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ORIGINAL PAPER

Bounds on the Identifying Codes in Trees

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Abstract
In this paper, we continue the study of identifying codes in graphs, introduced by Karpovsky et al. (IEEE Trans Inf Theory 44:599–611, 1998). A subset S of vertices in a graph G is an *identifying code* if for every pair of vertices x and y of G, the sets $N[x] \cap S$ and $N[y] \cap S$ are non-empty and different. The minimum cardinality of an identifying code in G is denoted by $M(G)$. We show that for a tree T with $n \geq 3$ vertices, ℓ leaves and s support vertices, $(2n - s + 3)/4 \leq M(T) \leq (3n + 2\ell - 1)/5$. Moreover, we characterize all trees achieving equality for these bounds.

Keywords Identifying code · Tree

Mathematics Subject Classification 05C69

1 Introduction

For notation and graph theory terminology in general we follow [10]. We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$, denoted by $N(v)$ (or $N_G(v)$ to refer it to G), is the set $\{u \in V \mid uv \in E\}$ and the *degree* of v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. A *leaf* of a tree T is a vertex of degree one, while a support vertex of T is a vertex

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adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to precisely one leaf. We denote the set of all support vertices of a tree T by $S(T)$ and the set of leaves by $L(T)$. We always denote $\ell = \ell(T) = |L(T)|$, and $s = s(T) = |S(T)|$. Whenever a tree T' (or T'', ...) is introduced, we let n', ℓ' (or n'', ℓ'',...) be its order, and number of leaves, respectively. A rooted tree T distinguishes one vertex u called the root. For each vertex $v \neq u$ of T we denote by T_v the sub-rooted tree rooted at v.

We also denote by ℓ_v the number of leaves adjacent to a support vertex v. We denote a path of order n by P_n (or $P_n : v_1 v_2 \ldots v_n$, where $V(P_n) = \{v_1, \ldots, v_n\}$ and v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$). The *distance* $d(x, y)$ between two vertices x and y is the length of a shortest path from x to y. The *diameter* $diam(G)$ of a graph G is the maximum distance over all pairs of vertices of G. For a rooted tree T and a vertex v, we denote by T_v the sub-rooted tree, rooted at v. A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* of a graph G is the graph obtained from G by subdividing each edge of G. The subdivision tree of a tree of order at least three, is called a *healthy spider*. A *wounded spider* is the graph formed by removing at least one leaf of a healthy spider. A *spider* is a tree which is either a healthy spider or a wounded spider.

A subset D of vertices in a graph G is an *identifying code* if for every two vertices x and y of G, the sets $N[x] \cap D$ and $N[y] \cap D$ are non-empty and different. The minimum cardinality of an identifying code in G is denoted by $M(G)$. Any identifying code with $M(G)$ elements is called a $M(G)$-set. Identifying codes were defined in [12] to model fault diagnosis in multiprocessor systems. In these systems, it may happen that some of the processors become faulty, in some sense that depends on the purpose of the system, and we wish to detect and replace such processors, so that the system can work properly. This concept was further studied in, for example, [1–3,5–9]. Bertrand et al. [2] obtained the minimum cardinality of an identifying code in a path.

Theorem 1 (Bertrand et al. [2]) *For a path P_n, $M(P_n) = (n+1)/2$ if $n \geq 1$ is odd, and $M(P_n) = (n+2)/2$ if $n \geq 4$ is even.*

Blidia et al. [3] obtained the following lower bound for the minimum cardinality of an identifying code of a tree.

Theorem 2 (Blidia et al. [3]) *If T is a tree of order $n \geq 4$, then $M(T) \geq 3(n + \ell - s + 1)/7$, and this bond is sharp for infinitely many values of n.*

In this paper we show that for a tree T with $n \geq 3$ vertices, l leaves and s support vertices, $(2n - s + 3)/4 \leq M(T) \leq (3n + 2\ell - 1)/5$. Moreover, we characterize all trees achieving these bounds. With our lower bound, the bound given in Theorem 2 is improved for trees of order n with ℓ leaves and s support vertices, where $2n + 9 \geq 12\ell - 5s$. The following is useful.

Observation 3 (Blidia et al. [3]) *If C is a $M(T)$-set in a tree T, then at most one vertex of L_x is not in C for each support vertex x, where L_x is the set consisting x and all of its leaves. Moreover, C contains at least ℓ vertices of $L(T) \cup S(T)$.*

We end this section by stating a variant of domination, namely *differentiating-total dominating set* which is similar to an identifying code. A *total dominating set* of a graph

G with no isolated vertex is a set S of vertices such that every vertex of G is adjacent to a vertex in S. A total dominating set S of a graph G is a *differentiating-total dominating* set if for every pair of distinct vertices u and v in $V(G)$, $N[u] \cap S \neq N[v] \cap S$. Let $\gamma_t^D(G)$ be the minimum cardinality of a differentiating-total dominating set of G. This variant has been introduced by Haynes, Henning and Howard [11], and further studied, for example in, [4,13]. The following follows immediately from the definition.

