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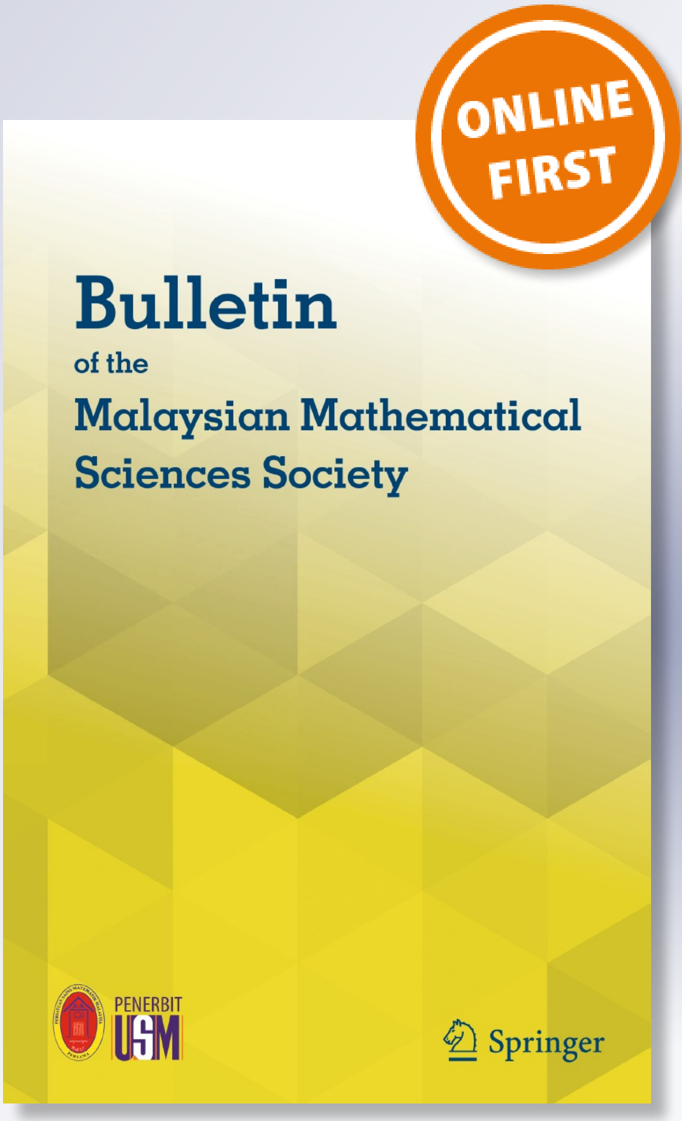

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A Classification of Cactus Graphs According to Their Total Domination Number

Majid Hajian[1] · Michael A. Henning[2] · Nader Jafari Rad[3]

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex in G is adjacent to some vertex in S. The total domination number, $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. A cactus is a connected graph in which every edge belongs to at most one cycle. Equivalently, a cactus is a connected graph in which every block is an edge or a cycle. Let G be a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves. Recently, the authors have proved that $\gamma_t(G) \geq \frac{1}{2}(n - \ell + 2) - k$. As a consequence of this bound, $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$ for some integer $m \geq 0$. In this paper, we characterize the class of cactus graphs achieving equality in this bound, thereby providing a classification of all cactus graphs according to their total domination number.

Keywords Total dominating sets · Total domination number · Cactus graphs

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1 Introduction

A *total dominating set*, abbreviated TD-set, of a graph G with no isolated vertex is a set S of vertices such that every vertex in G is adjacent to a vertex in S. The *total domination number*, denoted by $\gamma_t(G)$, of G is the minimum cardinality of a TD-set of G. We call a TD-set of cardinality $\gamma_t(G)$ a γ_t-set of G. For a recent book on total domination in graphs we refer the reader to [6]. A *cactus* is a connected graph in which every edge belongs to at most one cycle. Equivalently, a (non-trivial) cactus is a connected graph in which every block is an edge or a cycle. Our aim in this paper is to provide a characterization of all cactus graphs according to their total domination number.

For notation and graph theory terminology we generally follow [6]. The *order* of a graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$ is denoted by $n(G) = |V(G)|$ and its *size* by $m(G) = |E(G)|$. Two vertices v and w are *neighbors* in G if they are adjacent; that is, if $vw \in E(G)$. The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted $N_G(v)$, and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is denoted $d_G(v) = |N_G(v)|$.

For a set S of vertices in a graph G, the subgraph induced by S is denoted by $G[S]$. Further, the subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. Two vertices u and v in a graph G are *connected* if there exists a (u, v)-path in G. If every two vertices in G are connected, then the graph G is *connected*. The *distance* between two vertices u and v in a connected graph G is the minimum length of a (u, v)-path in G. The *diameter*, $\text{diam}(G)$, of G is the maximum distance among pairs of vertices in G. A *block* of G is a maximal connected subgraph of G which has no cut-vertex of its own. A *cycle edge* of a graph G is an edge that belongs to a cycle of G.

A *leaf* of a graph G is a vertex of degree 1 in G, while a *support vertex* of G is a vertex adjacent to a leaf. The set of all leaves of G is denoted by $L(G)$, and we let $\ell(G) = |L(G)|$ be the number of leaves in G. The set of all support vertices of G by $S(G)$. A tree T of order $n \geq 2$ is a *star* if $n = 2$ or $n \geq 3$ and T contains exactly one vertex that is not leaf. A *double star* is a tree with exactly two (adjacent) vertices that are not leaves. Further, if one of these vertices is adjacent to r leaves and the other to s leaves, then we denote the double star by $S(r, s)$. We denote the path and cycle on n vertices by P_n and C_n, respectively.

