The Longest (*s*, *t*)-paths of *C*-shaped Supergrid Graphs

Fatemeh Keshavarz-Kohjerdi[1], Ruo-Wei Hung[2,*], and Guo-Hao Qiu[2]

***Abstract*—A supergrid graph is a finite vertex-induced subgraph of the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if the difference of their *x* or *y* coordinates is not larger than 1. The Hamiltonian path (cycle) problem is to determine whether a graph contains a simple path (cycle) in which each vertex of the graph appears exactly once. These two problems are NP-complete for general graphs and they are also NP-complete for general supergrid graphs. Despite the many applications of the problem, they are still open for many classes, including solid supergrid graphs and supergrid graphs with some holes. A graph is called Hamiltonian connected if it contains a Hamiltonian path between any two distinct vertices. In this paper, first we will study the Hamiltonian cycle property of *C*-shaped supergrid graphs, which are a special case of rectangular supergrid graphs with a rectangular hole. Next, we will show that *C*-shaped supergrid graphs are Hamiltonian connected except few conditions. Finally, we will compute a longest (*s*, *t*)-path between two distinct vertices *s*, *t* in linear time. The Hamiltonian and longest (*s*, *t*)-paths of *C*-shaped supergrid graphs can be applied to compute the optimal stitching trace of computer embroidery machines, and construct the minimum printing trace of 3D printers with a *C*-like component being printed.**

***Index Terms*—Hamiltonicity, Hamiltonian connectivity, longest (*s*, *t*)-path, supergrid graphs, *C*-shaped supergrid graphs, computer embroidery machines, 3D printers.**

I. INTRODUCTION

A *Hamiltonian path* (*cycle*) in a graph is a simple path (cycle) in which each vertex of the graph appears exactly once. The *Hamiltonian path (cycle) problem* involves deciding whether or not a graph contains a Hamiltonian path (cycle). A graph is called *Hamiltonian* if it contains a Hamiltonian cycle. A graph G is said to be *Hamiltonian connected* if for each pair of distinct vertices u and v of G, there is a Hamiltonian path from u to v in G. The Hamiltonian path and cycle problems have numerous applications in different areas, including establishing transport routes, production launching, the on-line optimization of flexible manufacturing systems [1], computing the perceptual boundaries of dot patterns [33], pattern recognition [2], [34], [37], DNA physical mapping [12], fault-tolerant routing for 3D network-on-chip architectures [8], and so on. It is well known that the Hamiltonian path and cycle problems are NP-complete for general graphs [9], [22]. The same holds true for bipartite graphs [30], split graphs [10], circle graphs [7], undirected path graphs [3], grid graphs [21], triangular grid graphs [11], supergrid graphs [13], etc.

The *longest path problem*, i.e., the problem of finding a simple path with the maximum number of vertices, is one of the most important problems in graph theory. The Hamiltonian path problem is clearly a special case of the longest path problem. Despite the many applications of the problem, it is still open for some classes of graphs, including solid supergrid graphs and supergrid graphs with some holes [14], [15]. There are few classes of graphs in which the longest path problem is polynomial solvable [5], [20], [24], [32], [38]. In this paper, we focus on supergrid graphs. We will give the necessary and sufficient conditions for the Hamiltonian and Hamiltonian connected of C-shaped supergrid graphs. We then present a linear-time algorithm for finding a longest path between any two distinct vertices in a C-shaped supergrid graph.

The *two-dimensional integer grid graph* G^∞ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if the (Euclidean) distance between them is equal to 1. The *two-dimensional triangular grid graph* T^∞ is an infinite graph obtained from G^∞ by adding all edges on the lines traced from up-left to down-right. A *grid graph* is a finite, vertex-induced subgraph of G^∞ (see Fig. 1(a)). A *triangular grid graph* is a finite, vertex-induced subgraph of T^∞ (see Fig. 1(b)). Hung *et al.* [13] have introduced a new class of graphs, namely *supergrid graphs*. The *two-dimensional supergrid graph* S^∞ is an infinite graph obtained from T^∞ by adding all edges on the lines traced from up-right to down-left. A *supergrid graph* is a finite, vertex-induced subgraph of S^∞ (see Fig. 1(c)). A solid supergrid graph is a supergrid graph without holes. A *rectangular supergrid graph* is a supergrid graph bounded by a axis-parallel rectangle (see 2(a)). A *L-shaped* or *C-shaped* supergrid graph is a supergrid graph obtained from a rectangular supergrid graph by removing a rectangular supergrid graph from it to make a L-like or C-like shape (see 2(b) and 2(c)).

Previous related works are summarized as follows. Recently, Hamiltonian path (cycle) and Hamiltonian connected problems in grid, triangular grid, and supergrid graphs have received much attention. In [21], Itai *et al.* proved that the Hamiltonian path (cycle) problem on grid graphs is NP-complete. They also gave necessary and sufficient conditions for a rectangular grid graph having a Hamiltonian path between two given vertices. Note that rectangular grid graphs are not Hamiltonian connected. Zamfirescu *et al.* [39] gave sufficient conditions for a grid graph having a Hamiltonian cycle, and proved that all grid graphs of positive width have

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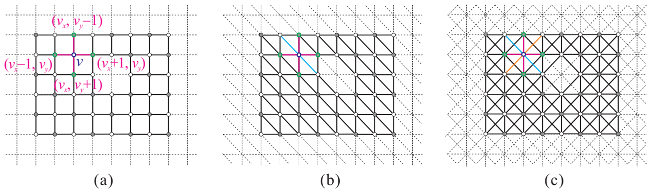

Fig. 1. (a) A grid graph, (b) a triangular grid graph, and (c) a supergrid graph, where circles represent the vertices and solid lines indicate the edges in the graphs.

