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On the complexity of some hop domination parameters

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Abstract

A hop Roman dominating function (HRDF) on a graph $G = (V, E)$ is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = 2$. The weight of an HRDF f is the sum of its values on V. The minimum weight of an HRDF on G is called the hop Roman domination number of G. An HRDF f is a hop Roman independent dominating function (HRIDF) if for any pair v, w with $f(v) > 0$ and $f(w) > 0$, $d(v, w) \neq 2$. The minimum weight of an HRIDF on G is called the hop Roman independent domination number of G. In this paper, we study the complexity of the hop independent dominating problem, the hop Roman domination function problem and the hop Roman independent domination function problem, and show that the decision problem for each of the above three problems is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Keywords: dominating set, hop dominating set, hop independent set, hop Roman dominating function, hop Roman independent dominating function, complexity
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1. Introduction

For notation and graph theory terminology not given here, we refer to [6]. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n(G) = |V(G)|$. The *open neighborhood* of a vertex v is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The *degree* of v, denoted

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by $\deg(v)$, is $|N_G(v)|$. A vertex of degree one in a tree is referred as a *leaf* and its unique neighbor as a *support vertex*. The *open neighborhood* of a subset $S \subseteq V$, is $N_G(S) = \bigcup_{v\in S} N_G(v)$, and the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is the minimum length of a (u, v)-path in G. A *bipartite graph* is a graph whose vertices can *chordal graph* is a graph that does not contain an induced cycle of length greater than 3. A *planar graph* is a graph which can be drawn in the plane without any edges crossing. A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) - S$ has a neighbor in S. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G.

Ayyaswamy and Natarajan [2] introduced the concept of hop domination in graphs. A subset S of vertices of a graph G is a *hop dominating set* (HDS) if every vertex outside S is at distance two from a vertex of S. The *hop domination number*, $\gamma_h(G)$, of G is the minimum cardinality of an HDS of G. An HDS of G of minimum cardinality is referred to a $\gamma_h(G)$-set. The concept of hop domination was further studied, for example, in [1, 4, 7, 9]. For a subset $S \subseteq V(G)$ and a vertex $v \in V(G)$, we say that v is hop dominated by S (or S hop dominates v) if either $v \in S$ or $v \notin S$ and $d(u, v) = 2$ for some vertex $u \in S$. Ayyaswamy and Natarajan [9] also introduced the concept of hop independent domination in graphs. A subset S of vertices of a graph G is a *hop independent dominating set* (HIDS) if S is a HDS and for any pair $v, w \in S$, $d(v, w) \neq 2$. The *hop independent domination number*, $\gamma_{hi}(G)$, of G is the minimum cardinality of an HIDS of G.

A function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. For an RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. The mathematical concept of Roman domination, was defined and discussed by Stewart [12], and ReVelle and Rosing [10], and was subsequently developed by Cockayne et al. [3].

Shabani [11] introduced the concept of hop Roman dominating functions. A *hop Roman dominating function* (HRDF) is a function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u with $f(u) = 2$ and $d(u, v) = 2$. The weight of an HRDF f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of an HRDF on G is called the *hop Roman domination number* of G and is denoted $\gamma_{hR}(G)$. An HRDF with minimum weight is referred as a $\gamma_{hR}(G)$-function. For an HRDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an HRDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. For a function $f = (V_0, V_1, V_2)$ and a vertex $v \in V(G)$, we say that v is hop Roman dominated by f (or f hop Roman dominates v), if either $v \in V_1 \cup V_2$ or there exist $u \in V_2$, such that $d(v, u) = 2$. An HRDF $f = (V_0, V_1, V_2)$ is a *hop Roman independent dominating function*(HRIDF) if for any pair $v, w \in V_1 \cup V_2$, $d(v, w) \neq 2$. The minimum weight of an HRIDF on G is called the *hop Roman independent domination number* of G and is denote $\gamma_{hRI}(G)$.

In this paper we study the complexity of the hop independent dominating problem, the hop Roman domination function problem and the hop Roman independent domination function problem, and show that the decision problem for each of the above three problems is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs. We use a transformation of the Vertex Cover Problem which was one of Karp's 21 NP-complete problems [8] (see also [5]). A *vertex cover* of a graph is a set of vertices such that each edge of the graph is incident with at least one vertex of the set. The Vertex Cover Problem is the following decision problem.

Vertex Cover Problem (VCP).
Instance: A non-empty graph G, and a positive integer k.
Question: Does G have a vertex cover of size at most k?

2. Hop Independent Dominating Problem

Consider the following decision problem:

Hop Independent Dominating Problem (HIDP).
Instance: A non-empty graph G, and a positive integer k.
Question: Does G have a hop independent dominating set of size at most k?

We show that the decision problem for HIDF is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