Let v be a vertex of a tree T. We call the vertex v a *bad leaf* of T if v is a leaf and no γ_t-set of T contains v. The vertex v is a *stable vertex* of T if v is not a support vertex and $\gamma_t(T - v) \geq \gamma_t(T)$. We remark that the total domination of a tree can be computed in linear time. In particular, to determine if a leaf of a tree is a bad leaf can be determined in linear time. The bad leaves of a tree can also be efficiently computed using results of Cockayne et al. [2].

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. The set of children of v is denoted by $C(v)$. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. In particular, every child of v is a descendant of v. A *grandchild* and a *great grandchild* of v in T are descendants of v at distance 2 and 3 from v, respectively. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$ and is denoted by T_v.

We use the standard notation $[k] = \{1, \ldots, k\}$.

2 Main Results

Our aim in this paper is to provide a characterization of all cactus graphs according to their total domination number. More precisely, we shall prove the following result where $\mathcal{G}_k^m$ is a family of graphs defined in Sect. 3 for each integer $k \geq 0$ and $m \geq 0$.

Theorem 1 *Let $m \geq 0$ be an integer. If G is a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$ if and only if $G \in \mathcal{G}_k^m$.*

We proceed as follows. In Sect. 3 we define the families $\mathcal{G}_k^m$ of graphs for each integer $k \geq 0$ and $m \geq 0$. Known results on the total domination number are given in Sect. 4. In Sect. 5 we present a proof of our main result.

3 The Families $\mathcal{G}_k^m$

In this section, we define the families $\mathcal{G}_k^m$ of graphs for each integer $k \geq 0$ and $m \geq 0$.

3.1 The Family $\mathcal{G}_k^0$

The family $\mathcal{G}_0^0$ of trees was defined by Chellali and Haynes [1] as follows. Let $\mathcal{G}_0^0$ be the class of trees T that can be obtained from a sequence $T_1, \ldots, T_\ell$ of trees, where $\ell \geq 1$ and where the tree T_1 is the path P_4 with support vertices x and y, and where the tree $T = T_\ell$. Further if $\ell \geq 2$, then for each $i \in [\ell]$, the tree T_i can be obtained from the tree T_{i-1} by applying one of the following three operations $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ defined below. For the initial tree T_1 (recall that T_1 is a path P_4 with support vertices x and y), we define $A(T_1) = \{x, y\}$, and for $i \geq 2$, we define the set $A(T_i)$ of vertices in T_i recursively according to the rules given below. Further, we define H to be a path P_4 with support vertices u and v.

- **Operation** $\mathcal{O}_1$. Add a new vertex to T_{i-1} and join it to a vertex of $A(T_{i-1})$. Let $A(T_i) = A(T_{i-1})$.
- **Operation** $\mathcal{O}_2$. Add a vertex disjoint copy of H to T_{i-1}, and add an edge from a leaf of H to a leaf of T_{i-1}. Let $A(T_i) = A(T_{i-1}) \cup \{u, v\}$.
- **Operation** $\mathcal{O}_3$. Add a vertex disjoint copy of H to T_{i-1}, and add a new vertex w and an edge from w to a support of H and a leaf of T_{i-1}. Let $A(T_i) = A(T_{i-1}) \cup \{u, v\}$.

For $k \geq 1$, Hajian et al. [4] recursively define the family $\mathcal{G}_i^0$ of graphs for each $i \in [k]$ by the following procedure.

- **Procedure A**: For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^0$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^0$ and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.

3.2 The Family $\mathcal{G}_k^1$

Hajian et al. [4] defined the family of trees $\mathcal{G}_0^1$ as follows. Let $\mathcal{T}_0^1$, $\mathcal{T}_0^2$ and $\mathcal{T}_0^3$ be the class of trees defined as follows.

- Let $\mathcal{T}_0^1$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding $a \geq 1$ new vertices and joining all of them to the same leaf of T'.
- Let $\mathcal{T}_0^2$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^0$ that contains a support vertex x all of whose neighbors, except for exactly one neighbor, are leaves in T' by removing all leaf-neighbors of x.
- Let $\mathcal{T}_0^3$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^0$ by adding a vertex disjoint copy of a double star Q and adding an edge from a leaf of Q to a vertex of degree at least 2 in T'.

Let $\mathcal{G}_0^1$ be the class of trees T that can be obtained from a sequence $T_1, \ldots, T_\ell$ of trees, where $\ell \geq 1$ and where the tree $T_1 \in \mathcal{T}_0^1 \cup \mathcal{T}_0^2 \cup \mathcal{T}_0^3$ and the tree $T = T_\ell$. Further, if $\ell \geq 2$, then for each $i \in [\ell]$, the tree T_i can be obtained from the tree T_{i-1} by applying operation $\mathcal{O}^*$ defined below.

- **Operation** $\mathcal{O}^*$. Add a vertex disjoint copy of a double star Q to T_{i-1} by adding an edge joining a leaf of Q and a leaf of T_{i-1}.

For $k \geq 1$, Hajian et al. [4] defined the family $\mathcal{G}_i^1$ of graphs for each $i \in [k]$ by the following two procedures.

- **Procedure B**: For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^1$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^1$ and the vertices x and y are leaves in $G_i - e$ that are connected by a unique path in $G_i - e$.
- **Procedure C**: For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^1$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^0$ and the vertices x and y are connected by a unique path in $G_i - e$. Further, exactly one of x and y is a leaf in $G_i - e$.

3.3 Family $\mathcal{G}_0^m, m \geq 2$

Recall that the families $\mathcal{G}_0^0$ and $\mathcal{G}_0^1$ are defined in Sects. 3.1 and 3.2, respectively. For $m \geq 2$, we recursively define the family $\mathcal{G}_0^m$ of graphs constructed from the families $\mathcal{G}_0^{m-1}$ and $\mathcal{G}_0^{m-2}$ as follows.