Observation 4 *Any differentiating-total dominating set in a graph is an identifying code.*

2 Lower Bound

In this section we give a lower bound on the identifying code of a tree, and characterize all trees achieving equality of the lower bound. The *corona* of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is a graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1, where $1 \leq i \leq |V(G_1)|$. If $G_2 = K_1$, then we denote $G_1 \circ G_2$ by $cor(G_1)$.

Theorem 5 *If T is a tree of order $n \geq 4$ with s support vertices, then $M(T) \geq (2n - s + 3)/4$, with equality if and only if $T = P_3$ or $T = cor(P_3)$.*

Proof We proceed by induction on the order n. If $3 \leq n \leq 5$, then it can be easily checked that $M(T) \geq (2n-s+3)/4$, and equality holds if and only if $T = P_3$. These are sufficient for the base step of the induction. Assume that the result holds for any tree T' of order $n' < n$. Now consider the tree T of order $n \geq 6$. Let C be a $M(T)$-set that contains as few leaves of T as possible. Assume that $u, v \in V-C$ and $uv \in E(T)$. Let T_u and T_v be the components of $T - uv$ with $u \in V(T_u)$ and $v \in V(T_v)$. Assume that $w \in V(T_u) \cap C$, then $N(w) - N(u) \neq \emptyset$. Hence $|V(T_u)| \geq |N(w)| \geq 3$ and similarly $|V(T_v)| \geq 3$. Since $u \notin C$, the set $C \cap V(T_u)$ is an identifying code for T_u and also the set $C \cap V(T_v)$ is an identifying code of T_v. Hence by the induction hypothesis, we have

$$\begin{aligned}
|C| &= |C \cap V(T_u)| + |C \cap V(T_v)| \\
&\geq (2|V(T_u)| - |S(T_u)| + 3)/4 + (2|V(T_v)| - |S(T_v)| + 3)/4 \\
&\geq (2n - (s+2) + 6)/4 > (2n - s + 3)/4.
\end{aligned}$$

Thus $V - C$ is an independent set.

Now assume that there exists a vertex $w \in V - C$, such that $\deg(w) \geq 3$. Let $\{x, y, z\} \subseteq N(w)$. Since $V - C$ is an independent set, we have $\{x, y, z\} \subseteq C$. Let T_x and T_w be the components of $T - xw$ with $x \in V(T_x)$ and $w \in V(T_w)$. Clearly $C \cap V(T_x)$ is an identifying code for T_x and $C \cap V(T_w)$ is an identifying code for T_w. By the inductive hypothesis,

$$\begin{aligned}
|C| &= |C \cap V(T_x)| + |C \cap V(T_w)| \\
&\geq (2|V(T_x)| - |S(T_x)| + 3)/4 + (2|V(T_w)| - |S(T_w)| + 3)/4 \\
&\geq (2n - (s+1) + 6)/4 > (2n - s + 3)/4.
\end{aligned}$$

Thus we assume for the next that $\deg(w) \leq 2$ for every vertex $w \in V - C$.

Assume that $T[C]$ is connected. Then $\deg(w) = 1$ for every vertex $w \in V - C$, and so $V - (L \cup S) \subseteq C$. By Observation 3, C contains at least ℓ vertices of $L \cup S$. Thus $|C| \geq |V - (L \cup S)| + |C \cap (L \cup S)| \geq |V - (L \cup S)| + l = n - s$. If $\ell \geq 4$, then $2n \geq 2\ell + 2s \geq 3s + \ell \geq 3s + 4$ and so $M(T) \geq n - s > (2n - s + 3)/4$, as desired. Thus we may assume that $\ell \leq 3$. If $\ell > s$, then $l = 3$ and $s = 2$, since $n \geq 6$. Hence $2n \geq 2\ell + 2s \geq 10$ and thus $M(T) \geq n - s = n - 2 > (2n + 1)/4 = (2n - s + 3)/4$, as desired. Thus we may assume that $\ell = s$. Similarly, we may assume that $n = \ell + s$. If $\ell = 2$, then $n = 4$, a contradiction. Thus assume that $\ell = 3$. Then $T = cor(P_3)$ and $M(T) = 3 = (2n - s + 3)/4$.

Next assume that $T[C]$ is not connected. Let C_1 be a largest component of $T[C]$. If $|V(C_1)| = 1$, then C is independent, $L \subseteq C$ and $\deg(w) = 2$ for every vertex $w \in V - C$. We show that $n = 2|C| - 1$. Let T^* be the graph of order $|C|$ with $V(T^*) = C$ such that $uv \in E(T^*)$ if and only if there exists a vertex $w \in V - C$ with $N_T(u) \cap N_T(v) = \{w\}$. Then $|E(T^*)| = |V - C|$ and since T is a tree, by the construction of T^*, it follows that T^* is a tree. Therefore we have, $n = |C| + |V - C|$ $= |C| + |E(T^*)| = |C| + |C| - 1 = 2|C| - 1$. Now, since T is not a star, we have $|C| = (n + 1)/2 > (2n - s + 3)/4$, as desired. Thus, assume that $|V(C_1)| \geq 2$. If $|V(C_1)| = 2$ and $C_1 = \{x, y\}$, then $N(x) \cap C = \{x, y\} = N(v) \cap C$, a contradiction. Hence $|V(C_1)| \geq 3$. Let $v \in V(C_1)$ be a vertex such that $N(v) - V(C_1) \neq \emptyset$, and $w \in N(v) - V(C_1)$ be a vertex with $\deg(w) = 2$. Let $N(w) = \{u, v\}$. Since $T[C_1]$ is connected, we find that $u \notin C_1$. Assume that $\deg(v) = 2$. Let $T' = T - wu$, and T_w and T_u be the components of $T - wu$ with $w \in V(T_w)$ and $u \in V(T_u)$. Clearly $C \cap V(T_w)$ is an identifying code for T_w and $C \cap V(T_u)$ is an identifying code for T_u. By the inductive hypothesis, we can easily obtain that $|C| = |C \cap V(T_w)| + |C \cap V(T_u)|$ $\geq (2|V(T_w)| - |S(T_w)| + 3)/4 + (2|V(T_u)| - |S(T_u)| + 3)/4 > (2n - s + 3)/4$. Thus we may assume that $\deg(v) \geq 3$.