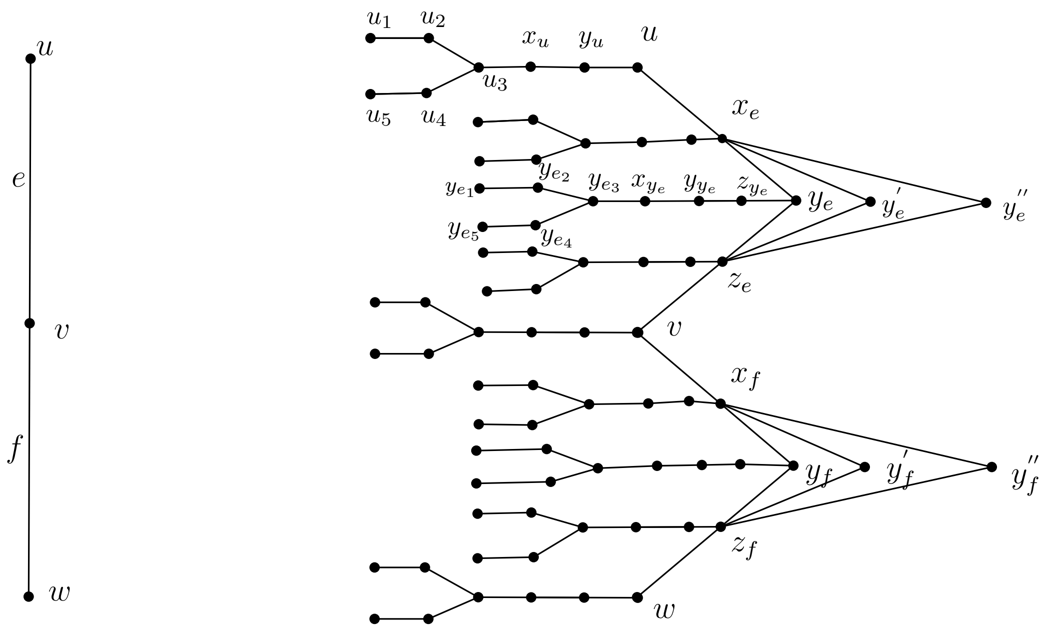
Theorem 2.1. *HIDP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the HIDP is NP, since it is easy to verify a "yes" instance of the HIDP in polynomial time. Now let us show how to transform the vertex cover problem to the HIDP so that one of them has a solution if and only if the other has a solution.

Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$, we subdivide the edge e three times. Let x_e, y_e, z_e be the subdivided vertices that were produced by subdividing e, where u is adjacent to x_e, v is adjacent to z_e, and y_e is adjacent to both x_e and z_e. For every vertex $v \in V(G) \cup \{x_e, z_e \mid e \in E(G)\}$ we add a P_5-path $P_5^v : v_1v_2...v_5$, and join v_3 to v, and then subdivide the edge v_3v two times. Let x_v and y_v be the subdivided vertices that were produced by subdividing the edge v_3v two times, where x_v is adjacent v_3 and y_v is adjacent to v. For every vertex $v \in \{y_e \mid e \in E(G)\}$ we add a P_5- path $P_5^v : v_1v_2...v_5$ and join v_3 to v, and subdivide the edge v_3v three times. Let x_v, y_v, z_v be the subdivided vertices that were produced by subdividing the edge v_3v three times, where x_v is adjacent to v_3, z_v is adjacent to v, and y_v is adjacent to both x_v and z_v. Finally for each edge $e = uv \in E(G)$ we add two new vertices $y_e^{'}$ and $y_e^{''}$ and join each of them to both x_e and z_e. The resulting graph H has order $n_H = 8n_G + 27m_G$ and size $m_H = 7n_G + 30m_G$. Figure 1 illustrates the graph H if G is a path P_3.

We show that G has a vertex cover of size at most k if and only if H has an HIDS of size at most $k + 2n_G + 6m_G$. Assume first that G has a vertex cover, S_G, of size at most k. Let

$$S_H = S_G \cup \{v_3, v_4 \mid v \in S_G\} \cup \{x_v, v_3 \mid v \in ((V(G) - S_G) \cup \{x_e, z_e, y_e \mid e \in E(G)\})\}.$$



$(a)G$ $(b)H$

Figure 1. The graphs G and H in the proof of Theorem 2.1

Clearly $d(a,b) \neq 2$ for any pair $a, b \in S_H$. We show S_H is an HIDS of size at most $k + 2n_G + 6m_G$. Clearly any vertex in $\{y_e, y'_e, y''_e : e \in E(G)\}$ is hop dominated by S_G. For any vertex $v \in S_G$, any vertex in $\{v_1, v_2, ..., v_5\} \cup \{x_v, y_v\}$ is hop dominated by $\{v_3, v_4\}$. For any vertex $v \in V(G) - S_G$, any vertex in $\{v_1, v_2, ..., v_5\} \cup \{x_v, y_v, v\}$ is hop dominated by $\{v_3, x_v\}$. For any edge $e \in E(G)$, any vertex in $\{y_{e1}, y_{e2}, ..., y_{e5}\} \cup \{x_{y_e}, y_{y_e}, z_{y_e}\}$ is hop dominated by $\{y_{e3}, x_{y_e}\}$. For any edge $e \in E(G)$, any vertex in $\{x_{e1}, x_{e2}, ..., x_{e5}\} \cup \{x_{x_e}, y_{x_e}, x_e\}$ is hop dominated by $\{x_{e3}, x_{x_e}\}$. Finally, for any edge $e \in E(G)$, any vertex in $\{z_{e1}, z_{e2}, ..., z_{e5}\} \cup \{x_{z_e}, y_{z_e}, z_e\}$ is hop dominated by $\{z_{e3}, x_{z_e}\}$. Consequently, S_H is an HIDS of size at most $k + 2n_G + 6m_G$.

Assume now that H has an HIDS, S_H, of size at most $k + 2n_G + 6m_G$. It is evident that for any vertex $v \in V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$, $|S_H \cap \{v_1, v_5, v_3\}| \geq 1$, and for any vertex $v \in V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$, $|S_H \cap \{v_2, v_4, x_v\}| \geq 1$. Furthermore, for every edge $e = uv \in E(G)$, we have $S_H \cap \big(\{v, u, y_e, y'_e, y''_e\} \cup \{y_v \mid v \in \{x_e, z_e, y_e \mid e \in E(G)\}\}\big) \neq \emptyset$. If $S_H \cap \{v, u\} = \emptyset$ for some edge $e = uv \in E(G)$, then we remove one vertex of $S_H \cap \big(\{y_e, y'_e, y''_e\} \cup \{y_v \mid v \in \{x_e, z_e, y_e \mid e \in E(G)\}\}\big)$ and add precisely one of the vertices u or v to S_H. Thus we may assume that $S_H \cap \{v, u\} \neq \emptyset$ for any edge $e = uv \in E(G)$. Thus $S_G = S_H \cap V(G)$ is a vertex cover for G of size at most $|S_H| - 2n_G - 6m_G$. Therefore G has a vertex cover of size at most k. □