- Let $\mathcal{T}_0^{m,1}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding $a \geq 1$ new vertices and joining all of them to precisely one bad leaf of T'. For $m = 2$, add the path P_2 to $\mathcal{T}_0^{2,1}$, and for simplicity and uniformity, denote the resulting class by $\mathcal{T}_0^{2,1}$. (That is, $\mathcal{T}_0^{2,1} \cup \{P_2\}$ is denoted by $\mathcal{T}_0^{2,1}$.)
- Let $\mathcal{T}_0^{m,2}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a double star Q and identifying a leaf of Q with a stable vertex of T'.
- Let $\mathcal{T}_0^{m,3}$ be the class of all trees T that can be obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a double star Q and adding an edge from a leaf of Q to a vertex of T' of degree at least 2.

Let $\mathcal{G}_0^m$ be the class of trees T that can be obtained from a sequence $T_1, \ldots, T_\ell$ of trees, where $\ell \geq 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2} \cup \mathcal{T}_0^{m,3}$ and the tree $T = T_\ell$. Further, if $\ell \geq 2$, then for each $i \in [\ell]\backslash\{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying operation $\mathcal{O}^*$ defined in Sect. 3.2.

3.4 Family $\mathcal{G}_k^m$, $m \geq 2$ and $k \geq 1$

For $m \geq 2$ and $k \geq 1$, construct a family $\mathcal{G}_k^m$ from $\mathcal{G}_{k-1}^{m-2}$, $\mathcal{G}_{k-1}^{m-1}$ and $\mathcal{G}_{k-1}^m$, recursively, as follows.

- **Procedure D**: For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^{m-2}$ and the vertices x and y are non-leaves in $G_i - e$ that are connected by a unique path in $G_i - e$ and $\gamma_t(G_i) = \gamma_t(G_i - e)$.
- **Procedure E**: For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^{m-1}$ and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma_t(G_i) = \gamma_t(G_i - e)$. Further, exactly one of x and y is a leaf in $G_i - e$.
- **Procedure F**: For $i \in [k]$, a graph G_i belongs to the family $\mathcal{G}_i^m$ if it contains an edge $e = xy$ such that the graph $G_i - e$ belongs to the family $\mathcal{G}_{i-1}^m$ and the vertices x and y are connected by a unique path in $G_i - e$ and $\gamma_t(G_i) = \gamma_t(G_i - e)$. Further, both x and y are leaves in $G_i - e$.

4 Known Results

In this section, we present some known results and observations. We begin with the following elementary properties of a total dominating set in a graph G.

Observation 1 *The following hold in a graph G with no isolated vertex.*

(a) *Every TD-set in G contains the set of support vertices of G.*
(b) *If G is connected and* $\mathrm{diam}(G) \geq 3$*, then there exists a γ_t-set of G that contains no leaf of G.*

Lower and upper bounds on the total domination number of a graph are well studied in the literature. A detailed discussion of such bounds can be found in the 2013 book [6] on total domination in graphs, and in particular in [8]. For subsequent recent papers on bounds on the total domination number we refer the reader to [3–5,9].

A discussion of lower and upper bounds on the total domination number of a tree can be found in [7]. Chellali and Haynes [1] were the first to establish a lower bound on the total domination number of a tree in terms of the order, number of leaves, and number of support vertices in the tree.

Theorem 2 [1] *If T is a tree of order $n \geq 2$ with ℓ leaves, then $\gamma_t(T) \geq (n - \ell + 2)/2$, with equality if and only if $T \in \mathcal{G}_0^0$.*

The authors [4] have recently generalized the Chellali–Haynes result to connected graphs.

Theorem 3 [4] *If G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then the following holds.*

(a) $\gamma_t(G) \geq \frac{1}{2}(n - \ell + 2) - k$, *with equality if and only if* $G \in \mathcal{G}_k^0$.
(b) $\gamma_t(G) = \frac{1}{2}(n - \ell + 3) - k$ *if and only if* $G \in \mathcal{G}_k^1$.

As an immediate consequence of Theorem 3, we have the following result.

Corollary 4 [4] *If G is a connected graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$ for some integer $m \geq 0$.*

5 Proof of Main Result

In this section, we present a proof of our main result, namely Theorem 1. For this purpose, we first prove Theorem 1 in the special case when $k = 0$, that is, when the cactus is a tree.

Theorem 5 *Let $m \geq 0$ be an integer. If T is a tree of order $n \geq 2$ with ℓ leaves, then $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$ if and only if $T \in \mathcal{G}_0^m$.*

Proof Let T be a tree of order $n \geq 2$ with ℓ leaves. We proceed by induction on $m \geq 0$, namely **first induction**, to show that $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$ if and only if $T \in \mathcal{G}_0^m$. If $m = 0$ and $m = 1$, then the result follows by Theorem 3(a) and Theorem 3(b), respectively. This establishes the base step of the induction. Let $m \geq 2$ and assume that if m' is an integer where $0 \leq m' < m$ and T' is a tree of order $n' \geq 2$ with ℓ' leaves, then $\gamma_t(T') = \frac{1}{2}(n - \ell' + 2 + m')$ if and only if $T' \in \mathcal{G}_0^{m'}$. Let T be a tree of order $n \geq 2$ with ℓ leaves. We show that $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$ if and only if $T \in \mathcal{G}_0^m$.

($\Longrightarrow$) Assume that $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$. We show that $T \in \mathcal{G}_0^m$. If $T = P_2$, then by definition of the family $\mathcal{T}_0^{2,1}$, we have $T \in \mathcal{T}_0^{2,1} \subseteq \mathcal{G}_0^2$. In this case when $T = P_2$, we note that $\gamma_t(T) = 2 = \frac{1}{2}(n - \ell + 2 + 2)$, and so $m = 2$ and $T \in \mathcal{G}_0^m$. If T is a star and $n \geq 3$, then by the definition of the family $\mathcal{T}_0^2$ we have $T \in \mathcal{T}_0^2 \subseteq \mathcal{G}_0^1$. Thus, by Theorem 3(b), $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + 1)$, and so $m = 1$ and $T \in \mathcal{G}_0^m$. If T is a double star, then by the definition of the family $\mathcal{G}_0^0$, we have $T \in \mathcal{G}_0^0$. Thus, by Theorem 3(a), $\gamma_t(T) = \frac{1}{2}(n - \ell + 2)$, and so $m = 0$ and $T \in \mathcal{G}_0^m$. Hence, we may assume that $\mathrm{diam}(T) \geq 4$, for otherwise $T \in \mathcal{G}_0^m$, as desired. In particular, $n \geq 5$.