Assume that $N(u) \cap C \neq \emptyset$. If $\deg(u) = 2$, then the result follows as before by letting $T' = T - wv$. Thus assume that $\deg(u) \geq 3$. Let T_v and T_u be the components of $T - w$ with $v \in V(T_v)$ and $u \in V(T_u)$. As before, $C \cap V(T_v)$ is an identifying code for T_v and $C \cap V(T_u)$ is an identifying code for T_u. By the inductive hypothesis, we have $|C| = |C \cap V(T_u)| + |C \cap V(T_v)| \geq (2|V(T_u)| - |S(T_u)| + 3)/4 + (2|V(T_v)|$ $- |S(T_v)| + 3)/4 \geq (2n - s + 6)/4 > (2n - s + 3)/4$. Thus we may assume that $N(u) \cap C = \emptyset$. Let T_u and T_v be the components of $T - w$, defined as before. If $|V(T_u)| = 1$, then $C' = (C - \{u\}) \cup \{w\}$ is a $M(T)$-set with $|L \cap C'| < |L \cap C|$, a contradiction by the choice of C (since C is a $M(T)$-set that contains as few leaves of T as possible.) Thus, $|V(T_u)| \geq 2$, and so we obtain that $|V(T_u)| \geq 3$. If T_u is a path, then by Theorem 1, we have,

$$\begin{aligned} M(T) = |C \cap V(T_u)| + |C \cap V(T_v)| &\geq M(T_u) + |C_v| \\ &\geq (n_u + 1)/2 + (2n_v - s_v + 3)/4 \\ &> (2n_u - s_u + 3)/4 + (2n_v - s_v + 3)/4 \\ &= (2(n_u + n_v) - (s_u + s_v) + 6)/4 \\ &= (2(n-1) - (s+1) + 6)/4 = (2n-s+3)/4. \end{aligned}$$

Thus we assume that T_u is not a path. Let $u' \in V(T_u)$ be a vertex such that $\deg(u') \geq 3$ and every internal vertex in the (u, u')-path has degree two. Let $d = d(u, u')$ and P be the (u, u')-path. Let $P : x_0 x_1 \ldots x_d$, where $x_0 = u$ and $x_d = u'$. Assume that $|N(x_i) \cap C| = 1$ for some $i \in \{0, 1, \ldots, d-1\}$. Without loss of generality, assume that $x_{i-1} \in N(x_i) \cap (V - C)$. Then the result follows easily as before by letting $T' = T - x_{i-1}x_{i-2}$. Thus we may assume that $N(x_i) \cap C = \emptyset$ for every $i = 0, 1, \ldots, d-1$. Since $V - C$ is independent and $u \in C$, we have $x_{2j} \in C$ for $j = 0, 1, \ldots, \lfloor d/2 \rfloor$. Since $x_d = u' \in C$, we find that d is even. We deduce that $|C \cap (V(P) - \{u'\})| = d/2$. Assume that $T_v, T_{u'}$ are the two components of $(T - (V(P) \cup \{w\})) \cup \{u'\}$, with $v \in V(T_v)$ and $u' \in V(T_{u'})$. As before, $C \cap V(T_v)$ is an identifying code for T_v and $C \cap V(T_{u'})$ is an identifying code for $T_{u'}$. By the inductive hypothesis,

$$\begin{aligned} |C| &= |C \cap V(T_{u'})| + |C \cap V(T_v)| + d/2 \\ &\geq (2|V(T_{u'})| - |S(T_{u'})| + 3)/4 + (2|V(T_v)| - |S(T_v)| + 3)/4 + d/2 \\ &= (2(n - d - 1) - s + 6)/4 + d/2 > (2n - s + 3)/4. \end{aligned}$$

This completes the proof. □

We note that if $2n + 9 \geq 12\ell - 5s$, then the lower bound of Theorem 5 is better than the bound given in Theorem 2.

3 Upper Bound

We begin this section with the following.

Theorem 6 (Ning et al. [13]) *If T is a tree of order $n \geq 3$ with ℓ leaves, then $\gamma_t^D(T) \leq 3(n + \ell)/5$, with equality if and only if $T = P_3$, or $T \in \mathcal{F}$.*

As noted in Observation 4, any differentiating-total dominating set is an identifying code, as well. Thus for any graph G, $M(G) \leq \gamma_t^D(G)$. Now from Theorem 6, we obtain the following upper bound.