Theorem 2.2. *HIDP is NP-complete for planar chordal graphs.*

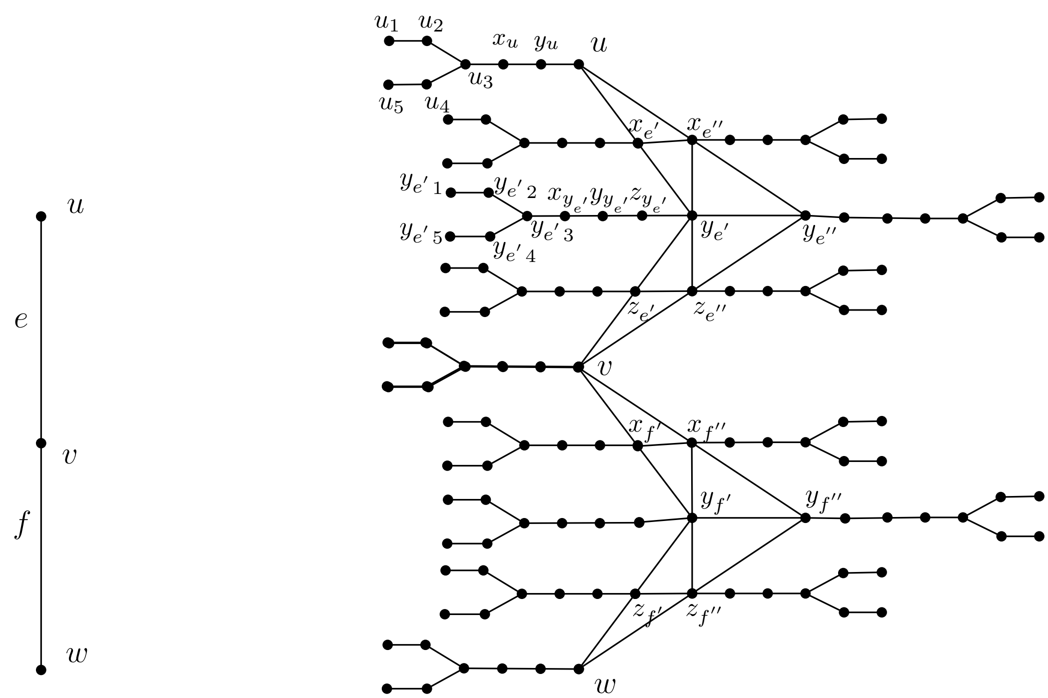
Proof. Clearly, the HIDP is in NP. Now let us show how to transform the vertex cover problem to the HIDP so that one of them has a solution if and only if the other has a solution.

Let G be a planar chordal graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows: We replace every edge $e = vu \in E(G)$ with a double edge $e' = uv$ and $e'' = uv$. Then we subdivide each of e' and e'' three times. Let $x_{e'}, y_{e'}, z_{e'}$ be the vertices that produced from subdividing the edge e', and $x_{e''}, y_{e''}, z_{e''}$ be the vertices that were produced by subdividing the edge e'', such that $x_{e'}$ and $x_{e''}$ are adjacent to u, $z_{e'}$ and $z_{e''}$ are adjacent to v, $y_{e'}$ is adjacent to $x_{e'}$ and $z_{e'}$, and $y_{e''}$ is adjacent to $x_{e''}$ and $z_{e''}$. Then we add edges $x_{e'}x_{e''}$, $y_{e'}y_{e''}$, $z_{e'}z_{e''}$, $y_{e'}x_{e''}$ and $y_{e'}z_{e''}$. Next for any vertex $v \in V(G) \cup \{x_{e'}, x_{e''}, z_{e'}, z_{e''} \mid e \in E(G)\}$, we add a P_5-path $P_5^v : v_1v_2...v_5$ and join v_3 to v and then subdivide the edge v_3v two times, and let x_v and y_v be the vertices that were produced by subdividing v_3v, where x_v is adjacent to v_3 and y_v is adjacent to v. For any vertex $v \in \{y_{e'}, y_{e''} \mid e \in E(G)\}$, we add a P_5-path $P_5^v : v_1v_2...v_5$ and join v_3 to v, and then subdivide the edge v_3v three times, and let x_v, y_v and z_v be the vertices that produced from subdividing v_3v, where x_v is adjacent to v_3, z_v is adjacent to v, and y_v is adjacent to both x_v and z_v. The resulting graph H is a planar chordal graph of order $n_H = 8n_G + 50m_G$ and size $7n_G + 57m_G$. Figure 2 illustrates the graph H if G is a path P_3.

We show that G has a vertex cover of size at most k if and only if H has an HIDS of size at most $k + 2n_G + 12m_G$. Assume first that G has a vertex cover, S_G, of size at most k. Let

$$S_H = S_G \cup \{v_3, v_4 \mid v \in S_G\} \cup \{v_3, x_v \mid v \in \big((V(G) - S_G) \cup \{x_{e'}, x_{e''}, z_{e'}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}\big)\}.$$

Clearly $d(a,b) \neq 2$ for any pair $a, b \in S_H$. We show S_H is an HIDS of size at most $k + 2n_G + 12m_G$. Clearly any vertex in $\{y_{e'}, y_{e''} : e \in E(G)\}$ is hop dominated by S_G. For