We now root the tree T at a vertex r at the end of a longest path P in T. Let u be a vertex at maximum distance from r, and so $d_T(u, r) = \mathrm{diam}(T)$. Necessarily, r and u are leaves. Let v be the parent of u, let w be the parent of v, let x be the parent of w, and let y be the parent of x. Possibly, $y = r$. Since u is a vertex at maximum distance from the root r, every child of v is a leaf. By Observation 1(b), there exists a γ_t-set, D say, of T that contains no leaf of T, implying that $\{v, w\} \subseteq D$. Let $d_T(v) = t_1$. $\square$

Claim 1 *If the vertex w has at least two neighbors in D, then $T \in \mathcal{G}_0^m$.*

Proof Suppose that $|N_T(w) \cap D| \geq 2$. As observed earlier, $v \in D$. Let v' be a neighbor of w, different from v, that belong to the set D. We now consider the tree T' obtained from T by deleting all leaf-neighbors of v. Let T' have order n', and let T' have ℓ' leaves. We note that $n' = n - (t_1 - 1)$ and $\ell' = \ell - (t_1 - 1) + 1 = \ell - t_1 + 2$. By Observation 1(b), there exists a γ_t-set, D' say, of T' that contains no leaf of T'. Since the vertex v is a leaf-neighbor of w in T', we note that $w \in D'$ and $v \notin D'$. The set D' can be extended to a TD-set of T by adding to it the vertex v, and so $\gamma_t(T) \le |D'| + 1 = \gamma_t(T') + 1$. Conversely, since the set $D \backslash \{v\}$ is a TD-set of T', we note that $\gamma_t(T') \le |D| - 1 = \gamma_t(T) - 1$. Consequently, $\gamma_t(T') = \gamma_t(T) - 1$. Thus,

$$\begin{aligned}
\gamma_t(T') &= \gamma_t(T) - 1 \\
&= \tfrac{1}{2}(n - \ell + 2 + m) - 1 \\
&= \tfrac{1}{2}(n - \ell + m) \\
&= \tfrac{1}{2}((n' + t_1 - 1) - (\ell' + t_1 - 2) + m) \\
&= \tfrac{1}{2}(n' - \ell' + 2 + (m - 1)).
\end{aligned}$$

Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^{m-1}$. If there is a γ_t-set of T' that contains the leaf v, then such a set is a TD-set of T', implying that $\gamma_t(T) \le \gamma_t(T')$, contradicting our earlier observation that $\gamma_t(T) = \gamma_t(T') + 1$. Hence, the vertex v is a bad leaf of T, implying that $T \in \mathcal{T}_0^{m,1} \subseteq \mathcal{G}_0^m$. □

By Claim 1, we may assume that the vertex w has exactly one neighbor in D, for otherwise $T \in \mathcal{G}_0^m$ as desired. As observed earlier, the vertex v belongs to D. Thus, $N_T(w) \cap D = \{v\}$. In particular, $x \notin D$. Recall that the γ_t-set D of T contains no leaf of T. By Observation 1(a), the set D contains the set $S(T)$ of support vertices of T. Thus, if $d_T(w) \ge 3$, then every child of w different from v is a leaf in T. Let $d_T(w) = t_2$. We note that $t_2 \ge 2$. Recall that x is the parent of w in the rooted tree T.

Claim 2 *If $d_T(x) \ge 3$, then $T \in \mathcal{G}_0^m$.*

Proof Suppose that $d_T(x) \ge 3$. We now consider the tree T' obtained from T by deleting all vertices in the maximal subtree of T induced by $D[w]$; that is, $T' = T - V(T_w)$. If $t_2 = 2$, then the subtree T_w is a star with v as its central vertex. If $t_2 \ge 3$, then by our earlier observations, the subtree T_w is a double star with v and w as the two vertices that are not leaves in the double star. Let T' have order n', and let T' have ℓ' leaves. We note that $n' = n - (t_1 + t_2 - 1)$ and $\ell' = \ell - (t_1 - 1) - (t_2 - 2) = \ell - t_1 - t_2 + 3$.

We show that $\gamma_t(T') = \gamma_t(T) - 2$. Every γ_t-set of T' can be extended to a TD-set of T by adding to it the vertices v and w, implying that $\gamma_t(T) \le \gamma_t(T') + 2$. We prove next the reverse inequality. By supposition, $d_T(x) \ge 3$. By our earlier observations, $x \notin D$. Thus, since the set D contains the set $S(T)$ of support vertices of T, we note that x is not a support vertex of T. Thus, no child of x is a leaf. Let w' be an arbitrary child of x different from w. If every child of w' is a leaf, then since D contains no leaf of T this implies that both w' and x belong to D, a contradiction. Therefore, at least one child of w' is not a leaf. Let v' be an arbitrary child of w' that is not a leaf. By maximality of the path P, every child of v' is a leaf. Thus, by Observation 1, both v' and w' belong to the set D, implying that the set $D \backslash \{v, w\}$ is a TD-set of T'. Thus, $\gamma_t(T') \le |D| - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t(T') = \gamma_t(T) - 2$. Thus,

$$\begin{aligned}\gamma_t(T') &= \gamma_t(T) - 2\\ &= \tfrac{1}{2}(n - \ell + 2 + m) - 2\\ &= \tfrac{1}{2}(n - \ell + m - 2)\\ &= \tfrac{1}{2}((n' + t_1 + t_2 - 1) - (\ell' + t_1 + t_2 - 3) + m - 2)\\ &= \tfrac{1}{2}(n' - \ell' + 2 + (m - 2)).\end{aligned}$$

Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^{m-2}$. Every γ_t-set of $T' - x$ can be extended to a TD-set of T by adding to it the vertices v and w, implying that $\gamma_t(T') + 2 = \gamma_t(T) \le \gamma_t(T' - x) + |\{v, w\}| = \gamma_t(T' - x) + 2$. Consequently, we must have equality throughout this inequality chain. Hence, $\gamma_t(T') = \gamma_t(T' - x)$. Thus, the vertex x is a stable vertex of T'. Let Q be obtained from the double star T_w by adding to it a new vertex x' and the edge $x'w$. We note that Q is a double star with v and w as the two vertices that are not leaves in Q. Since T can be obtained from the tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of the double star Q and identifying the leaf x' of Q with the stable vertex x of T', the tree T belongs to the family $\mathcal{T}_0^{m,2}$. Thus, $T \in \mathcal{T}_0^{m,2} \subseteq \mathcal{G}_0^m$. □

By Claim 2, we may assume that $d_T(x) = 2$, for otherwise $T \in \mathcal{G}_0^m$ as desired. By our earlier observations, the subtree T_x is a double star with v and w as the two vertices of T_x that are not leaves. We note that $T_x \in \mathcal{G}_0^0$.

Claim 3 *If* $\mathrm{diam}(T) = 4$, *then* $T \in \mathcal{G}_0^m$.

Proof Suppose that $\mathrm{diam}(T) = 4$. Thus, the vertex y is the root r of the tree T, and so $T - r = T_x$. Thus, T is obtained from the tree $T_x \in \mathcal{G}_0^0$ by adding the new vertex z and joining it with an edge to the leaf x of T_x. Hence, $T \in \mathcal{T}_0^1 \subseteq \mathcal{G}_0^1$. Thus, $T \in \mathcal{G}_0^m$ where $m = 1$. □

By Claim 3, we may assume that $\mathrm{diam}(T) \ge 5$, for otherwise $T \in \mathcal{G}_0^m$ as desired. We now consider the tree T' obtained from T by deleting all vertices in the maximal subtree of T induced by $D[x]$; that is, $T' = T - V(T_x)$. As observed earlier, the subtree T_x is a double star with v and w as the two vertices that are not leaves in the double star. Further, $T_x \in \mathcal{G}_0^0$. Let T' have order n', and let T' have ℓ' leaves. We note that $n' = n - t_1 - t_2$.

We show that $\gamma_t(T') = \gamma_t(T) - 2$. Every γ_t-set of T' can be extended to a TD-set of T by adding to it the vertices v and w, implying that $\gamma_t(T) \le \gamma_t(T') + 2$. We prove next the reverse inequality. By our earlier observations, $x \notin D$. Thus, the restriction of the set D to the tree T' is a TD-set of T', implying that $\gamma_t(T') \le |D \setminus \{v, w\}| = |D| - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t(T') = \gamma_t(T) - 2$.

Claim 4 *If* $d_T(y) = 2$, *then* $T \in \mathcal{G}_0^m$.

Proof Suppose that $d_T(y) = 2$. In this case, $\ell' = \ell - (t_1 - 1) - (t_2 - 2) + 1 = \ell - t_1 - t_2 + 4$. Thus,

$$\begin{aligned}
\gamma_t(T') &= \gamma_t(T) - 2 \\
&= \tfrac{1}{2}(n - \ell + 2 + m) - 2 \\
&= \tfrac{1}{2}(n - \ell + m - 2) \\
&= \tfrac{1}{2}((n' + t_1 + t_2) - (\ell' + t_1 + t_2 - 4) + m - 2) \\
&= \tfrac{1}{2}(n' - \ell' + 2 + m).
\end{aligned}$$

Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^m$. As observed earlier, T_x is a double star with v and w as the two vertices that are not leaves in the double star. Thus, the tree T can be obtained from the tree T' by adding to T' the double star T_x and adding the edge xy joining the leaf x of T_x and the leaf y of T'. Hence, T can be obtained from the tree $T' \in \mathcal{G}_0^m$ by applying operation $\mathcal{O}^*$, implying that $T \in \mathcal{G}_0^m$. □

By Claim 4, we may assume that $d_T(y) \geq 3$, for otherwise $T \in \mathcal{G}_0^m$ as desired. Recall that $T' = T - V(T_x)$. Thus, $d_{T'}(y) = d_T(y) - 1 \geq 2$, and so y is not a leaf of T'. In this case, $\ell' = \ell - (t_1 - 1) - (t_2 - 2) = \ell - t_1 - t_2 + 3$. Thus,

$$\begin{aligned}
\gamma_t(T') &= \gamma_t(T) - 2 \\
&= \tfrac{1}{2}(n - \ell + m - 2) \\
&= \tfrac{1}{2}((n' + t_1 + t_2) - (\ell' + t_1 + t_2 - 3) + m - 2) \\
&= \tfrac{1}{2}(n' - \ell' + 2 + (m - 1)).
\end{aligned}$$

Applying the inductive hypothesis to the tree T', we have $T' \in \mathcal{G}_0^{m-1}$. The tree T can be obtained from the tree $T' \in \mathcal{G}_0^{m-1}$ by adding to T' the double star T_x and adding the edge xy joining the leaf x of T_x and the vertex y of degree at least 2 in T'. Thus, the tree T belongs to the family $\mathcal{T}_0^{m,3}$. Hence, $T \in \mathcal{T}_0^{m,3} \subseteq \mathcal{G}_0^m$. This completes the necessity part of the proof of Theorem 5.