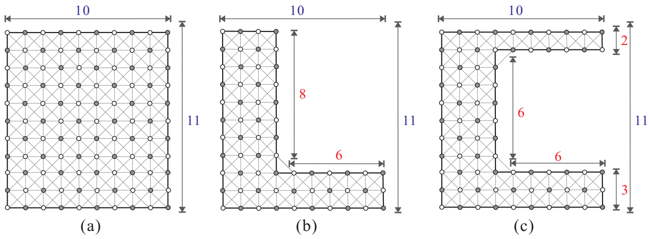

Fig. 2. (a) A rectangular supergrid graph, (b) a L-shaped supergrid graph, and (c) a C-shaped supergrid graph, where circles represent the vertices and solid lines indicate the edges in the graphs.

Hamiltonian line graphs. Later, Chen *et al.* [6] improved the Hamiltonian path algorithm of [21] on rectangular grid graphs and presented a parallel algorithm for the Hamiltonian path problem with two given endpoints in rectangular grid graphs. Also there is a polynomial-time algorithm for finding Hamiltonian cycles in solid grid graphs [31]. In [36], Salman introduced alphabet grid graphs and determined classes of alphabet grid graphs which contain Hamiltonian cycles. Keshavarz-Kohjerdi and Bagheri gave necessary and sufficient conditions for the existence of Hamiltonian paths in alphabet grid graphs, and presented linear-time algorithms for finding Hamiltonian paths with two given endpoints in these graphs [23]. They also presented a linear-time algorithm for computing the longest path between two given vertices in rectangular grid graphs [24], gave a parallel algorithm to solve the longest path problem in rectangular grid graphs [25], and solved the Hamiltonian path and longest path problems in some classes of grid graphs [26], [27], [28], [29]. Reay and Zamfirescu [35] proved that all 2-connected, linear-convex triangular grid graphs except one special case contain Hamiltonian cycles. The Hamiltonian cycle (path) on triangular grid graphs has been shown to be NP-complete [11]. They also proved that all connected, locally connected triangular grid graphs (with one exception) contain Hamiltonian cycles.

Recently, we proved that the Hamiltonian cycle and path problems on supergrid graphs are NP-complete [13]. We also showed that every rectangular supergrid graph always contains a Hamiltonian cycle, and proved linear-convex supergrid graphs to be Hamiltonian [14]. Very recently, we verified the Hamiltonian connectivity of rectangular, shaped, alphabet, and L-shaped supergrid graphs [15], [16], [17], [18]. We also proposed a linear-time algorithm for the Hamiltonian connected problem on alphabet supergrid graphs [17]. The Hamiltonian connectivity of L-shaped supergrid graphs has been verified in [18], [19]. The L-alphabet and C-alphabet supergrid graphs in [17] are special cases of L-shaped and C-shaped supergrid graphs, respectively. Note that C-shaped supergrid graphs contain L-shaped supergrid graphs as their subgraphs.

In this paper, we consider the Hamiltonian, Hamiltonian connectivity, and longest path of C-shaped supergrid graphs, which are special case of rectangular supergrid graph with a rectangular hole. This can be considered as the first attempts to solve these problems in supergrid graphs with some holes.

The rest of the paper is organized as follows. In Section II, some notations and observations are given. Previous results are also introduced. Section III gives the necessary and sufficient conditions for the Hamiltonicity and Hamiltonian connectivity of C-shaped supergrid graphs. That is, we show that C-shaped supergrid graphs are always Hamiltonian and Hamiltonian connected except few conditions. In Section IV, we present a linear-time algorithm to compute a longest path between any two distinct vertices in a C-shaped supergrid graph. Finally, we make concluding remarks in Section V.

II. Terminologies and Background Results

In this section, we will introduce some terminologies and symbols. Some observations and previously established results for the Hamiltonicity and Hamiltonian connectivity of rectangular and L-shaped supergrid graphs are also presented. For graph-theoretic terminology not defined in this paper, the reader is referred to [4].

The *two-dimensional integer grid graph* G^∞ is an infinite graph whose vertex set consists of all points of the Euclidean plane with integer coordinates and in which two vertices are adjacent if the (Euclidean) distance between them is equal to 1. A *grid graph* is a finite vertex-induced subgraph of G^∞. For a node v in the plane with integer coordinates, let v_x and v_y represent the x and y *coordinates* of node v, respectively, denoted by $v = (v_x, v_y)$. If v is a vertex in a grid graph, then its possible adjacent vertices include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, and $(v_x, v_y + 1)$ (see Fig. 1(a)). The *two-dimensional triangular grid graph* T^∞ is an infinite graph obtained from G^∞ by adding all edges on the lines traced from up-left to down-right. A *triangular grid graph* is a finite vertex-induced subgraph of T^∞. If v is a vertex in a triangular grid graph, then its possible neighboring vertices include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, and $(v_x + 1, v_y + 1)$ (see Fig. 1(b)). Thus, triangular grid graphs contain grid graphs as subgraphs. The triangular grid graphs defined above are isomorphic to the original triangular grid graphs in [11] but these graphs are different when considered as geometric graphs.