$(a)G$ $(b)H$

Figure 2. The graphs G and H in the proof of Theorem 2.2

any vertex $v \in S_G$, any vertex in $\{v_1, v_2, ..., v_5\} \cup \{x_v, y_v\}$ is hop dominated by $\{v_3, v_4\}$. For any vertex $v \in V(G) - S_G$, any vertex in $\{v_1, v_2, ..., v_5\} \cup \{x_v, y_v, v\}$ is hop dominated by $\{v_3, x_v\}$. For any edge $e \in E(G)$, any vertex in $\{y_{e'1}, y_{e'2}, ..., y_{e'5}\} \cup \{x_{y_{e'}}, y_{y_{e'}}, z_{y_{e'}}\}$ is hop dominated by $\{y_{e'3}, x_{y_{e'}}\}$, and any vertex in $\{y_{e''1}, y_{e''2}, ..., y_{e''5}\} \cup \{x_{y_{e''}}, y_{y_{e''}}, z_{y_{e''}}\}$ is hop dominated by $\{y_{e''3}, x_{y_{e''}}\}$. For any edge $e \in E(G)$, any vertex in $\{x_{e'1}, x_{e'2}, ..., x_{e'5}\} \cup \{x_{x_{e'}}, y_{x_{e'}}, x_{e'}\}$ is hop dominated by $\{x_{e'3}, x_{x_{e'}}\}$, and any vertex in $\{x_{e''1}, x_{e''2}, ..., x_{e''5}\} \cup \{x_{x_{e''}}, y_{x_{e''}}, x_{e''}\}$ is hop dominated by $\{x_{e''3}, x_{x_{e''}}\}$. For any edge $e \in E(G)$, any vertex in $\{z_{e'1}, z_{e'2}, ..., z_{e'5}\} \cup \{x_{z_{e'}}, y_{z_{e'}}, z_{e'}\}$ is hop dominated by $\{z_{e'3}, x_{z_{e'}}\}$, and any vertex in $\{z_{e''1}, z_{e''2}, ..., z_{e''5}\} \cup \{x_{z_{e''}}, y_{z_{e''}}, z_{e''}\}$ is hop dominated by $\{z_{e''3}, x_{z_{e''}}\}$. Consequently, S_H is a hop independent dominating set of size at most $k + 2n_G + 12m_G$.

Assume now that H has an HIDS, S_H, of size at most $k + 2n_G + 12m_G$. It is evident that for any vertex $v \in V(G) \cup \{x_{e'}, x_{e''}, z_{e'}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$, $|S_H \cap \{v_1, v_5, v_3\}| \geq 1$, and for any vertex $v \in V(G) \cup \{x_{e'}, x_{e''}, z_{e'}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$, $|S_H \cap \{v_2, v_4, x_v\}| \geq 1$. Furthermore, for every edge $e = uv \in E(G)$, we have $S_H \cap \Big(\{v, u, y_{e''}, y_{e'}\} \cup \{y_v \mid v \in \{x_{e'}, x_{e''}, z_{e'}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}\} \cup \{z_{y_{e'}}, z_{y_{e''}} : e \in E(G)\}\Big) \neq \emptyset$. If $S_H \cap \{v, u\} = \emptyset$ for some edge $e = uv \in E(G)$, then we remove one vertex of $S_H \cap \Big(\{y_{e''}, y_{e'}\} \cup \{y_v \mid v \in$

$\{x_{e'}, x_{e''}, z_{e'}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}\} \cup \{z_{y_{e'}}, z_{y_{e''}} : e \in E(G)\}\big)$ and add precisely one of the vertices u or v to S_H. Thus we may assume that $S_H \cap \{v, u\} \neq \emptyset$, for any edge $e = uv \in E(G)$. Thus $S_G = S_H \cap V(G)$ is a vertex cover for G of size at most $|S_H| - 2n_G - 12m_G$. Therefore G has a vertex cover of size at most k. □

3. Hop Roman Dominating Function Problem

We next study the complexity issue of the hop Roman domination function problem. Consider the following decision problem:

> **Hop Roman Dominating Function Problem** (HRDFP).
> **Instance**: A non-empty graph G, and a positive integer k.
> **Question**: Does G have a hop Roman dominating function of weight at most k?

We show that the decision problem for the HRDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 3.1. *HRDFP is NP-complete for planar bipartite graphs.*

Proof. Clearly, the HRDFP is in NP, since it is easy to verify a "yes" instance of the HRDFP in polynomial time. Now let us show how to transform the vertex cover problem to the HRDFP so that one of them has a solution if and only if the other one has a solution.

Let G be a connected planar bipartite graph of order n_G and size $m_G \geq 2$, and let H be the graph obtained from G as follows: We convert each edge $e = vu \in E(G)$ into a triple-edge $e_1 = vu$, $e_2 = vu$ and $e_3 = vu$, and then subdivide each of edges e_1, e_2 and e_3 three times. Let the vertices $x_{e_i}, y_{e_i}, z_{e_i}$ be the vertices that were produced from subdividing the edge e_i, for $i = 1, 2, 3$, where the vertex x_{e_i} is adjacent to v, the vertex z_{e_i} is adjacent to u and $d(y_{e_i}, u) = d(y_{e_i}, v) = 2$, for $i = 1, 2, 3$. For each edge $e = vu \in E(G)$ we add a new vertex e_{vu} and a P_5-path $P_5^v : v_1^e v_2^e ... v_5^e$, and join the vertex e_{vu} to u, v and v_3^e. The resulting graph H has order $n_H = n_G + 15m_G$ and size $m_H = 19m_G$. Figure 3 illustrates the graph H if G is a path P_3. We note that since G is connected and planar, so H is connected and planar. Further, by construction, H is bipartite. Thus, H is a connected planar bipartite graph.

We show that G has a vertex cover of size at most k if and only if H has an HRDF of weight $2k + 4m_G$. Assume first that G has a vertex cover, S_G, of size at most k. Let

$$S_H = S_G \cup \bigcup_{e=uv \in E(G)} \{v_3^e, e_{vu}\}.$$

We show that $f = (V(H) - S_H, \emptyset, S_H)$ is an HRDF for H of weight at most $2k + 4m_G$. For every edge $e = vu \in E(G)$, the vertex v_3^e hop Roman dominates the vertices v_1^e, v_5^e, u and v in H, while the vertex e_{vu} hop Roman dominates the vertices v_2^e, v_4^e and the neighbors of u and v in H. For any $e \in E(G)$, since S_G is a vertex cover in G, the vertices y_{e_1}, y_{e_2} and y_{e_3}, are hop Roman dominated by S_G in H. Therefore, the function f is an HRDF for H of weight at most $2k + 4m_G$.