($\Longleftarrow$) Conversely, assume that $T \in \mathcal{G}_0^m$, where $m \geq 2$ and where recall that T is a tree of order $n \geq 2$ with ℓ leaves. As shown earlier, if T is a star and $n \geq 3$, then $T \in \mathcal{G}_0^1$, while if T is a double star, then $T \in \mathcal{G}_0^0$. In both cases we contradict the assumption that $T \in \mathcal{G}_0^m$ where $m \geq 2$. Hence, T is neither a star of order $n \geq 3$ nor a double star. Thus, if $T \neq P_2$, then $\mathrm{diam}(T) \geq 4$. By definition of the family $\mathcal{G}_0^m$, the tree T is obtained from a sequence $T_1, \ldots, T_q$ of trees, where $q \geq 1$ and where the tree $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2} \cup \mathcal{T}_0^{m,3}$ and the tree $T = T_q$. Further, if $q \geq 2$, then for each $i \in [q] \backslash \{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying operation $\mathcal{O}^*$ defined in Sect. 3.2. We proceed by induction on $q \geq 1$, namely **second induction**, to show that $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$.

Claim 5 *If $q = 1$, then $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$.*

Proof Suppose that $q = 1$. Thus, $T = T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2} \cup \mathcal{T}_0^{m,3}$. We consider each of the three possibilities in turn. □

Claim 5.1 *If $T \in \mathcal{T}_0^{m,1}$, then $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$.*

Proof Suppose that $T \in \mathcal{T}_0^{m,1}$. If $T = P_2$, then as observed earlier $T \in \mathcal{G}_0^2$. In this case, $m = 2$ and $\gamma_t(T) = 2 = \frac{1}{2}(n - \ell + 2 + 2) = \frac{1}{2}(n - \ell + 2 + m)$. Hence, we may assume that $T \neq P_2$, for otherwise the desired result follows. Thus, $n \geq 3$ and the tree T is neither a star nor a double star; we note that $\mathrm{diam}(T) \geq 4$. The tree $T \in \mathcal{T}_0^{m,1}$ can be obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding $a \geq 1$ new vertices and joining all of them to precisely one bad leaf, x say, of T'. Thus, the vertex x does not belong to any γ_t-set of T'. Let T' have order n', and let T' have ℓ' leaves. We note that $n' = n - a$ and $\ell' = \ell - a + 1$. Applying the first induction hypothesis to the tree $T' \in \mathcal{G}_0^{m-1}$, we have $\gamma_t(T') = \frac{1}{2}(n' - \ell' + 2 + (m - 1))$.

We show next that $\gamma_t(T) = \gamma_t(T') + 1$. Every γ_t-set of T' can be extended to a TD-set of T by adding to it the vertex x, and so $\gamma_t(T) \leq \gamma_t(T') + 1$. By Observation 1(b), there exists a γ_t-set, D say, of T that contains no leaf of T, implying that $x \in D$ and that no leaf of x belongs to D. Thus, the set D is a TD-set of T' that contains the vertex x. Since no γ_t-set of T' contains the vertex x, we note that $\gamma_t(T) = |D| \geq \gamma_t(T') + 1$. Consequently, $\gamma_t(T) = \gamma_t(T') + 1$. Thus,

$$\begin{aligned}
\gamma_t(T) &= \gamma_t(T') + 1 \\
&= \tfrac{1}{2}(n' - \ell' + 1 + m) + 1 \\
&= \tfrac{1}{2}((n - a) - (\ell - a + 1) + 1 + m) + 1 \\
&= \tfrac{1}{2}(n - \ell + 2 + m).
\end{aligned}$$

This completes the proof of Claim 5.1. □

Claim 5.2 *If $T \in \mathcal{T}_0^{m,2}$, then $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$.*

Proof Suppose that $T \in \mathcal{T}_0^{m,2}$. Thus, the tree T can be obtained from a tree $T' \in \mathcal{G}_0^{m-2}$ by adding a vertex disjoint copy of a double star Q and identifying a leaf of Q with a stable vertex, x say, of T'. Let u and v be the two vertices in Q that are not leaves, and so $S(Q) = \{u, v\}$. Since x is a stable vertex of T', we note that x is not a support vertex and $\gamma_t(T' - x) \geq \gamma_t(T')$. Let T' have order n', and let T' have ℓ' leaves. Further, let Q have t leaves, and so $|L(Q)| = t$ and $n(Q) = t + 2$. We note that $n' = n - t - 1$ and $\ell' = \ell - t + 1$. Applying the first induction hypothesis to the tree $T' \in \mathcal{G}_0^{m-2}$, we have $\gamma_t(T') = \frac{1}{2}(n' - \ell' + 2 + (m - 2)) = \frac{1}{2}(n' - \ell' + m)$.

We show next that $\gamma_t(T) = \gamma_t(T') + 2$. Every γ_t-set of T' can be extended to a TD-set of T by adding to it the vertices u and v, implying that $\gamma_t(T) \leq \gamma_t(T') + 2$. Recall that $\mathrm{diam}(T) \geq 4$. By Observation 1(b), there exists a γ_t-set, D say, of T that contains no leaf of T, implying that $\{u, v\} \subseteq D$ and that neither leaf-neighbor of u nor v in T belongs to D. Thus, the set $D \setminus \{u, v\}$ is a TD-set of $T' - x$, implying that $\gamma_t(T') \leq \gamma_t(T' - x) \leq |D| - 2 = \gamma_t(T) - 2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$. Thus,

$$\begin{aligned}
\gamma_t(T) &= \gamma_t(T') + 2 \\
&= \tfrac{1}{2}(n' - \ell' + m) + 2 \\
&= \tfrac{1}{2}((n - t - 1) - (\ell - t + 1) + m) + 2 \\
&= \tfrac{1}{2}(n - \ell + 2 + m).
\end{aligned}$$

This completes the proof of Claim 5.2. □

Claim 5.3 *If $T \in \mathcal{T}_0^{m,3}$, then $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$.*