The *two-dimensional supergrid graph* S^∞ is the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are adjacent if the difference of their x or y coordinates is not larger than 1. A *supergrid graph* is a finite vertex-induced subgraph of S^∞. The possible adjacent vertices of a vertex $v = (v_x, v_y)$ in a supergrid graph hence include $(v_x, v_y - 1)$, $(v_x - 1, v_y)$, $(v_x + 1, v_y)$, $(v_x, v_y + 1)$, $(v_x - 1, v_y - 1)$, $(v_x + 1, v_y + 1)$, $(v_x + 1, v_y - 1)$, and $(v_x - 1, v_y + 1)$ (see Fig. 1(c)). Thus, supergrid graphs contain grid graphs and triangular grid graphs as subgraphs. Notice that grid and triangular grid graphs are not subclasses of supergrid graphs, and the converse is also true: these classes of graphs have common elements (points) but in general they are distinct since the edge sets of these graphs are different. It is clear that all

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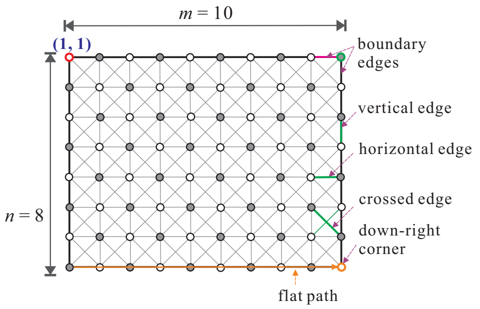

Fig. 3. A rectangular supergrid graph $R(m, n)$, where $m = 10$, $n = 8$, and the bold dashed lines indicate vertical and horizontal separations.

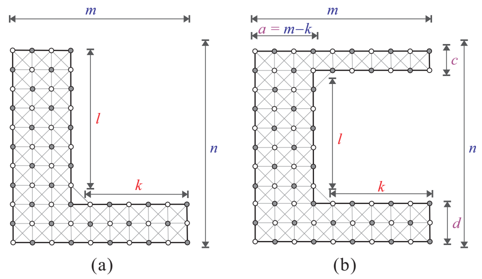

Fig. 4. The structure of (a) L-shaped supergrid graph $L(m, n; k, l)$, where $k = 6$, $l = 8$, $m - k = 4$, and $n - l = 3$ and (b) C-shaped supergrid graph $C(m, n; k, l; c, d)$, where $k = l = 6$, $c = 2$, $d = n - l - c = 3$, and $a = m - k = 4$.

grid graphs are bipartite [21] but triangular grid graphs and supergrid graphs are not bipartite. For a vertex $v = (v_x, v_y)$ in a supergrid graph, we color vertex v to be *white* if $v_x + v_y \equiv 0 \pmod 2$; otherwise, v is colored to be *black*. Then there are eight possible neighbors of vertex v including four white vertices and four black vertices.

A *rectangular supergrid graph*, denoted by $R(m, n)$, is a supergrid graph whose vertex set is $V(R(m, n)) = \{v = (v_x, v_y) | 1 \leqslant v_x \leqslant m \text{ and } 1 \leqslant v_y \leqslant n\}$. That is, $R(m, n)$ contains m columns and n rows of vertices in S^∞. The size of $R(m, n)$ is defined to be mn, and $R(m, n)$ is called n-rectangle. Let $v = (v_x, v_y)$ be a vertex in $R(m, n)$. The vertex v is called the *upper-left* (resp., *upper-right*, *down-left*, *down-right*) *corner* of $R(m, n)$ if for any vertex $w = (w_x, w_y) \in R(m, n)$, $w_x \geqslant v_x$ and $w_y \geqslant v_y$ (resp., $w_x \leqslant v_x$ and $w_y \geqslant v_y$, $w_x \geqslant v_x$ and $w_y \leqslant v_y$, $w_x \leqslant v_x$ and $w_y \leqslant v_y$). The edge (u, v) is said to be *horizontal* (resp., *vertical*) if $u_y = v_y$ (resp., $u_x = v_x$), and is called *crossed* if it is neither a horizontal nor a vertical edge. There are four boundaries in a rectangular supergrid graph $R(m, n)$ with $m, n \geqslant 2$. The edge in the boundary of $R(m, n)$ is called *boundary edge*. A path is called *flat* of $R(m, n)$ if it visits all vertices and edges of the same boundary in $R(m, n)$ and its length equals to the number of vertices in the visited boundary. For example, Fig. 3 shows a rectangular supergrid graph $R(10, 8)$ which is called 8-rectangle and contains $2 \times (9 + 7) = 32$ boundary edges. Fig. 3 also indicates the types of edges and corners. In the figures we will assume that $(1, 1)$ are coordinates of the upper-left corner in a rectangular supergrid graph $R(m, n)$, except we explicitly change this assumption.

A *L-shaped supergrid graph*, denoted by $L(m, n; k, l)$, is a supergrid graph obtained from a rectangular supergrid graph $R(m, n)$ by removing its subgraph $R(k, l)$ from the upper-right corner, where $m, n > 1$ and $k, l \geqslant 1$. Then, $m - k \geqslant 1$ and $n - l \geqslant 1$. A *C-shaped supergrid graph* $C(m, n; k, l; c, d)$ is a supergrid graph obtained from a rectangular supergrid graph $R(m, n)$ by removing its subgraph $R(k, l)$ from its node coordinated as $(m, c + 1)$ while $R(m, n)$ and $R(k, l)$ have exactly one border side in common, where $m \geqslant 2$, $n \geqslant 3$, $k, l \geqslant 1$, $c \geqslant 1$, $d = n - l - c \geqslant 1$, and $m - k \geqslant 1$. The structures of $L(m, n; k, l)$ and $C(m, n; k, l; c, d)$ are explained in Fig. 4(a) and Fig. 4(b), respectively.