Assume next that $f = (V_0^f, V_1^f, V_2^f)$ is an HRDF for H of weight $2k + 4m_G$. Without loss of generality we assume that f is a γ_{hR}-function of H.

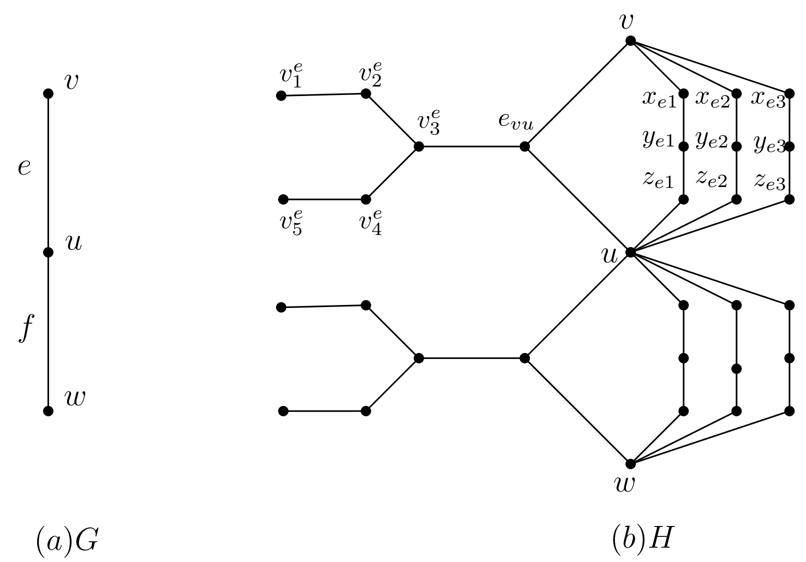


Figure 3. The graph G and H in the proof of Theorem 3.1

If for an edge $e \in E(G)$, $f(v_1^e)+f(v_3^e)+f(v_5^e) \leq 1$, then v_1^e or v_5^e is not hop Roman dominated by f, a contradiction. Therefore, $f(v_1^e)+f(v_3^e)+f(v_5^e) \geq 2$ for every edge $e \in E(G)$. If for an edge $e \in E(G)$, $f(v_2^e)+f(v_4^e)+f(e_{vu}) \leq 1$, then v_2^e or v_4^e is not hop Roman dominated by f, a contradiction. Therefore, $f(v_2^e)+f(v_4^e)+f(e_{vu}) \geq 2$ for every edge $e \in E(G)$.

If for some edge $e \in E(G)$, $f(v_2^e)+f(v_4^e)+f(e_{vu}) \geq 3$, then function g defined by $g(v_2^e)=0$, $g(v_4^e)=0$, $g(e_{vu})=2$ and $g(v)=f(v)$ otherwise, is an HRDF for H of weight less than $\gamma_{hR}(H)$, a contradiction. Thus, for every $e \in E(G)$, $f(v_2^e)+f(v_4^e)+f(e_{uv})=2$. Similarly we can show that $f(v_1^e)+f(v_3^e)+f(v_5^e)=2$ for every edge $e \in E(G)$.

If there exists an edge $e \in E(G)$ such that $f(z_{e_i})>0$ or $f(x_{e_i})>0$, for some $i=1,2,3$, then the function g defined by $g(v_2^e)=g(v_4^e)=0$, $g(e_{vu})=2$, $g(z_{e_i})=g(x_{e_i})=0$ for each $i=1,2,3$, and $g(z)=f(z)$ otherwise, is an HRDF for H of weight $g(V)<f(V)$, a contradiction. Thus, for every $e \in E(G)$, $f(z_{e_i})=f(x_{e_i})=0$, for each $i=1,2,3$.

If there exists an edge $e=uv \in E(G)$ such that $f(y_{e_i})>0$ for each $i=1,2,3$, then the function g defined by $g(y_{e_1})=g(y_{e_2})=g(y_{e_3})=0$, $g(u)=2$ and $g(z)=f(z)$ otherwise, is an HRDF with $g(V)<f(V)$, a contradiction. Thus, for every edge $e=uv \in E(G)$ there exists an integer $i \in \{1,2,3\}$ such that $f(y_{e_i})=0$. Since y_{e_i} is hop Roman dominated by f, we obtain that $f(v)=2$ or $f(u)=2$. Then we can see that $f(y_{e_j})=0$ for each $j \in \{1,2,3\}-\{i\}$. Therefore, every edge $e=uv \in E(G)$ has at least one endpoint in V_2^f. Hence, $S_G=V_2^f \cap V(G)$ is a vertex cover of G of size at most $\frac{1}{2}\big(\gamma_{hR}(H)-4m_G\big)$. Thus, G has a vertex cover of size at most k. □

Theorem 3.2. *HRDFP is NP-complete for planar chordal graphs.*

Proof. Clearly, the HRDFP is in NP. Now let us show how to transform the vertex cover problem