Proof Suppose that $T \in \mathcal{T}_0^{m,3}$. Thus, the tree T can be obtained from a tree $T' \in \mathcal{G}_0^{m-1}$ by adding a vertex disjoint copy of a double star Q and adding an edge from a leaf, say y, of Q to a vertex, z say, of T' of degree at least 2. We note that $d_T(y) = 2$. Let u and v be the two vertices in Q that are not leaves, and so $S(Q) = \{u, v\}$. Let T' have order n', and let T' have ℓ' leaves. Further, let Q have t leaves, and so $|L(Q)| = t$ and $n(Q) = t+2$. We note that $n' = n-t-2$ and $\ell' = \ell-t+1$. Applying the first induction hypothesis to the tree $T' \in \mathcal{G}_0^{m-1}$, we have $\gamma_t(T') = \frac{1}{2}(n'-\ell'+2+(m-1)) = \frac{1}{2}(n'-\ell'+1+m)$.

We show next that $\gamma_t(T) = \gamma_t(T') + 2$. Every γ_t-set of T' can be extended to a TD-set of T by adding to it the vertices u and v, implying that $\gamma_t(T) \le \gamma_t(T') + 2$. By Observation 1(b), there exists a γ_t-set, D say, of T that contains no leaf of T, implying that $\{u, v\} \subseteq D$ and that no leaf-neighbor of u nor v in T belongs to D. If $y \in D$, then we can replace y in D with an arbitrary neighbor of z in T' to produce a new γ_t-set of T. Hence, we may assume that $y \notin D$. With this assumption, the set $D\setminus\{u, v\}$ is a TD-set of T', implying that $\gamma_t(T') \le |D|-2 = \gamma_t(T)-2$. Consequently, $\gamma_t(T) = \gamma_t(T') + 2$. Thus,

$$\begin{aligned}\gamma_t(T) &= \gamma_t(T') + 2\\ &= \tfrac{1}{2}(n' - \ell' + 1 + m) + 2\\ &= \tfrac{1}{2}((n - t - 2) - (\ell - t + 1) + 1 + m) + 2\\ &= \tfrac{1}{2}(n - \ell + 2 + m).\end{aligned}$$

This completes the proof of Claim 5.3. □

By Claims 5.1, 5.2 and 5.3, if $T \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2} \cup \mathcal{T}_0^{m,3}$, then $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$. This completes the proof of Claim 5. □

By Claim 5, if $q = 1$, then $\gamma_t(T) = \frac{1}{2}(n - \ell + 2 + m)$. This establishes the base step of the second induction. Let $q \ge 2$ and assume that if q' is an integer where $1 \le q' < q$ and if $T' \in \mathcal{G}_0^m$ is a tree obtained from a sequence $T_1, \ldots, T_{q'}$ of trees, where $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2} \cup \mathcal{T}_0^{m,3}$ and where the tree $T' = T_{q'}$ and if $q' \ge 2$, then for each $i \in [q']\setminus\{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying operation $\mathcal{O}^*$.

Recall that the tree T is obtained from a sequence $T_1, \ldots, T_q$ of trees, where $T_1 \in \mathcal{T}_0^{m,1} \cup \mathcal{T}_0^{m,2} \cup \mathcal{T}_0^{m,3}$ and the tree $T = T_q$, and where for each $i \in [q]\setminus\{1\}$, the tree T_i can be obtained from the tree T_{i-1} by applying operation $\mathcal{O}^*$. In particular, the tree T is obtained from the tree T_{q-1} by adding to it a vertex disjoint copy of a double star Q and adding an edge joining a leaf of Q and a leaf of T_{q-1}. Let u and v be the two vertices in Q that are not leaves, and so $S(Q) = \{u, v\}$. Let $T' = T_{q-1}$, and let T' have order n' and ℓ' leaves. Further, let Q have t leaves, and so $|L(Q)| = t$ and $n(Q) = t+2$. We note that $n' = n-t-2$ and $\ell' = \ell-(t-1)+1 = \ell-t+2$. Applying the second induction hypothesis to the tree $T_{q-1} \in \mathcal{G}_0^m$, we have $\gamma_t(T') = \frac{1}{2}(n' - \ell' + 2 + m)$. Proceeding analogously as in the proof of the last paragraph of Claim 5.3, we have that $\gamma_t(T) = \gamma_t(T') + 2$. Thus,

$$\begin{aligned}\gamma_t(T) &= \gamma_t(T') + 2 \\ &= \tfrac{1}{2}(n' - \ell' + 2 + m) + 2 \\ &= \tfrac{1}{2}((n - t - 2) - (\ell - t + 2) + 2 + m) + 2 \\ &= \tfrac{1}{2}(n - \ell + 2 + m).\end{aligned}$$

This completes the proof of Theorem 5. □

We are now in a position to prove Theorem 1. Recall its statement.

Theorem 1 *Let $m \geq 0$ be an integer. If G is a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves, then $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$ if and only if $G \in \mathcal{G}_k^m$.*

Proof Let $m \geq 0$ be an integer, and let G be a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves. We proceed by induction on k to show that $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$ if and only if $G \in \mathcal{G}_k^m$. If $k = 0$, then the result follows from Theorem 5. This establishes the base case. Let $k \geq 1$ and assume that if G' is a cactus graph of order $n' \geq 2$ with k' cycles and ℓ' leaves where $0 \leq k' < k$, then $\gamma_t(G') = \frac{1}{2}(n' - \ell' + 2 + m') - k'$ if and only if $G \in \mathcal{G}_{k'}^{m'}$. Let G be a cactus graph of order $n \geq 2$ with $k \geq 0$ cycles and ℓ leaves. We show that $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$ if and only if $G \in \mathcal{G}_k^m$. If $m = 0$, then the result follows by Theorem 3(a), while if $m = 1$, then the result follows by Theorem 3(b). Hence, we may assume that $m \geq 2$.