Let $G = (V, E)$ be a supergrid graph with vertex set $V(G)$ and edge set $E(G)$. Let S be a subset of vertices in G, and let u and v be two vertices in G. We write $G[S]$ for the subgraph of G *induced* by S, $G - S$ for the subgraph $G[V - S]$, i.e., the subgraph induced by $V - S$. In general, we write $G - v$ instead of $G - \{v\}$. We say that u is *adjacent* to v, and u and v are *incident* to edge (u, v), if $(u, v) \in E(G)$. The notation $u \sim v$ (resp., $u \nsim v$) means that vertices u and v are adjacent (resp., non-adjacent). A vertex w *adjoins* edge (u, v) if $w \sim u$ and $w \sim v$. For two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$, if $u_1 \sim u_2$ and $v_1 \sim v_2$, then we say that e_1 and e_2 are *parallel*, denoted by $e_1 \approx e_2$. For a vertex $v \in V(G)$, the *degree* of v in G, denoted by $deg(v)$, is the number of vertices adjacent to v. A path P of length $|P|$ in G, denoted by $v_1 \to v_2 \to \cdots \to v_{|P|-1} \to v_{|P|}$, is a sequence $(v_1, v_2, \cdots, v_{|P|-1}, v_{|P|})$ of vertices such that $(v_i, v_{i+1}) \in E(G)$ for $1 \leqslant i < |P|$, and all vertices except $v_1, v_{|P|}$ in it are distinct. The first and last vertices visited by P are denoted by $start(P)$ and $end(P)$, respectively. We will use $v_i \in P$ to denote "P visits vertex v_i" and use $(v_i, v_{i+1}) \in P$ to denote "P visits edge (v_i, v_{i+1})". A path from v_1 to v_k is denoted by (v_1, v_k)-path. In addition, we use P to refer to the set of vertices visited by path P if it is understood without ambiguity. A cycle is a path C with $|V(C)| \geqslant 4$ and $start(C) = end(C)$. Two paths (or cycles) P_1 and P_2 of graph G are called *vertex-disjoint* if $V(P_1) \cap V(P_2) = \emptyset$. If $end(P_1) \sim start(P_2)$, then two vertex-disjoint paths P_1 and P_2 can be concatenated into a path, denoted by $P_1 \Rightarrow P_2$.

Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$, $\mathcal{C}$ be a cycle of $R(m, n)$, and let H be a boundary of $R(m, n)$, where H is a subgraph of $R(m, n)$. The restriction of $\mathcal{C}$ to H is denoted by $\mathcal{C}_{|H}$. If $|\mathcal{C}_{|H}| = 1$, i.e. $\mathcal{C}_{|H}$ is a flat path on H, then $\mathcal{C}_{|H}$ is called *flat face* on H. If $|\mathcal{C}_{|H}| > 1$ and $\mathcal{C}_{|H}$ contains at least one boundary edge of H, then $\mathcal{C}_{|H}$ is called *concave face* on H. A Hamiltonian cycle of $R(m, 3)$ is called *canonical* if it contains three flat faces on two shorter boundaries and one longer boundary, and it contains one concave face on the other boundary, where the shorter boundary consists of three vertices. And, a Hamiltonian cycle of $R(m, n)$ with $n = 2$ or $n \geqslant 4$ is said to be *canonical* if it contains three flat faces on three boundaries, and it contains one concave face on the other boundary. The following lemma states the result in [13] concerning the Hamiltonicity of rectangular supergrid graphs.

Lemma 1. *(See [13]) Let $R(m, n)$ be a rectangular supergrid graph with $m \geqslant n \geqslant 2$. Then, the following statements hold true:*
(1) if $n = 3$, then $R(m, 3)$ contains a canonical Hamiltonian cycle;
(2) if $n = 2$ or $n \geqslant 4$, then $R(m, n)$ contains four canonical

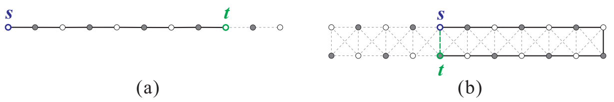

Fig. 5. Rectangular supergrid graphs in which there is no Hamiltonian (s,t)-path for (a) $R(m,1)$, and (b) $R(m,2)$, where solid lines indicate the longest path between s and t.

Hamiltonian cycles with concave faces being on different boundaries.

Let (G,s,t) denote the supergrid graph G with two specified distinct vertices s and t. Without loss of generality, we will assume that $s_x \leqslant t_x$ in the rest of the paper. We denote a Hamiltonian path between s and t in G by $HP(G,s,t)$. We say that $HP(G,s,t)$ does exist if there is a Hamiltonian (s,t)-path in G. From Lemma 1, we know that $HP(R(m,n),s,t)$ does exist if $m,n \geqslant 2$ and (s,t) is an edge in the constructed Hamiltonian cycle of $R(m,n)$.

Definition 1. Assume that G is a connected supergrid graph and V_1 is a subset of the vertex set $V(G)$. V_1 is a *vertex cut* if $G - V_1$ is disconnected. A vertex $v \in V(G)$ is a *cut vertex*, if $G - \{v\}$ is disconnected. For an example, in Fig. 5(b) $\{s,t\}$ is a vertex cut, and in Fig. 5(a) t is a cut vertex.

In [15], we showed that $HP(R(m,n),s,t)$ does not exist if the following condition hold:

(F1) s or t is a cut vertex, or $\{s,t\}$ is a vertex cut (see Fig. 5(a) and Fig. 5(b)).