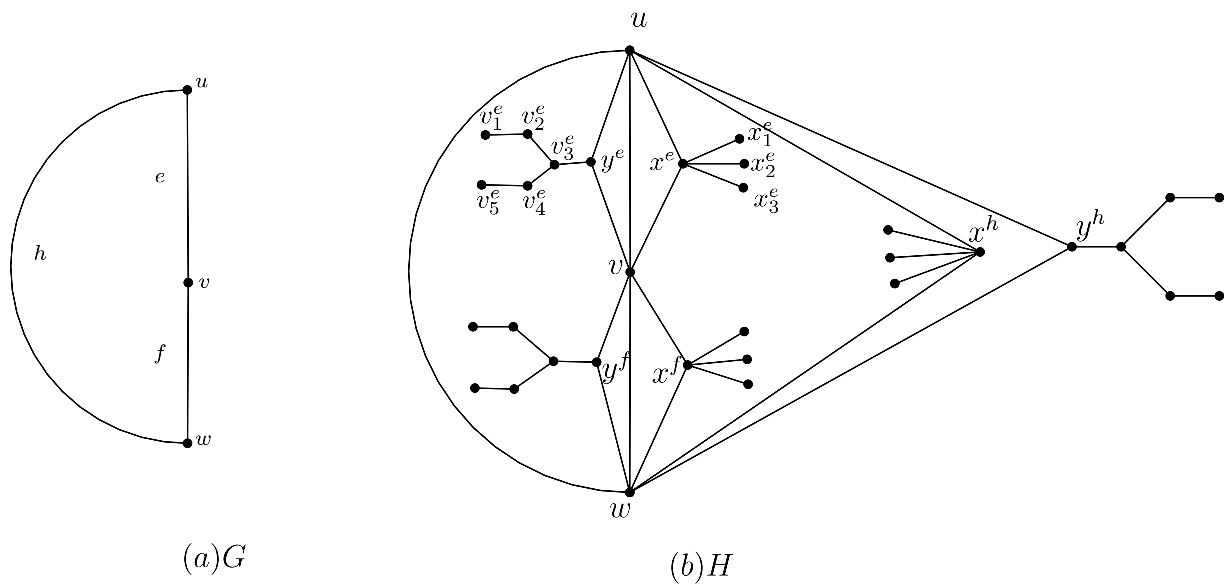


Figure 4. The graphs G and H in the proof of Theorem 3.2

to the HRDFP so that one of them has a solution if and only if the other has a solution.

Let G be a connected planar chordal graph of order n_G and size $m_G \geq 2$. Let H be the graph obtained from G as follows. For each edge $e = uv \in E(G)$ we add two new vertices x^e and y^e and join each of them to both u and v. We then add three leaves x_1^e, x_2^e and x_3^e to x^e. We next add a P_5-path $v_1^e v_2^e v_3^e v_4^e v_5^e$, and join v_3^e to y^e. The resulting graph H has order $n_H = n_G + 10m_G$ and size $m_H = 13m_G$. Figure 4 illustrates the graph H if G is a cycle C_3.

We show that G has a vertex cover of size at most k if and only if H has an HRDF of weight at most $2k + 4m_G$. Assume first that G has a vertex cover, S_G, of size at most k. Let

$$S_H = S_G \cup \bigcup_{e=uv \in E(G)} \{v_3^e, y^e\}.$$

For every edge $e = uv \in E(G)$, the vertex v_3^e hop dominates the vertices v_1^e, v_5^e, u and v in H, while the vertex y^e hop dominates the vertices v_2^e, v_4^e and x^e. Since S_G is a vertex cover in G, for any edge $e \in E(G)$, the vertices x_1^e, x_2^e and x_3^e are hop dominated by S_G. Therefore, the set S_H is an HDS for G and $f = (V(H) - S_H, \emptyset, S_H)$ is a HRDF of weight at most $2k + 4m_G$ for H.

Assume next that $f = (V_0^f, V_1^f, V_2^f)$ is an HRDF for H of weight at most $2k + 4m_G$. Without loss of generality, we may assume that f is a $\gamma_{hR}(H)$-function. If for an edge $e \in E(G)$, $f(v_1^e) + f(v_3^e) + f(v_5^e) \leq 1$, then v_1^e or v_5^e is not hop Roman dominated by f, a contradiction. Therefore, $f(v_1^e) + f(v_3^e) + f(v_5^e) \geq 2$ for every edge $e \in E(G)$. If for an edge $e \in E(G)$, $f(v_2^e) + f(v_4^e) + f(y^e) \leq 1$, then v_2^e or v_4^e is not hop Roman dominated by f, a contradiction. Therefore, $f(v_2^e) + f(v_4^e) + f(y^e) \geq 2$ for every edge $e \in E(G)$.

If for an edge $e \in E(G)$, $f(v_2^e) + f(v_4^e) + f(y^e) \geq 3$, then the function g defined by $g(v_2^e) =$

$g(v_4^e)=0, g(y^e)=2$ and $g(v)=f(v)$ otherwise, is an HRDF with $g(V)<f(V)$, a contradiction. Thus, for every $e\in E(G)$, $f(v_2^e)+f(v_4^e)+f(y^e)=2$. Also, we can show that for every $e\in E(G)$, $f(v_1^e)+f(v_3^e)+f(v_5^e)=2$.

If there exists an edge $e=uv\in E(G)$ such that $f(x_i^e)>0$ for each $i=1,2,3$, then the function g defined by $g(x_i^e)=0$ for each $i=1,2,3$, $g(u)=2$, and $g(z)=f(z)$ otherwise, is an HRDF with $g(V)<f(V)$, a contradiction. Thus, for every edge $e=uv\in E(G)$, there exists an integer $i\in\{1,2,3\}$ such that $f(x_i^e)=0$. If for an edge $e\in E(G)$, $f(x_1^e)+f(x_2^e)+f(x_3^e)\geq 3$, then we can obtain an HRDF g with $g(V)<f(V)$, a contradiction. Thus $f(x_1^e)+f(x_2^e)+f(x_3^e)\leq 2$ for each $e\in E(G)$.