Corollary 7 *For any tree T of order $n \geq 4$, $M(T) \leq 3(n + \ell)/5$.*

Our aim in this section is to improve Corollary 7 for any tree T of order $n \geq 3$. We show that for any tree T of order $n \geq 3$ with ℓ leaves, $M(T) \leq (3n + 2\ell - 1)/5$, and characterize all trees achieving equality for this bound. We first present some necessary lemmas. The following is easily verified.

Lemma 8 *If T is a tree obtained from a tree T' of order $n' \geq 3$ by adding a leaf to T', then $M(T) \leq M(T') + 1$.*

Proof Assume that T' is a tree of order $n' \geq 3$ and T is obtained from T' by adding a new vertex w to T' with edge wv, where $v \in V(T')$. Let C' be a $M(T')$-set. If $v \notin C'$, then $C' \cup \{w\}$ is an identifying code for T, and so $M(T) \leq M(T') + 1$. Thus assume that $v \in C'$. If $N_{T'}(v) \cap C' \neq \emptyset$, then $C' \cup \{w\}$ is an identifying code for T, and so the result follows. Thus assume that $N_{T'}(v) \cap C' = \emptyset$. Let $u \in N_{T'}(v)$. Then $C' \cup \{u\}$ is an identifying code for T, and so $M(T) \leq M(T') + 1$. □

Lemma 9 *Let T^* be a spider tree with central vertex w such that* $\deg(w) \geq 2$*, and T' be an arbitrary tree. Let T be a tree obtained from T' and T^* by joining w to a vertex $v \in V(T')$, and $T'' = T' - v$. If* $\deg(v) \geq 3$ *and* $M(T') \leq (3n(T') + 2\ell(T') - 1)/5$ *or* $\deg(v) = 2$ *and* $M(T'') \leq (3n(T'') + 2\ell(T'') - 1)/5$*, then* $M(T) < (3n(T) + 2\ell(T) - 1)/5$.

Proof Each vertex of $N(w)$ is a leaf or a support vertex of degree two, since T^* is a spider tree. Let r_1 be the number of leaves of T^* at distance one from w and r_2 be the number of leaves of T^* at distance two from w. Let C' be a $M(T')$-set. If $\deg(v) \geq 3$, then $C = C' \cup N_{T*}[w]$ is an identifying code for T. Thus we obtain that $M(T) \leq M(T') + r_2 + r_1 + 1 \leq (3n' + 2\ell' - 1)/5 + r_2 + r_1 + 1 < (3n + 2\ell - 1)/5$, since $n' = n - 2r_2 - r_1 - 1$ and $\ell' = \ell - r_1 - r_2$. Thus assume that $\deg(v) = 2$. If C'' is a $M(T'')$-set, then $C = C'' \cup N_{T*}[w]$ is an identifying code for T. Hence $M(T) \leq M(T'') + r_2 + r_1 + 1 \leq (3n'' + 2\ell'' - 1)/5 + r_2 + r_1 + 1 < (3n + 2\ell - 1)/5$, since $n'' = n - 2r_2 - r_1 - 2$ and $\ell'' \leq \ell - r_1 - r_2 + 1$. □

Lemma 10 *Let T' be an arbitrary tree and $v \in V(T')$ be a vertex with* $\deg(v) \geq 2$*, and T be a tree obtained from T' by joining v to a leaf of a path P_3. If $M(T') \leq (3n(T') + 2\ell(T') - 1)/5$, then $M(T) < (3n(T) + 2\ell(T) - 1)/5$.*

Proof Assume that T is obtained from T' by joining v to the leaf w of a path $P_3 = wzy$. Let C' be a $M(T')-$set. If $v \in C'$, then $C' \cup \{w, z\}$ is an identifying code for T and if $v \notin C'$, then $C' \cup \{w, y\}$ is an identifying code for T. Hence $M(T) \leq M(T') + 2 \leq (3n(T') + 2\ell(T') - 1)/5 + 2 \leq (3(n - 3) + 2(\ell - 1) - 1)/5 + 2 < (3n + 2\ell - 1)/5$. □

We are now ready to present the main result of this section.

Theorem 11 *For any tree T, of order $n \geq 3$ with ℓ leaves, $M(T) \leq (3n + 2\ell - 1)/5$. Equality holds if and only if $T = P_4$.*

Proof We use an induction on the order n of T to prove the upper bound. The base step is obvious for $n = 3$ and $n = 4$. Assume that for any tree T' of order $n' < n$, with l' leaves, $M(T') \leq (3n' + 2\ell' - 1)/5$. Now consider the tree T of order $n > 4$. Assume that T has a strong support vertex. Let v be a strong support vertex, and u be a leaf adjacent to v. Let $T' = T - u$. By Lemma 8, $M(T) \leq M(T') + 1$. By the inductive hypothesis, $M(T) \leq M(T') + 1 \leq (3n(T') + 2\ell(T') - 1)/5 + 1 = (3(n - 1) + 2(\ell - 1) - 1)/5 + 1 = (3n + 2\ell - 1)/5$. Thus we may assume for the next that T has no strong support vertex.