Let G be any supergrid graphs. The following lemma showing that $HP(G,s,t)$ does not exist if (G,s,t) satisfies condition (F1) can be verified by the arguments in [26].

Lemma 2. *(See [26]) Let G be a supergrid graph with two vertices s and t. If (G,s,t) satisfies condition* (F1)*, then $HP(G,s,t)$ does not exist.*

The Hamiltonian (s,t)-path P of $R(m,n)$ constructed in [15] satisfies that P contains at least one boundary edge of each boundary, and is called *canonical*.

Lemma 3. *(See [15]) Let $R(m,n)$ be a rectangular supergrid graph with $m,n \geqslant 1$, and let s and t be its two distinct vertices. If $(R(m,n),s,t)$ does not satisfy condition* (F1)*, then there exists a canonical Hamiltonian (s,t)-path of $R(m,n)$, i.e., $HP(R(m,n),s,t)$ does exist.*

Consider that $(R(m,n),s,t)$ does not satisfy condition (F1). Let $w = (1,1)$, $z = (2,1)$, and $f = (3,1)$ be three vertices of $R(m,n)$ with $m \geqslant 3$ and $n \geqslant 2$. In [19], we have proved that there exists a Hamiltonian (s,t)-path Q of $R(m,n)$ such that $(z,f) \in Q$ if the following condition (F2) holds; and $(w,z) \in Q$ otherwise.

(F2) $n = 2$ and $\{s,t\} \in \{\{w,z\}, \{(1,1), (2,2)\}, \{(2,1), (1,2)\}\}$, or $n \geqslant 3$ and $\{s,t\} = \{w,z\}$.

The above result is presented as follows and can be used in proving our result.

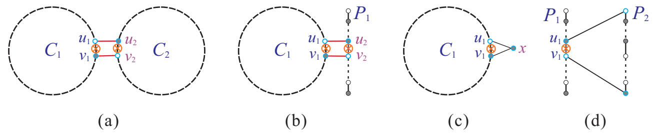

Fig. 6. A schematic diagram for (a) Statement (1), (b) Statement (2), (c) Statement (3), and (d) Statement (4) of Proposition 1, where bold dashed lines indicate the cycles (paths) and $\otimes$ represents the destruction of an edge while constructing a cycle or path.

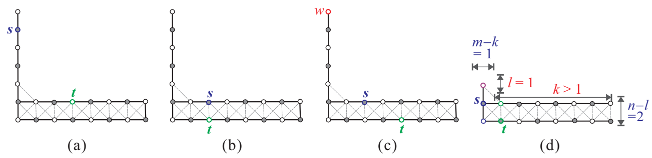

Fig. 7. L-shaped supergrid graph in which there is no Hamiltonian (s,t)-path for (a) s is a cut vertex, (b) $\{s,t\}$ is a vertex cut, (c) there exists a vertex w such that $deg(w) = 1$, $w \neq s$, and $w \neq t$, and (d) $m - k = 1$, $n - l = 2$, $l = 1$, $k \geqslant 2$, and $\{s,t\} = \{(1,2),(2,3)\}$.

Lemma 4. *(See [19]) Let $R(m,n)$ be a rectangular supergrid graph with $m \geqslant 3$ and $n \geqslant 2$, s and t be its two distinct vertices, and let $w = (1,1)$ and $z = (2,1)$. If $(R(m,n),s,t)$ does not satisfy condition* (F1)*, then there exists a canonical Hamiltonian (s,t)-path Q of $R(m,n)$ such that $(z,f) \in Q$ if $(R(m,n),s,t)$ does satisfy condition* (F2)*; and $(w,z) \in Q$ otherwise.*

We then give some observations on the relations among cycle, path, and vertex. These propositions will be used in proving our results and are given in [13], [14], [15].

Proposition 1. *(See [13], [14], [15]) Let C_1 and C_2 be two vertex-disjoint cycles of a graph G, let C_1 and P_1 be a cycle and a path, respectively, of G with $V(C_1) \cap V(P_1) = \emptyset$, and let x be a vertex in $G - V(C_1)$ or $G - V(P_1)$. Then, the following statements hold true:*
(1) If there exist two edges $e_1 \in C_1$ and $e_2 \in C_2$ such that $e_1 \approx e_2$, then C_1 and C_2 can be combined into a cycle of G (see Fig. 6(a)*).*
(2) If there exist two edges $e_1 \in C_1$ and $e_2 \in P_1$ such that $e_1 \approx e_2$, then C_1 and P_1 can be combined into a path of G (see Fig. 6(b)*).*
(3) If vertex x adjoins one edge (u_1,v_1) of C_1 (resp., P_1), then C_1 (resp., P_1) and x can be combined into a cycle (resp., path) of G (see Fig. 6(c)).
(4) If there exists one edge $(u_1,v_1) \in C_1$ such that $u_1 \sim start(P_1)$ and $v_1 \sim end(P_1)$, then C_1 and P_1 can be combined into a cycle C of G (see Fig. 6(d)*).*

In addition to condition (F1) (as depicted in Fig. 7(a) and 7(b)), in [19], we showed that $HP(L(m,n;k,l),s,t)$ does not exist whenever one of the following conditions is satisfied.

(F3) assume that G is a supergrid graph, there exists a vertex $w \in G$ such that $deg(w) = 1$, $w \neq s$, and $w \neq t$ (see Fig. 7(c)).