For any $e\in E(G)$ with $f(x_1^e)+f(x_2^e)+f(x_3^e)=0$, we have $f(u)=2$ or $f(v)=2$. Assume that $f(x_1^e)+f(x_2^e)+f(x_3^e)>0$ for some edge $e\in E(G)$. Suppose that $f(x_1^e)+f(x_2^e)+f(x_3^e)=1$. Without loss of generality, assume that $f(x_1^e)=1$. Then $f(u)=2$ or $f(v)=2$, and so we can change $f(x_1^e)$ to 0, to obtain an HRDF with weight less than $\gamma_{hR}(H)$, a contradiction by the choice of f. Thus $f(x_1^e)+f(x_2^e)+f(x_3^e)>1$, and so $f(x_1^e)+f(x_2^e)+f(x_3^e)=2$. As before, we can see that $f(x_i^e)\neq 1$ for $i=1,2,3$. Therefore, there exists an integer $i\in\{1,2,3\}$ such that $f(x_i^e)=2$. Then $f(u)=f(v)=0$. Now we replace x_i^e with u or v in V_2^f. We conclude that $S_G=V_2^f\cap V(G)$ is a vertex cover of G of size at most $\frac{1}{2}\big(\gamma_{hR}(H)-4m_G\big)$. Thus, G has a vertex cover of size at most k. □

4. Hop Roman Independent Dominating Function Problem

We next study the complexity issue of the hop Roman independent domination function problem. Consider the following decision problem:

> **Hop Roman Independent Dominating Function Problem** (HRIDFP).
> **Instance**: A non-empty graph G, and a positive integer k.
> **Question**: Does G have a hop Roman independent dominating function of weight at most k?

We show that the decision problem for HRIDFP is NP-complete even when restricted to planar bipartite graphs or planar chordal graphs.

Theorem 4.1. *HRIDFP is NP-complete for planar bipartite graphs.*

Proof. Let G be a graph of order n_G and size m_G, and let H be the connected planar bipartite graph constructed in the proof of Theorem 2.1. We show that G has a vertex cover of size at most k if and only if H has an HRIDF of weight at most $2k+4n_G+12m_G$. Assume that G has a vertex cover, S_G, of size at most k. Let $f=(V(H)-S_H,\emptyset,S_H)$, where S_H is the HIDS constructed in the proof of Theorem 2.1. Then f is an HRIDF for H of weight $2k+4n_G+12m_G$. Therefore, H has an HRIDF of weight at most $2k+4n_G+12m_G$.

Assume next that $f=(V_0^f,V_1^f,V_2^f)$ is an HRIDF in H of weight at most $2k+4n_G+12m_G$. We show that G has a vertex cover of size at most k. Without loss of generality, we may assume that f is a $\gamma_{hRI}(H)-$ function. If for a vertex $v\in V(G)\cup\big\{x_e,z_e,y_e\mid e\in E(G)\big\}$, $f(v_1)+f(v_3)+f(v_5)\leq 1$, then v_1 or v_5 is not hop Roman dominated by f, a contradiction. Therefore,

$f(v_1) + f(v_3) + f(v_5) \geq 2$ for any vertex of $V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$. If for a vertex $v \in V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$, $f(v_2) + f(v_4) + f(x_v) \leq 1$, then v_2 or v_4 is not hop Roman dominated by f, a contradiction. Therefore, $f(v_2) + f(v_4) + f(x_v) \geq 2$ for any vertex of $V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$. If for a vertex of $V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$, $f(v_2) + f(v_4) + f(x_v) \geq 3$, then the function g defined by $g(v_2) = g(v_4) = 0, g(x_v) = 2$ and $g(v) = f(v)$ otherwise, is an HRDF with $g(V) < f(V)$, a contradiction. Thus, for any vertex of $V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$, $f(v_2) + f(v_4) + f(x_v) = 2$. Also, we can show that for any vertex of $V(G) \cup \{x_e, z_e, y_e \mid e \in E(G)\}$, $f(v_1) + f(v_3) + f(v_5) = 2$.

If there exists an edge $e = uv \in E(G)$ such that $f(y_e) > 0$, $f(y_e^{'}) > 0$ and $f(y_e^{''}) > 0$, then the function g defined by $g(y_e) = g(y_e^{'}) = g(y_e^{''}) = 0, g(u) = 2$ and $g(v) = f(v)$ otherwise, is an HRDF with $g(V) < f(V)$, a contradiction. Thus, for every edge $e = uv \in E(G)$, there exist at least one vertex $v \in \{y_e, y_e^{'}, y_e^{''}\}$ such that $f(v) = 0$. If for an edge $e \in E(G)$, $f(y_e) + f(y_e^{'}) + f(y_e^{''}) \geq 3$, then there exists an HRDF g with $g(V) < f(V)$, a contradiction. Thus $f(y_e) + f(y_e^{'}) + f(y_e^{''}) \leq 2$ for each $e \in E(G)$. For any $e \in E(G)$ with $f(y_e) + f(y_e^{'}) + f(y_e^{''}) = 0$, we have $f(u) = 2$ or $f(v) = 2$. Assume that $f(y_e) + f(y_e^{'}) + f(y_e^{''}) > 0$ for some edge $e \in E(G)$. Suppose that $f(y_e) + f(y_e^{'}) + f(y_e^{''}) = 1$. Without loss of generality, assume that $f(y_e) = 1$. Then $f(u) = 2$ or $f(v) = 2$, and so we can change $f(y_e)$ to 0, to obtain an HRIDF with weight less than $\gamma_{hRI}(H)$, a contradiction by the choice of f. Thus, $f(y_e) + f(y_e^{'}) + f(y_e^{''}) > 1$, and so $f(y_e) + f(y_e^{'}) + f(y_e^{''}) = 2$. As before, we can see that $f(y_e) \neq 1, f(y_e^{'}) \neq 1$ and $f(y_e^{''}) \neq 1$. Therefore, there exist a vertex $z \in \{y_e, y_e^{'}, y_e^{''}\}$ such that $f(z) = 2$. Then $f(u) = 0$ and $f(v) = 0$. Without loss of generality, assume that $f(y_e) = 2$. Now replace y_e with u or v in V_2^f. We conclude that $S_G = V_2^f \cap V(G)$ is a vertex cover of G of size at most $\frac{1}{2}\big(\gamma_{hRI}(H) - 4n_G - 12m_G\big)$. Thus, G has a vertex cover of size at most k. □

