold dashed lines indicate the separation operations and $\otimes$ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path.

of W can be merged into HC_6, as depicted in Fig. 14(c). Let HC'_6 be the embedded Hamiltonian cycle of $Y_6 \cup W$. By Lemma 2, Y_1 contains a Hamiltonian (s,p)-path P_1. Then, $P' = P_1 \Rightarrow P_4$ forms a canonical Hamiltonian (s,t)-path of $Y_1 \cup (Y_4 - W)$. By Statement (2) of Proposition 6, we can construct a canonical Hamiltonian cycle HC'_2 of $Y_2 \cup Y_5$. By Statement (1) of Proposition 6, HC'_2, HC'_6, and HC_3 can be combined into a cycle C^*. Then, there exist two edges $e' \in P'$ and $e^* \in C^*$ such that $e' \approx e^*$. By Statement (2) of Proposition 6, P' and C^* can be combined into a Hamiltonian (s,t)-path of $Y(m,n)$ with $m = 5$ or 6. The construction of such a Hamiltonian (s,t)-path is shown in Fig. 14(c).

In any case, we have constructed a Hamiltonian (s,t)-path of $Y(m,n)$, where $6 \geqslant m \geqslant 5$. Thus, $HP(Y(m,n),s,t)$ does exist under that Y_4 and Y_5 are 1-parallelograms. ∎

For the case that Y_4 and Y_5 are 2-parallelograms, we get that $\lceil \frac{m}{2} \rceil - 2 = 2$, and hence $m = 7$ or 8. The following lemma shows the Hamiltonian connectivity of $Y(m,n)$ under that $8 \geqslant m \geqslant 7$, and can be verified by similar arguments in proving Lemma 15.

Lemma 16. *Let $Y(m,n)$ be an Y-alphabet supergrid graph such that $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$, and let s,t be two distinct vertices of $Y(m,n)$. Let Y_i's, $1 \leqslant i \leqslant 6$, be partitioned subgraphs of $Y(m,n)$ as defined in Fig. 13(a). If Y_4 and Y_5 are 2-parallelograms, then $Y(m,n)$ contains a Hamiltonian (s,t)-path, and hence $HP(Y(m,n),s,t)$ does exist.*

Proof: By similar arguments in proving Lemma 15, the lemma can be proved. ∎

Finally, we consider that $m \geqslant 9$. Then, $Y_4 = Y_5 = P(m, \lceil \frac{m}{2} \rceil - 2)$ is a κ-parallelogram with $\kappa \geqslant 3$. By similar arguments in proving Lemma 14, we verify the Hamiltonian connectivity of $Y(m,n)$ under $m \geqslant 9$ as follows.

Lemma 17. *Let $Y(m,n)$ be an Y-alphabet supergrid graph such that $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$, and let s and t be two distinct vertices of $Y(m,n)$. Let Y_i's, $1 \leqslant i \leqslant 6$, be partitioned subgraphs of $Y(m,n)$ as defined in Fig. 13(a). If $m \geqslant 9$, then $Y(m,n)$ contains a Hamiltonian (s,t)-path, and hence $HP(Y(m,n),s,t)$ does exist.*

Proof: By Lemmas 1 and 4, Y_i, $1 \leqslant i \leqslant 6$, contains a canonical Hamiltonian cycle HC_i such that its flat faces are placed to face its neighboring partitioned subgraphs. By Lemmas 2 and 5, $HP(Y_i,s_i,t_i)$ does exist and Y_i contains a canonical Hamiltonian (s_i,t_i)-path P_i for $1 \leqslant i \leqslant 6$, if (Y_i,s_i,t_i) does not satisfy conditions (F6)–(F8), where s_i and t_i are any two distinct vertices of Y_i. On the other hand, HC_1 and HC_4 (resp., HC_2 and HC_5) can be easily combined into a canonical Hamiltonian cycle HC'_1 (resp., HC'_2) of $Y_1 \cup Y_4$ (resp., $Y_2 \cup Y_5$). By the same arguments in proving Lemma 14, we consider the following cases:

Case 1: $s,t \in Y_i$ for $1 \leqslant i \leqslant 6$. Let $s,t \in Y_\gamma$. Then, Y_γ contains a canonical Hamiltonian (s,t)-path P_γ if (Y_i,s,t) does not satisfy conditions (F6)–(F8). For the partitioned subgraph Y_ι, $\iota \neq \gamma$, Y_ι contains a canonical Hamiltonian cycle HC_ι such that its flat faces are placed to face its neighboring partitioned subgraphs. Note that if $Y_\gamma = Y_6$ is a trapezoid, we can construct a canonical Hamiltonian (s,t)-path of Y_6 by the same arguments in proving Case 1.2 of Lemma 14. There are two subcases:

Case 1.1: (Y_γ,s,t) satisfies condition (F6), (F7), or (F8). In this subcase, $Y_\gamma = Y_4$, Y_5, or Y_6. In [20], we can construct a canonical Hamiltonian (s,t)-path P'_γ of $Y_\gamma - \{w\}$, where w is a corner of Y_γ with $s \sim w$ and $t \sim w$. Let Y_ι be a neighboring partitioned subgraph of Y_γ. Then, there exists one edge (u,v) in HC_ι such that $u \sim w$ and $v \sim w$. By Statement (3) of Proposition 6, HC_ι and w can be merged into a canonical Hamiltonian cycle HC'_ι of $Y_\iota \cup \{w\}$. Then, we can find some parallel edges in P'_γ, HC'_ι, and $HC_\jmath$'s, where $\jmath \neq \gamma$ and $\jmath \neq \iota$. By Statements (1) and (2) of Proposition 6, P'_γ, HC'_ι, and $HC_\jmath$'s can be combined into a Hamiltonian (s,t)-path of $Y(m,n)$.

Case 1.2: (Y_γ,s,t) does not satisfy conditions (F6)–(F8). In this subcase, we can easily compute some parallel edges in P_γ and HC_ι's, where $1 \leqslant \iota \leqslant 6$ and $\iota \neq \gamma$. By Statements (1) and (2) of Proposition 6, P_γ and HC_ι's can be combined into a Hamiltonian (s,t)-path of $Y(m,n)$.

Case 2: $s \in Y_i$ and $t \in Y_j$ for $1 \leqslant i \leqslant 6$ and $i \neq j$. Depending on whether Y_i and Y_j are adjacent neighbors, we consider the following subcases:

Case 2.1: Y_i and Y_j are adjacent partitioned subgraphs. Let $p \in Y_i$ and $q \in Y_j$ such that $p \sim q$, and (Y_i,s,p) and (Y_j,q,t) do not satisfy conditions (F6)–(F8). The vertices p, q can be easy to compute. By Lemmas 2 and 5, Y_i

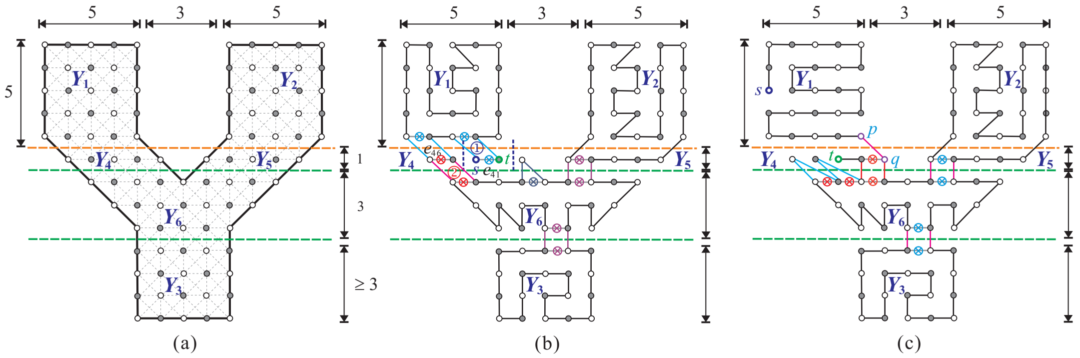


Fig. 14. (a) The partitioned subgraphs of $Y(m,n)$ for $m=5$, (b) the construction of $HP(Y(5,n),s,t)$ under that $s,t\in Y_4$ and (Y_4,s,t) satisfies condition (F4), and (c) the construction of $HP(Y(5,n),s,t)$ under that $s\in Y_1$ and $t\in Y_4$, where bold dashed lines indicate the separation operations and $\otimes$ represents the destruction of an edge while constructing a Hamiltonian (s,t)-path.

and Y_j contain canonical Hamiltonian (s,p)-path P_i and Hamiltonian (q,t)-path P_j. Then, $P=P_i\Rightarrow P_j$ forms a canonical Hamiltonian (s,t)-path of $Y_i\cup Y_j$. Let Y_γ be the partitioned subgraph such that $Y_\gamma\neq Y_i$ and $Y_\gamma\neq Y_j$. Then, we can easily find some parallel edges in P and HC_γ's. By Statements (1) and (2) of Proposition 6, P and HC_γ's can be combined into a Hamiltonian (s,t)-path of $Y(m,n)$.