(F4) $m - k = 1$, $n - l = 2$, $l = 1$, $k \geq 2$, and $\{s,t\} = \{(1,2),(2,3)\}$ or $\{(1,3),(2,2)\}$ (see Fig. 7(d)).

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We then verify the following theorem in [19].

Theorem 1. *(See [19]) Let $L(m,n;k,l)$ be a L-shaped supergrid graph with vertices s and t. If $(L(m,n;k,l),s,t)$ does not satisfy conditions* (F1)*,* (F3)*, and* (F4)*, then $L(m,n;k,l)$ contains a Hamiltonian (s,t)-path, i.e., $HP(L(m,n;k,l),s,t)$ does exist.*

The following theorem shows the Hamiltonicity of L-shaped supergrid graphs and has been proved in [18].

Theorem 2. *(See [18]) Let $L(m,n;k,l)$ be a L-shaped supergrid graph. Then, $L(m,n;k,l)$ contains a Hamiltonian cycle if it does not satisfy condition* (F5).

Where condition (F5) is defined as follows:

(F5) there exists a vertex w in $L(m,n;k,l)$ such that $deg(w)=1$.

In the following, we use $\hat{L}(G,s,t)$ to denote the length of longest paths between s and t and $\hat{U}(G,s,t)$ to indicate the upper bound on the length of longest paths between s and t, where G is a rectangular, L-shaped, or C-shaped supergrid graph. By the length of a path we mean the number of vertices of the path. In [15] and [19], we showed that a longest (s,t)-path of a rectangular or L-shaped supergrid graph can be computed in linear time.

Theorem 3. *(See [15], [19]) Given a rectangular supergrid graph $R(m,n)$ with $mn \geqslant 2$ or L-shaped supergrid graph $L(m,n;k,l)$, and two distinct vertices s and t in $R(m,n)$ or $L(m,n;k,l)$, a longest (s,t)-path can be found in $O(mn)$-linear time.*

III. The Necessary and Sufficient Conditions for the Hamiltonian and Hamiltonian Connected of C-shaped Supergrid Graphs

In this section, we will give necessary and sufficient conditions for C-shaped supergrid graphs to have a Hamiltonian cycle and Hamiltonian (s,t)-path. First, we will verify the Hamiltonicity of C-shaped supergrid graphs. If $a(=m-k)=1$ or there exists a vertex $w\in V(C(m,n;k,l;c,d))$ such that $deg(w)=1$, then $C(m,n;k,l;c,d)$ contains no Hamiltonian cycle. Therefore, $C(m,n;k,l;c,d)$ is not Hamiltonian if condition (F6) is satisfied, where (F6) is defined below.

(F6) $a(=m-k)=1$ or there exists a vertex $w\in V(C(m,n;k,l;c,d))$ such that $deg(w)=1$.

By using Lemma 1 and Proposition 1, we can prove the following theorem. Due to the space limitation, we omit its proof.

Theorem 4. *$C(m,n;k,l;c,d)$ contains a Hamiltonian cycle if and only if it does not satisfy condition* (F6).

Next, we give necessary and sufficient conditions for the existence of a Hamiltonian (s,t)-path in $C(m,n;k,l;c,d)$. In addition to condition (F1) (as depicted in Fig. 8(a)–8(b)) and (F3) (as depicted in Fig. 8(c)), if $(C(m,n;k,l;c,d),s,t)$ satisfies one of the following conditions, then it contains no Hamiltonian (s,t)-path.

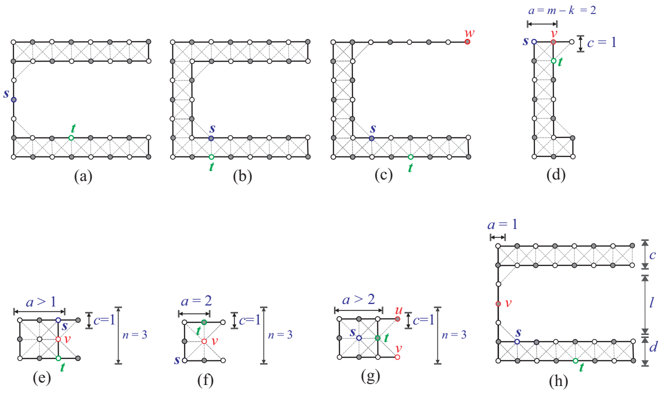

Fig. 8. Some C-shaped supergrid graphs in which there is no Hamiltonian (s,t)-path.

(F7) $m=3$, $a=m-k=2$, and $[(c=1$ and $\{s,t\}=\{(1,1),(2,2)\}$ or $\{(1,2),(2,1)\})$ or $(d=1$ and $\{s,t\}=\{(1,n),(2,n-1)\}$ or $\{(1,n-1),(2,n)\})]$ (see Fig. 8(d)).

(F8) $n=3$, $k=c=d=1$, and
1) $a\geqslant 2$ and $s_x=t_x=m-1$ (see Fig. 8(e)); or
2) $a=2$, $s_x=1$, $t_x=2$, and $|s_y-t_y|=2$ (see Fig. 8(f)); or
3) $a>2$, $s_x<m-1$, and $t=(m-1,2)$ (see Fig. 8(g)).

(F9) $a=m-k=1$, and $(s_y,t_y\leqslant c$ or $s_y,t_y>c+l)$ (see Fig. 8(h)).

By using Lemma 1, Lemma 3, Lemma 4, and Theorem 1, we prove the necessary and sufficient conditions for $HP(C(m,n;k,l;c,d),s,t)$ does exist in the following three lemmas. Due to the space limitation, the proofs are omitted.