Let $d = diam(T)$. Since $n > 4$ and T has no strong support vertex, we have $d \geq 4$. We root T at a leaf x_0 of a diametrical path $x_0x_1 \ldots x_d$ from x_0 to a leaf x_d farthest from x_0. Thus, $\deg(x_{d-1}) = \deg(x_1) = 2$.

Assume that $d = 4$. Note that $\deg(x_3) = 2$. If $\deg(x_2) = 2$, then $T = P_5$, and $M(T) = 3 < (3n+2\ell-1)/5$. Thus assume that $\deg(x_2) > 2$. If x_2 is a support vertex, then T has $\deg(x_2)-1$ support vertices of degree two. Then $N[x_2]-L$ is an identifying code for T, implying that $M(T) \le \deg(x_2) < (3n+2\ell-1)/5$, since $n = 2\deg(x_2)$ and $\ell = \deg(x_2)$. Thus assume that x_2 is not a support vertex. Then T has $\deg(x_2)$ support vertices of degree two, and we can see that $N[x_2]$ is an identifying code for T, implying that $M(T) \le \deg(x_2)+1 < (3n+2\ell-1)/5$, since $n = 2\deg(x_2)+1$ and $\ell = \deg(x_2)$.

Assume next that $d = 5$. Note that $\deg(x_4) = 2$. Assume that $\deg(x_3) = 2$. If $\deg(x_2) = 2$, then $T = P_6$, and $M(T) = 4 < (3n+2\ell-1)/5$. Thus assume that $\deg(x_2) > 2$. Since $d = 5$, any vertex of $N(x_2)-\{x_3\}$ is a leaf or a support vertex of degree two. Assume that x_2 is a support vertex. There is a unique leaf adjacent to x_2. Then $S(T) \cup \{x_3\}$ is an identifying code for T, implying that $M(T) \le |S(T)|+1$ $= \deg(x_2)+1 < (3n+2\ell-1)/5$, since $n = 2\deg(x_2)+1$ and $\ell = \deg(x_2)$. Thus assume that x_2 is not a support vertex. Then $S(T) \cup \{x_2, x_3\}$ is an identifying code for T, implying that $M(T) \le |S(T)|+2 = \deg(x_2)+2 < (3n+2\ell-1)/5$, since $n = 2\deg(x_2)+2$ and $\ell = \deg(x_2)$. Thus assume that $\deg(x_3) \ge 3$, and similarly, $\deg(x_2) \ge 3$. Since T has no strong support vertex, any child of x_3 is a leaf or a support vertex of degree two. Let r_1 be the number of leaves of T_{x_3} at distance one from x_3 and r_2 be the number of leaves of T_{x_3} at distance two from x_3. Note that $\deg(x_3) = r_1 + r_2 + 1$. Let $T' = T - T_{x_3}$, then $n' = n - r_1 - 2r_2 - 1$ and $\ell' = \ell - r_1 - r_2$. If C' is a $M(T')$-set, then $C = C' \cup N_{T_{x_3}}[x_3]$ is an identifying code for T. Hence $M(T) \le M(T') + r_2 + r_1 + 1 \le (3n' + 2\ell' - 1)/5 + r_2 + r_1 + 1$ $= (3n + 2\ell - 3r_2 + 1)/5 < (3n+2\ell-1)/5$, since $r_2 \ge 1$.

Thus assume that $d \ge 6$. If $\deg(x_{d-2}) \ge 3$, then by the inductive hypothesis and Lemma 9, $M(T) < (3n+2\ell-1)/5$, since $T_{x_{d-2}}$ is a spider. Thus assume that $\deg(x_{d-2}) = 2$. If $\deg(x_{d-3}) \ge 3$, then from the inductive hypothesis and Lemma 10, we obtain $M(T) < (3n+2\ell-1)/5$. We thus assume that $\deg(x_{d-3}) = 2$.

We next proceed according to the value of $\deg(x_{d-4})$. First assume that $\deg(x_{d-4})$ $= 2$. Assume that $\deg(x_{d-5}) \ge 3$. Let $T' = T - T_{x_{d-4}}$, and C' be a $M(T')$–set. Then $C' \cup \{x_{d-1}, x_{d-2}, x_{d-3}\}$ is an identifying code for T. By the inductive hypothesis, $M(T) \le M(T') + 3 \le (3(n-5) + 2(\ell-1) - 1)/5 + 3 < (3n+2\ell-2)/5$. We thus assume that $\deg(x_{d-5}) = 2$. Assume that $\deg(x_{d-6}) \ge 3$. Let y_0'' be a leaf of $T_{x_{d-6}} - \{x_d, x_{d-1}, \ldots, x_{d-5}\}$ at maximum distance from x_{d-6}, and $y_0''y_1'' \ldots y_t''x_{d-6}$ be the shortest path from y_0'' to x_{d-6} in $T_{x_{d-6}} - \{x_d, x_{d-1}, \ldots, x_{d-5}\}$. Clearly $t \le 5$. With a similar argument as before, we can assume that $t \in \{0, 3, 5\}$, and $\deg(y_i'') = 2$ for $i = 1, \ldots, t-1$ if $t \in \{3, 5\}$. Let A be the set vertices of $T_{x_{d-6}}$ at even distance from x_{d-6}. Let $T' = T - T_{x_{d-6}}$, and C' be a $M(T')$-set. Then $C' \cup A \cup \{x_{d-6}, x_{d-5}\}$ is an identifying code for T. Let r_i be the number of leaves of $T_{x_{d-6}}$ at distance i from x_{d-6} for $i \in \{1, 4, 6\}$. By the inductive hypothesis,