($\Longrightarrow$) Assume that $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$. We show that $G \in \mathcal{G}_k^m$. For this purpose, we first prove the following claim. □

Claim 6 *The graph G contains a cycle edge e such that $\gamma_t(G - e) = \gamma_t(G)$.*

Proof Let $C\colon v_1 v_2 \ldots v_\ell v_1$ be a cycle in G, and let S be a γ_t-set of G. If $V(C) \subseteq S$, then let $e = v_1 v_2$. In this case, the set S is a TD-set of $G - e$, and so $\gamma_t(G - e) \leq |S| = \gamma_t(G)$. Since removing a cycle edge from a graph cannot decrease the total domination of the graph, we note that $\gamma_t(G) \leq \gamma_t(G - e)$. Consequently, $\gamma_t(G) = \gamma_t(G - e)$. Hence, we may assume that at least one vertex of the cycle C does not belong to the set S. Renaming vertices of C if necessary, we may assume that $v_2 \notin S$. If $v_1 \notin S$, then letting $e = v_1 v_2$ we note that the set S is a TD-set of $G - e$, and so as before $\gamma_t(G) = \gamma_t(G - e)$. Hence, we may assume that $v_1 \in S$. Analogously, we may assume that $v_3 \in S$. But then as before, letting $e = v_1 v_3$ the set S is a TD-set of $G - e$, implying that $\gamma_t(G) = \gamma_t(G - e)$. □

By Claim 6, the graph G contains a cycle edge e such that $\gamma_t(G - e) = \gamma_t(G)$. Let $e = uv$, and consider the graph $G' = G - e$. Let G' have order n' with $k' \geq 0$ cycles and ℓ' leaves. We note that $n' = n$. Further, since G is a cactus graph, $k' = k - 1$. Removing the cycle edge e from G produces at most two new leaves, namely the ends of the edge e, implying that $\ell' - 2 \leq \ell \leq \ell'$. By Corollary 4, $\gamma_t(G') = \frac{1}{2}(n' - \ell' + 2 + m') - k'$ for some integer $m' \geq 0$. Applying the inductive hypothesis to the cactus graph G', we have that $G' \in \mathcal{G}_{k'}^{m'} = \mathcal{G}_{k-1}^{m'}$. We note that $\frac{1}{2}(n - \ell + 2 + m) - k = \gamma_t(G) = \gamma_t(G') = \frac{1}{2}(n' - \ell' + 2 + m') - k'$, implying that $m - \ell = m' - \ell' + 2$. Since G is a cactus, the vertices u and v are connected in $G' = G - e$ by a unique path.

Suppose that $\ell = \ell'$. In this case, neither u nor v is a leaf of G', implying that both u and v have degree at least 2 in G'. Further, the equation $m - \ell = m' - \ell' + 2$

simplifies to $m' = m - 2$. Thus, $G' \in \mathcal{G}_{k-1}^{m-2}$. Hence, the graph G is obtained from G' by Procedure D, and therefore $G \in \mathcal{G}_k^m$.

Suppose that $\ell = \ell' - 1$. In this case, exactly one of u and v is a leaf of G'. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m - 1$. Thus, $G' \in \mathcal{G}_{k-1}^{m-1}$. Hence, the graph G is obtained from G' by Procedure E, and therefore $G \in \mathcal{G}_k^m$.

Suppose that $\ell = \ell' - 2$. In this case, both u and v are leaves in G'. Further, the equation $m - \ell = m' - \ell' + 2$ simplifies to $m' = m$. Thus, $G' \in \mathcal{G}_{k-1}^m$. Hence, the graph G is obtained from G' by Procedure F, and therefore $G \in \mathcal{G}_k^m$. This completes the necessity part of the proof of Theorem 1.

($\Longleftarrow$) Conversely, assume that $G \in \mathcal{G}_k^m$. Recall that by our earlier assumptions, $m \ge 2$ and $k \ge 1$. Thus, the graph G is obtained from either a graph $G' \in \mathcal{G}_{k-1}^{m-2}$ by Procedure D or from a graph $G' \in \mathcal{G}_{k-1}^{m-1}$ by Procedure E or from a graph $G' \in \mathcal{G}_{k-1}^m$ by Procedure F. In all three cases, let G' have order n' with $k' \ge 0$ cycles and ℓ' leaves. Further, in all cases we note that $n' = n$ and $k' = k - 1$. We consider the three possibilities in turn.

Suppose firstly that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^{m-2}$ by Procedure D. In this case, $\ell = \ell'$ and $\gamma_t(G) = \gamma_t(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^{m-2}$, we have $\gamma_t(G) = \gamma_t(G') = \frac{1}{2}(n' - \ell' + 2 + (m - 2)) - (k - 1) = \frac{1}{2}(n - \ell + 2 + m) - k$.

Suppose next that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^{m-1}$ by Procedure E. In this case, $\ell = \ell' - 1$ and $\gamma_t(G) = \gamma_t(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^{m-1}$, we have $\gamma_t(G) = \gamma_t(G') = \frac{1}{2}(n' - \ell' + 2 + (m - 1)) - (k - 1) = \frac{1}{2}(n - \ell + 2 + m) - k$.

Suppose finally that G is obtained from a graph $G' \in \mathcal{G}_{k-1}^m$ by Procedure F. In this case, $\ell = \ell' - 2$ and $\gamma_t(G) = \gamma_t(G')$. Applying the inductive hypothesis to the graph $G' \in \mathcal{G}_{k-1}^m$, we have $\gamma_t(G) = \gamma_t(G') = \frac{1}{2}(n' - \ell' + 2 + m) - (k - 1) = \frac{1}{2}(n - \ell + 2 + m) - k$. In all three cases, $\gamma_t(G) = \frac{1}{2}(n - \ell + 2 + m) - k$. This completes the proof of Theorem 1. $\square$

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