Lemma 5. *If $HP(C(m,n;k,l;c,d),s,t)$ exists, then $(C(m,n;k,l;c,d),s,t)$ does not satisfy conditions* (F1)*,* (F3)*,* (F7)*,* (F8)*, and* (F9).

Lemma 6. *Let $C(m,n;k,l;c,d)$ be a C-shaped supergrid graph with $a=m-k=1$, and let s and t be its two distinct vertices such that $(C(m,n;k,l;c,d),s,t)$ does not satisfy conditions* (F1)*,* (F3)*,* (F7)*,* (F8)*, and* (F9)*. Then, $C(m,n;k,l;c,d)$ contains a Hamiltonian (s,t)-path, i.e., $HP(C(m,n;k,l;c,d),s,t)$ does exist.*

Lemma 7. *Let $C(m,n;k,l;c,d)$ be a C-shaped supergrid graph with $a=m-k\geqslant 2$, and let s and t be its two distinct vertices such that $(C(m,n;k,l;c,d),s,t)$ does not satisfy conditions* (F1)*,* (F3)*,* (F7)*,* (F8)*, and* (F9)*. Then, $C(m,n;k,l;c,d)$ contains a Hamiltonian (s,t)-path, i.e., $HP(C(m,n;k,l;c,d),s,t)$ does exist.*

From Lemmas 5–7, it immediately follows that the following theorem holds true.

Theorem 5. *Let $C(m,n;k,l;c,d)$ be a C-shaped supergrid graph with vertices s and t. $C(m,n;k,l;c,d)$ contains a Hamiltonian (s,t)-path, i.e., $HP(C(m,n;k,l;c,d),s,t)$ does exist if and only if $(C(m,n;k,l;c,d),s,t)$ does not satisfy conditions* (F1)*,* (F3)*,* (F7)*,* (F8)*, and* (F9).

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IV. The Longest (s,t)-path in C-shaped Supergrid Graphs

From Theorem 5, it follows that if $(C(m,n;k,l;c,d),s,t)$ satisfies one of the conditions (F1), (F3), (F7), (F8), and (F9), then $(C(m,n;k,l;c,d),s,t)$ contains no Hamiltonian (s,t)-path. So in this section, first for these cases we give upper bounds on the lengths of longest paths between s and t. Then, we show that the derived upper bound is equal to the length of longest paths between s and t. Notice that the isomorphic cases are omitted. The following lemmas give the bounds. Due to the space limitation, the proofs of the following four lemmas are omitted.

Lemma 8. *Let $a = 1$ and $w = (1, c+1)$. Suppose that $(C(m,n;k,l;c,d),s,t)$ satisfies one of the conditions* (F1) *and* (F9)*. Then, the following statements hold true:*
(FC1) *If $s_y, t_y > c$, then the length of any path between s and t cannot exceed $\hat{L}(G_1,s,t)$, where $G_1 = L(m,n-c;k,l)$.*
(FC2) *If ($s_y \leqslant c$ and $t_y > c+l$) or ($t_y \leqslant c$ and $s_y > c+l$), without loss of generality assume that $s_y \leqslant c$, then the length of any path between s and t cannot exceed $\hat{L}(G_1,s,z) + \hat{L}(G_2,w,t)$, where $G_1 = R(m,c)$, $G_2 = L(m,n-c;k,l)$, and $z = (1,c)$ if $s \neq (1,c)$; otherwise $z = (2,c)$.*

Lemma 9. *Assume that $a > 1$, $\{s,t\}$ is a vertex cut, and ($k > 1$, $s_y, t_y \geqslant n-1$, and $a+1 \leqslant s_x = t_x \leqslant m-1$) or ($c+1 \leqslant s_y = t_y \leqslant c+l$)]. Then, the following statements hold true:*
(FC3) *If $a = 2$ and $s_y = t_y$, then the length of any path between s and t cannot exceed $\max\{\hat{L}(G_1,s,t), \hat{L}(G_2,s,t)\}$, where $G_1 = L(m,n';k,l')$, $G_2 = L(s_y,m;k',l'')$, $n' = n-s_y+1$, $l' = n'-d$, $k' = l-l'+1$, and $l'' = k$.*
(FC4) *If $c = 1$, $d = 2$, and $s_x = t_x$, then the length of any path between s and t cannot exceed $\max\{\hat{L}(G_1,s,t), \hat{L}(G_2,s,t)\}$, where $G_1 = C(m',n;k',l;c,d)$, $G_2 = R(m-m'+1,d)$, $m' = s_x$, and $k' = m'-a$.*
(FC5) *If $c > 1$, $d = 2$, and $s_x = t_x$, then the length of any path between s and t cannot exceed $\hat{L}(G_1,s,t) + |G_2| = \hat{L}(G_1,s,t) + k \times c$, where $G_1 = L(m,n;k,l+c)$ and $G_2 = R(k,c)$.*

Lemma 10. *Let $a > 1$ and $c = 1$. Then, the following statements hold true:*
(FC6) *If $a = 2$, $s_y, t_y \leqslant 2$, $s_y \neq t_y$, and $s_x \neq t_x$, then the length of any path between s and t cannot exceed $\hat{L}(L(m,n;k,l+c),s,t)$.*
(FC7) *If $(C(m,n;k,l;c,d),s,t)$ satisfies condition* (F8)) *or ($s_x = t_x = m-1$ and $s_y, t_y \leqslant 2$), then the length of any path between s and t cannot exceed $\hat{L}(G',s,t)$, where $G' = L(m,n;k,l+c)$.*