$$\begin{aligned} M(T) &\le M(T') + |A| + 2 \\ &= M(T') + 3r_6 + 2r_4 + 2 \\ &\le (3(n-6r_6-4r_4-r_1-1) + 2(\ell - r_6 - r_4 - r_1 + 1) - 1)/5 + 3r_6 + 2r_4 + 2 \\ &< (3n+2\ell-2)/5. \end{aligned}$$

We thus assume that $\deg(x_{d-6}) = 2$. If $T = P_8$ or $T = P_9$, then $M(T) = 5 < (3n + 2\ell - 1)/5$. Thus $n > 9$. Let $T' = T - T_{x_{d-6}}$, and C' be a $M(T')$-set. If $x_{d-7} \in C'$, then $C' \cup \{x_{d-1}, x_{d-2}, x_{d-3}, x_{d-5}\}$ is an identifying code of T, and if $x_{d-7} \notin C'$, then $C' \cup \{x_d, x_{d-2}, x_{d-4}, x_{d-6}\}$ is a identifying code of tree T. By the inductive hypothesis, $M(T) \leq M(T') + 4 \leq (3(n - 7) + 2\ell - 1)/5 + 4 < (3n + 2\ell - 1)/5$.

Next assume that $\deg(x_{d-4}) \geq 3$. Let y_0 be a leaf of $T_{x_{d-4}} - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}\}$ at maximum distance from (x_{d-4}), and $y_0 y_1 \ldots y_q x_{d-4}$ be the shortest path from y_0 to x_{d-4}. Clearly $q \leq 3$.

Assume that $q = 3$. Since y_0 plays the same role of x_d, we may assume that $\deg(y_i) = 2$ for $i = 1, 2, 3$. By Lemmas 9 and 10, we may assume that $T_{x_{d-4}}$ has no leaf at distance 3 from x_{d-4}. Assume that $T_{x_{d-4}}$ has a leaf b at distance 2 from x_{d-4}, and let a be the support vertex adjacent to b. Evidently, $\deg(a) = 2$. Let $T' = T - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}\}$, and C' be a $M(T')$-set. If $x_{d-4} \notin C'$, then $N(a) \cap C' = N(b) \cap C'$, a contradiction. Hence $x_{d-4} \in C'$ and so $C' \cup \{x_d, x_{d-2}\}$ is an identifying code for T. Then by the inductive hypothesis, we obtain $M(T) \leq |C'| + 2 \leq (3(n - 4) + 2(\ell - 1) - 1)/5 + 2 < (3n + 2\ell - 1)/5$. We thus assume that $T_{x_{d-4}}$ has no leaf at distance 2 from x_{d-4}. Let r_1 be the number of leaves of $T_{x_{d-4}}$ adjacent to x_{d-4}, and r_4 be the number of leaves of $T_{x_{d-4}}$ at distance four from x_{d-4}. Note that $r_1 \leq 1$. Let A be the set of all vertices of $T_{x_{d-4}}$ at distance 2 or 4 from x_{d-4}. Assume that $r_1 = 1$. Let $T' = T - T_{x_{d-4}}$. If C' is a $M(T')$-set, then $C' \cup A \cup \{x_{d-3}, x_{d-4}\}$ is an identifying code for T. Then by the inductive hypothesis, $M(T) \leq M(T') + |A| + 2 \leq (3(n - 4r_4 - 2) + 2(\ell - r_4) - 1)/5 + 2r_4 + 2 = ((3n + 2\ell - 2) - 4r_4 - 3r_1 + 7)/5 < (3n + 2\ell - 2)/5$, since $r_4 \geq 2$. Thus assume that $r_1 = 0$. If $\deg(x_{d-5}) \geq 2$, then the result follows as before by letting $T' = T - T_{x_{d-4}}$. Thus assume that $\deg(x_{d-5}) = 2$. Let $T' = T - T_{x_{d-5}}$, and C' be a $M(T')$-set. Then $(C' \cup A) \cup \{x_{d-3}, x_{d-4}\}$ is an identifying code for T. Then $M(T) \leq M(T') + |A| + 2 \leq (3(n - 4r_4 - 2) + 2(\ell - r_4 + 1) - 1)/5 + 2r_4 + 2 = ((3n + 2\ell - 2) - 4r_4 + 6)/5 < (3n + 2\ell - 2)/5$.

Assume next that $q = 2$. Then $\deg(y_1) = 2$. If $\deg(y_2) \geq 3$, then the inductive hypothesis and Lemma 9 imply that, $M(T) < (3n + 2\ell - 1)/5$. Thus assume that $\deg(y_2) = 2$. Then by the inductive hypothesis and Lemma 10, we have $M(T) < (3n + 2\ell - 1)/5$.