Case 2.2: Y_i and Y_j are not adjacent. Let $Y'_1=Y_1\cup Y_4$ and $Y'_2=Y_2\cup Y_5$. By the constructions in [19] and [20], we can construct a canonical Hamiltonian (p_1,q_1)-path and (p_2,q_2)-path of Y'_1 and Y'_2, respectively, where $p_1,q_1\in Y'_1$ and $p_2,q_2\in Y'_2$. Let $s\in Y'_i$ and $t\in Y'_j$, where $Y'_i,Y'_j\in\{Y'_1,Y'_2,Y_3,Y_6\}$. If Y'_i and Y'_j are adjacent, then $HP(Y(m,n),s,t)$ can be constructed by the same arguments in proving Case 2.1; otherwise, it can be constructed by the same arguments in proving Case 2.2 of Lemma 14.

We have considered any case to construct a Hamiltonian (s,t)-path of $Y(m,n)$ for $m\geqslant 9$, and hence the lemma holds true. ■

It immediately follows from Lemmas 14–17 that the following lemma holds true.

Lemma 18. *Let $Y(m,n)$ be an Y-alphabet supergrid graph such that $m\geqslant n+1\geqslant 4$ and $5n-4\geqslant 3m-2$, and let s and t be two distinct vertices of $Y(m,n)$. Then, $Y(m,n)$ contains a Hamiltonian (s,t)-path, and hence $HP(Y(m,n),s,t)$ does exist.*

By Lemmas 13 and 18, we conclude the following theorem.

Theorem 19. *Let $B(m,n)$ be an N-alphabet or Y-alphabet supergrid graph with $m\geqslant n+1\geqslant 4$, and let s and t be two distinct vertices of $B(m,n)$. Then, $B(m,n)$ contains a Hamiltonian (s,t)-path, and hence $HP(B(m,n),s,t)$ does exist.*

We have verified the Hamiltonian connectivity of L-, C-, F-, E-, N-, and Y-alphabet supergrid graphs. The structures of the other alphabet supergrid graphs are shown in Fig. 15, and their Hamiltonian connectivity can be verified similarly. We finally conclude the following corollary.

Corollary 20. *Let $AB(m,n)$ be an alphabet supergrid graph which is an induced subgraph of $R(3m-2,5n-4)$, where $m\geqslant n+1\geqslant 4$ and $5n-4\geqslant 3m-2$, and let s and t be two distinct vertices of $AB(m,n)$. Then, $AB(m,n)$ contains a Hamiltonian (s,t)-path, and hence $HP(AB(m,n),s,t)$ does exist.*

V. Concluding Remarks

Based on the Hamiltonicity and Hamiltonian connectivity of rectangular supergrid graphs, we prove that L-, C-, F-, and E-alphabet supergrid graphs are Hamiltonian connected. These types of alphabet supergrid graphs can be partitioned into disjoint rectangular supergrid subgraphs. In addition, many other alphabet supergrid graphs can not be decomposed into disjoint rectangles. However, they can be partitioned into disjoint shaped supergrid subgraphs, including rectangles, triangles, parallelograms, and trapezoids. By using the Hamiltonicity and the Hamiltonian connectivity of shaped supergrid graphs, we verify the Hamiltonian connectivity of N- and Y-alphabet supergrid graphs in which they can be partitioned into disjoint rectangles, parallelograms, and trapezoids. The other types of alphabet supergrid graphs can be verified to be Hamiltonian connected similarly. We leave their proofs to interested readers. On the other hand, the Hamiltonian cycle problem on solid grid graphs was known to be polynomial solvable. However, it remains open for solid supergrid graphs in which there exists no hole.