Lemma 11. *Assume that $k, a > 1$, $c = 1$, and $(C(m,n;k,l;c,d),s,t)$ does not satisfy* (FC6)*. Let $w = (a+1,1)$. Then, the following statements hold true:*
(FC8) *If $t_x > 1$, $t_y = 1$, and [($s_x \leqslant a$) or ($s_x > a$ and $s_y > c+l$)], then the length of any path between s and t cannot exceed $\hat{L}(G_1,s,z) + \hat{L}(G_2,w,t)$, where $G_1 = L(m,n;k,l+c)$, $G_2 = R(k,c)$, and $z = (a,1)$ if $s \neq (a,1)$; otherwise $z = (a,2)$.*
(FC9) *If $s_x, t_x > 1$ and $s_y = t_y = 1$, then the length of any path between s and t cannot exceed $t_x - s_x + 1$.*
(FC10) *If $deg(s) > 1$, $deg(t) > 1$, and $(C(m,n;k,l;c,d),s,t)$ does not satisfy condition* (F1)*, then the length of any path between s and t cannot exceed $\hat{L}(G_1,s,t)+1$, where $G_1 = L(m,n;k,l+c)$.*

It is easy to show that any $(C(m,n;k,l;c,d),s,t)$ must satisfy one of conditions (C1), (FC1), (FC2), (FC3), (FC4), (FC5), (FC6), (FC7), (FC8), (FC9), and (FC10). If $(C(m,n;k,l;c,d),s,t)$ satisfies (C1), then $\hat{U}(C(m,n;k,l;c,d),s,t)$ is $mn-kl$. Otherwise, $\hat{U}(C(m,n;k,l;c,d),s,t)$ can be computed using Lemmas 8–11. Where (C1) is defined as follows:

(C1) $(C(m,n;k,l;c,d),s,t)$ does not satisfy any of conditions (F1), (F3), (F7), (F8), and (F9).

We then conclude the upper bounds $\hat{U}(C(m,n;k,l;c,d),s,t)$ as follows:

$$\hat{U}(C(m,n;k,l;c,d),s,t) =$$
$$\begin{cases} \hat{L}(G_1,s,t), & \text{if (FC1)}, \\ \hat{L}(G_1,s,z) + \hat{L}(G_2,w,t), & \text{if (FC2) or (FC8)}, \\ \max\{\hat{L}(G_1,s,t), \hat{L}(G_2,s,t\}, & \text{if (FC3) or (FC4)}, \\ \hat{L}(G_1,s,t) + k \times c, & \text{if (FC5)}, \\ \hat{L}(L(m,n;k,l+c),s,t), & \text{if (FC6) or (FC7)}, \\ t_x - s_x + 1, & \text{if (FC9)}, \\ \hat{L}(G_1,s,t)+1, & \text{if (FC10)}, \\ mn-kl, & \text{if (C1)}. \end{cases}$$

Finally, we show how to obtain a longest (s,t)-path for C-shaped supergrid graphs in Lemma 12. Due to the space limitation, its proof is omitted. Notice that if $(C(m,n;k,l;c,d),s,t)$ satisfies (C1), then by Theorem 5, it contains a Hamiltonian (s,t)-path.

Lemma 12. *If $(C(m,n;k,l;c,d),s,t)$ satisfies one of conditions* (FC1)–(FC10)*, then $\hat{L}(C(m,n;k,l;c,d),s,t) = \hat{U}(C(m,n;k,l;c,d),s,t)$.*

We finally conclude the following theorem.

Theorem 6. *Given a C-shaped supergrid $C(m,n;k,l;c,d)$ and two distinct vertices s and t in $C(m,n;k,l;c,d)$, a longest (s,t)-path can be constructed in $O(mn)$-linear time.*

The algorithm is formally presented as Algorithm IV.1.

V. Concluding Remarks

Based on the Hamiltonicity and Hamiltonian connectivity of rectangular and L-shaped supergrid graphs, we can prove C-shaped supergrid graphs to be Hamiltonian and Hamiltonian connected except few conditions. On the other hand, we give a linear-time algorithm to find the longest (s,t)-path in C-shaped supergrid graph with two distinct vertices s, t. Whether the result can be applied to O-shaped supergrid graphs which are rectangular supergrid graphs with a rectangular hole. We leave it to interesting readers. On the other hand, the Hamiltonian cycle problem on solid grid graphs was known to be polynomial solvable. However, it

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Algorithm IV.1: The longest (s,t)-path algorithm

Input: A C-shaped supergrid graph $C(m,n;k,l;c,d)$ with $mn \geqslant 2$, and two distinct vertices s and t in $C(m,n;k,l;c,d)$.

Output: The longest (s,t)-path.

1. **if** $a(=m-k)=1$ **then output** $HP(C(m,n;k,l;c,d),s,t)$ constructed from Lemma 6; // $(C(m,n;k,l;c,d),s,t)$ does not satisfy the forbidden conditions (F1), (F3), (F7), (F8), and (F9);
2. **if** $a(=m-k)>1$ **then output** $HP(C(m,n;k,l;c,d),s,t)$ constructed from Lemma 7; // $(C(m,n;k,l;c,d),s,t)$ does not satisfy the forbidden conditions (F1), (F3), (F7), and (F8);
3. **if** $(C(m,n;k,l;c,d),s,t)$ satisfies one of the forbidden conditions (F1), (F3), (F7), (F8), and (F9), **then output** the longest (s,t)-path based on Lemma 12;

remains open for solid supergrid graphs in which there exists no hole.

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