Now assume that $q = 1$. Then $\deg(y_1) = 2$. Let $T' = T - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}\}$, and C' be a $M(T')-$set. If $x_{d-4} \notin C'$, then $N(y_0) \cap C' = N(y_1) \cap C'$, a contradiction. Hence $x_{d-4} \in C'$ and so $C' \cup \{x_d, x_{d-2}\}$ is an identifying code for T. Then $M(T) \leq M(T') + 2 \leq (3(n - 4) + 2(\ell - 1) - 1)/5 + 2 < (3n + 2\ell - 2)/5$.

It remains to assume that $q = 0$. Observe that x_{d-4} is a weak support vertex. Furthermore, $\deg(x_{d-4}) = 3$. Assume that $\deg(x_{d-5}) \geq 3$. Let $T' = T - T_{x_{d-4}}$, and C' be a $M(T')$-set. Clearly $n(T') \geq 3$. Then $C' \cup \{x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}\}$ is an identifying code for T, and so by the inductive hypothesis, $M(T) \leq M(T') + 4 \leq (3n(T') + 2\ell(T') - 1)/5 + 4 < (3n + 2\ell - 2)/5$, since $n(T') = n - 6$ and $\ell(T') = \ell - 2$. Thus assume that $\deg(x_{d-5}) = 2$. If $d = 6$, then T is a tree obtained from the path $P_7 : x_0 x_1 \ldots x_6$ by adding a leaf to x_2, and note that $M(T) = 5 < (3n + 2\ell - 1)/5$. We thus assume that $d \geq 7$.

Assume that $\deg(x_{d-6}) = 2$. If $d = 7$, then T is obtained from a path $P_8 : x_0 x_1 \ldots x_7$ by adding a leaf to x_3, and so we can see that $M(T) \leq 6$

$< (3n+1\ell-2)/5$. If $d = 8$, then $\deg(x_2) = 2$, since x_0 can plays the same role of x_d. So T is obtained from a path $P_9 : x_0x_1 \ldots x_8$ by adding a leaf to x_4, and we observe that $M(T) \le 6 < (3n+1\ell-2)/5$. Thus assume that $d \ge 9$. Let $T' = T - T_{x_{d-6}}$, and let C' be a $M(T')$-set. Then $C' \cup \{x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}, x_{d-5}\}$ is an identifying code for T. By the inductive hypothesis, $M(T) \le M(T') + 5 \le (3n(T') + 2\ell(T') - 1)/5 + 4$ $< (3n + 2\ell - 2)/5$, since $n(T') = n - 8$ and $\ell(T') \le \ell - 1$.

Thus assume that $\deg(x_{d-6}) \ge 3$. Let $T' = T - T_{x_{d-5}}$, and C' be a $M(T')$-set. If $x_{d-6} \in C'$, then $C' \cup \{x_d, x_{d-2}, x_{d-4}, x_{d-5}\}$ is an identifying code for T, and by the inductive hypothesis, $M(T) \le M(T') + 4$ $\le (3n(T') + 2\ell(T') - 1)/5 + 4 < (3n + 2\ell - 2)/5$, since $n(T') = n - 7$ and $\ell(T') = \ell - 2$. Thus we assume that $x_{d-6} \notin C'$. Hence x_{d-6} does not have a child which is a support vertex of degree two.

Let y'_0 be a leaf of $T_{x_{d-6}} - \{y_0, x_d, x_{d-1}, \ldots, x_{d-5}\}$ at maximum distance from x_{d-6}, and $y'_0y'_1 \ldots y'_t x_{d-6}$ be the shortest path from y'_0 to x_{d-6}. Clearly, $t \le 5$. As noted earlier, $t \ne 1$. We proceed depending on t.

Assume that $t = 5$. Since y'_0 plays the same role of x_d, we may assume that $\deg(y'_i) = 2$ for $i = 1, 2, 3, 5$, and y'_4 is a support vertex of degree three. Let $T' = T - T_{y'_5}$, and C' be a $M(T')$-set. If $x_{d-6} \notin C'$, then $|C' \cup \{x_d, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}, x_{d-5}, y_0\}| \ge 5$. Then $C'' = (C' - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}, x_{d-5}, y_0\}) \cup \{x_d, x_{d-2}, x_{d-4}, x_{d-5}, x_{d-6}\}$ is a $M(T')$-set. Thus we may assume that $x_{d-6} \in C'$. Then $C = C' \cup \{y'_1, y'_2, y'_3, y'_4\}$ is an identifying code for T. By the inductive hypothesis, $M(T) \le M(T') + 4 \le (3n(T') + 2\ell(T') - 1)/5 + 4 \le (3(n-7) + 2(\ell-2) - 1)/5 + 4 < (3n + 2\ell - 2)/5$.