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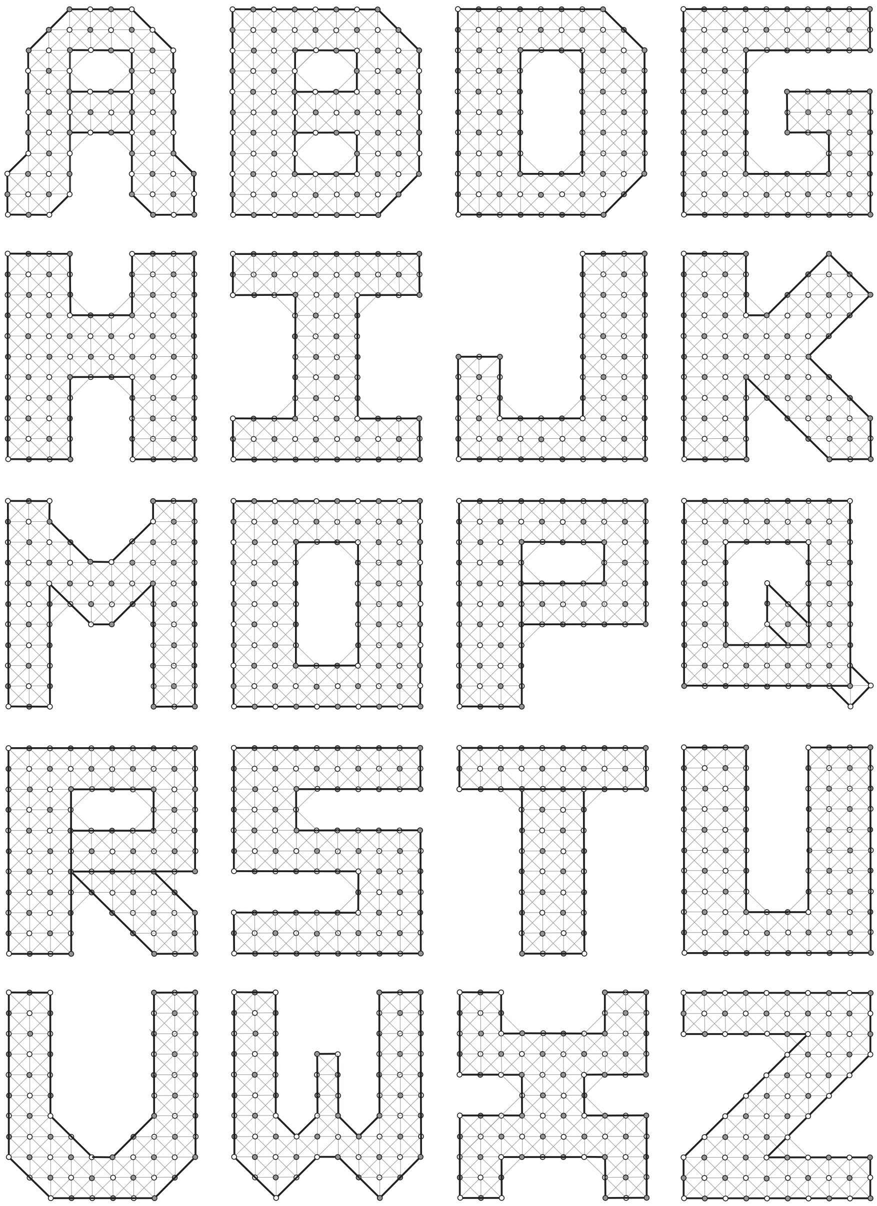

Fig. 15. The structures of the other alphabet supergrid graphs not verified in the paper, where they are defined as induced subgraphs of $R(3m-2, 5n-4)$ such that $m \geqslant n+1 \geqslant 4$ and $5n-4 \geqslant 3m-2$. Note that $m = 4$ and $n = 3$ in this figure.

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Ruo-Wei Hung received his B.S. degree in computer science and information engineering from Tunghai University, Taichung, Taiwan in 1979, and M.S. degree in computer science and information engineering from National Chung Cheng University, Chiayi, Taiwan in 1981. He received his Ph.D. degree in computer science and information engineering from National Chung Cheng University, Chiayi, Taiwan in 2005. Now, he is a Professor in the Department of Computer Science and Information Engineering, Chaoyang University of Technology, Wufeng District, Taichung, Taiwan. His current research interests include computer algorithms, graph theory, computer networking, sensor networks, embedded systems, and mobil devices applications.

Fatemeh Keshavarz-Kohjerdi received his Ph.D. degree in department of computer engineering and IT from Amirkabir University of Technology, Tehran, Iran in 2016. Now, he is a Assistant Professor in the Department of Computer Science, Shahed University, Tehran, Iran. His current research interests include computer algorithms, graph theory, and computational geometry.