Theorem 4.2. *HRIDFP is NP-complete for planar chordal graphs.*

Proof. Let G be a graph of order n_G and size m_G, and let H be the connected planar chordal graph constructed in the proof of Theorem 2.2. We show that G has a vertex cover of size at most k if and only if H has an HRIDF of weight at most $2k + 4n_G + 24m_G$. Assume that G has a vertex cover, S_G, of size at most k. Let $f = (V(H) - S_H, \emptyset, S_H)$, where S_H is the HIDS constructed in the proof of Theorem 2.2. Then f is an HRIDF for H of weight $2k + 4n_G + 24m_G$. Therefore, H has an HRIDF of weight at most $2k + 4n_G + 24m_G$.

Assume next that $f = (V_0^f, V_1^f, V_2^f)$ is an HRIDF for H of weight at most $2k+4n_G+24m_G$. We show that G has a vertex cover of size at most k. Without loss of generality, we may assume that f is a $\gamma_{hRI}(H)-$ function. If for a vertex $v \in V(G) \cup \{x_{e'}, z_{e'}, x_{e''}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$, $f(v_1) + f(v_3) + f(v_5) \leq 1$, then v_1 or v_5 is not hop Roman dominated by f, a contradiction. Therefore, $f(v_1) + f(v_3) + f(v_5) \geq 2$ for any vertex of $V(G) \cup \{x_{e'}, z_{e'}, x_{e''}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$. If for a vertex $v \in V(G) \cup \{x_{e'}, z_{e'}, x_{e''}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$, $f(v_2) + f(v_4) + f(x_v) \leq 1$, then v_2 or v_4 is not hop Roman dominated by f, a contradiction. Therefore, $f(v_2) + f(v_4) + f(x_v) \geq 2$ for any vertex of $V(G) \cup \{x_{e'}, z_{e'}, x_{e''}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$. If $f(v_2) + f(v_4) + f(x_v) \geq 3$, then the function g defined by $g(v_2) = g(v_4) = 0, g(x_v) = 2$ and $g(v) = f(v)$ otherwise, is an HRIDF for H implying that $g(V) < f(V)$, a contradiction. Thus, for any vertex of $V(G) \cup \{x_{e'}, z_{e'}, x_{e''}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$, $f(v_2) + f(v_4) + f(x_v) = 2$. Also, we can show that for any vertex of $V(G) \cup \{x_{e'}, z_{e'}, x_{e''}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}$, $f(v_1) + f(v_3) + f(v_5) = 2$.

For any vertex $z \in \{y_v \mid v \in \{x_{e'}, z_{e'}, x_{e''}, z_{e''}, y_{e'}, y_{e''} \mid e \in E(G)\}\}$, $f(z) = 0$, since otherwise we can find an HRIDF g with $g(V) < f(V)$, a contradiction. Similarly for any vertex $z \in \big\{z_v \mid v \in \{y_{e'}, y_{e''} \mid e \in E(G)\}\big\}$, $f(z) = 0$. If for an edge $e = uv \in E(G)$, $f(u) + f(y_{e'}) + f(y_{e''}) \leq 1$ and $f(v) + f(y_{e'}) + f(y_{e''}) \leq 1$, then $y_{e'}$ or $y_{e''}$ is not hop Roman dominated by f, a contradiction. Thus for any edge $e = uv \in E(G)$, either $f(u) + f(y_{e'}) + f(y_{e''}) \geq 2$ or $f(v) + f(y_{e'}) + f(y_{e''}) \geq 2$. If for an edge $e = uv \in E(G)$, $f(u) + f(y_{e'}) + f(y_{e''}) \geq 3$ or $f(v)+f(y_{e'})+f(y_{e''}) \geq 3$, then there exists a function g with $g(V) < f(V)$, a contradiction. Thus for every edge $e \in E(G)$, $f(u)+f(y_{e'})+f(y_{e''}) \leq 2$ and $f(v)+f(y_{e'})+f(y_{e''}) \leq 2$. Since for any edge $e = uv \in E(G)$, either $f(u)+f(y_{e'})+f(y_{e''}) \geq 2$ or $f(v)+f(y_{e'})+f(y_{e''}) \geq 2$, we deduce that for any edge $e = uv \in E(G)$, either $f(u)+f(y_{e'})+f(y_{e''}) = 2$ or $f(v)+f(y_{e'})+f(y_{e''}) = 2$. Since for every edge $e \in E(G)$, $d(y_{e'}, y_{e''}) = 1$, we have $f(y_{e'}) \neq 2$ and $f(y_{e''}) \neq 2$, and so either $f(y_{e'}) = f(y_{e''}) = 1$ or $f(y_{e'}) = f(y_{e''}) = 0$. For any edge $e = uv \in E(G)$ with $f(y_{e'}) = f(y_{e''}) = 0$ we have $f(u) = 2$ or $f(v) = 2$. Assume that $f(y_{e'}) = f(y_{e''}) = 1$ for some edge $e = uv \in E(G)$. Then $f(u) = f(v) = 0$, and we can change both $f(y_{e'})$ and $f(y_{e''})$ to 0 and one of $f(u)$ or $f(v)$ to 2 to obtain a $\gamma_{hRI}(H)$-function g such that $g(u) = 2$ or $g(v) = 2$. We conclude that $S_G = V_2^f \cap V(G)$ is a vertex cover for graph G of size at most $\frac{1}{2}\big(\gamma_{hRI}(H) - 4n_G - 24m_G\big)$. Therefore, G has a vertex cover of size at most k. □

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