If $t = 4$, then by Lemmas 9 and 10 we may assume that $\deg(y'_1) = \deg(y'_2)$ $= \deg(y'_3) = 2$. Assume that $\deg(y'_4) = 2$. Let $T' = T - T_{y'_4}$, and C' be a $M(T')$−set. Then $C' \cup \{y'_1, y'_2, y'_3\}$ is an identifying code for T, and by the inductive hypothesis, $M(T) \le M(T') + 3 \le (3n(T') + 2\ell(T') - 1)/5 + 3 = (3(n-5) + 2(\ell-1) - 1)/5 + 3$ $< (3n + 2\ell - 2)/5$. Thus assume that $\deg(y'_4) \ge 3$. By Lemmas 9 and 10, we may assume that there is no leaf in $T_{y'_4}$ at distance 3 from y'_4. Thus any leaf of $T_{y'_4}$ is at distance 1, 2 or 4 from y'_4. Let r_i be the number of leaves of $T_{y'_4}$ at distance i from y'_4 for $i = 1, 2, 4$. Since T has no strong support vertex, $r_1 \le 1$. Note that for every leaf x of $T_{y'_4}$ at distance 2 or 4 from y'_4, any internal vertex w of the path from x to y'_4 ($w \notin \{x, y'_4\}$) has degree two. Let A be the set of vertices $T_{y'_4}$ at distance 2 or 4 from y'_4. Let $T' = T - T_{y'_4}$, and C' be a $M(T')$-set. Then $C' \cup A \cup \{y'_3, y'_4\}$ is an identifying code for T. Thus by the inductive hypothesis,

$$
\begin{aligned}
M(T) &\le M(T') + |A| + 2 \\
&= M(T') + 2r_4 + r_2 + 2 \\
&\le (3n(T') + 2\ell(T') - 1)/5 + 2r_4 + 2 \\
&\le (3(n-4r_4-2r_2-r_1-1) + 2(\ell - r_4 - r_2 - r_1) - 1)/5 + 2r_4 + r_2 + r_1 + 1 \\
&\le (3n + 2\ell - 1 + (-4r_4 - 3r_2 - 5r_1 + 7))/5 \\
&< (3n + 2\ell - 1))/5.
\end{aligned}
$$

Next assume that $t = 3$. By Lemmas 9 and 10, we may assume that $\deg(y'_1) = \deg(y'_2) = \deg(y'_3) = 2$. Let $T' = T - T_{y'_3}$ and C' be a $M(T')$-set. If $x_{d-6} \notin C'$, then clearly $|C' \cup \{x_d, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}, x_{d-5}, y_0\}| \geq 5$, and so $C'' = (C' - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}, x_{d-4}, x_{d-5}, y_0\}) \cup \{x_d, x_{d-2}, x_{d-4}, x_{d-5}, x_{d-6}\}$ is a $M(T')$-set. Thus we may assume that $x_{d-6} \in C'$. Then $C = C' \cup \{y'_0, y'_2\}$ is an identifying code for T. By the inductive hypothesis, $M(T) \leq M(T') + 2 \leq (3n(T') + 2\ell(T') - 1)/5 + 3 = (3(n-4) + 2(\ell - 1) - 1)/5 + 2 < (3n + 2\ell - 2)/5$.

Next assume that $t = 2$. We may assume that $\deg(y'_1) = 2$. Now the result follows by the inductive hypothesis and Lemmas 9 and 10. Since x_{d-6} does not have a child which is a support vertex of degree two, we have $t \neq 1$.

Hence $t = 0$. Then x_{d-6} is a support vertex of degree 3. Let $T' = T - T_{x_{d-6}}$. If $n(T') = 1$, then $n = 10$, and $\{x_d, x_{d-2}, x_{d-4}, x_{d-5}, x_{d-6}, y'_0\}$ is an identifying code for T, and so $M(T) = 6 < (3n + 2\ell - 1)/5$. If $n(T') = 2$, then $n = 11$, and $\{x_d, x_{d-2}, x_{d-4}, x_{d-5}, x_{d-6}, x_{d-7}\}$ is an identifying code for T, and so $M(T) = 6 < (3n + 2\ell - 1)/5$. Thus we assume that $n(T') \geq 3$. Let C' be a $M(T')$-set. Then $C' \cup \{x_d, x_{d-2}, x_{d-4}, x_{d-5}, x_{d-6}\}$ is an identifying code for T. By the inductive hypothesis, $M(T) \leq M(T') + 5 \leq (3n(T') + 2\ell(T') - 1)/5 + 4 < (3n + 2\ell - 2)/5$. Thus the upper bound is proved.

Now we prove the second part of the theorem. Let T be a tree of order $n \geq 3$, with $M(T) = (3n + 2\ell - 1)/5$. Suppose that $diam(T) \geq 4$. If T has no strong support vertex, then as it is seen above, $M(T) < (3n+2\ell-1)/5$, a contradiction. Thus assume that T has some strong support vertex. Let T' be a tree obtained from T by removing $\ell_v - 1$ leaves of every strong support vertex v. Clearly T' has no strong support vertex, and as before, $M(T') < (3n(T') + 2\ell(T') - 1)/5 = (3(n - \ell + s) + 2s - 1)/5$. By Lemma 8, $M(T) \leq M(T') + \ell - s < (3(n-\ell+s) + 2s - 1)/5 + \ell - s = (3n+2\ell-1)/5$, a contradiction. We deduce that $diam(T) \leq 3$. If $diam(T) = 2$, then T is a star, and thus $M(T) = n - 1 < (3n + 2\ell - 1)/5$. Thus $diam(T) = 3$. Then T is a double-star. Now it can be easily seen that the assumption $M(T) = (3n + 2\ell - 1)/5$ implies that $T = P_4$. The converse is obvious. $\square$

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