

# Multistep methods for system of nonlinear Volterra integral equations of the first kind

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## Abstract

This paper is concerned with deriving numerical methods based on multistep methods for solving the system of nonlinear Volterra integral equations (NVIEs) of the first kind. The accuracy and efficiency of the method is verified by solving some examples.

**Keywords:** System of nonlinear Volterra integral equations, Integral equations of the first kind, Multistep methods.

**Mathematics Subject Classification:** 26A33; 44A; 33E

## 1 Introduction

In general, the integral equations of the first kind are the ill-posed problems, it means that, a small perturbation in the right hand side of the equation causes a large perturbation in the solution [1]. But many of this equations can not be usually solved exactly and they should be solved by numerical methods. Therefore, giving suitable numerical methods for these equations is very worthwhile. Up to now, many works have been done about them. For example, the block-by-block methods have been developed in [2], the numerical solution of these equations has been presented in [1] by taking the Chebyshev expansion for solution and using the Galerkin method, the Fredholm integral equations have been considered by regularised collocation method in [3].

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In [4], two methods have been proposed, which are based on the Chebyshev interpolation of the solution and integrand term for the numerical solution of the VIEs of the first kind. A Galerkin method in [5] and an algorithm based on the regularization in [6] have been studied to solve the first kind Fredholm integral equations. To see other numerical methods for the integral equations of the first kind, we refer to [7, 8]. On the other hand, the multistep methods have been developed to solve integral equations in some interesting works. For example, Holyhead, Mckee and Taylor in [9] and Holyhead, Mckee in [10] studied the multistep methods and their stability and convergence. Also, in [11] Andrade and Mckee discussed on the optimal accuracy linear multistep methods for the first kind VIEs.

In this paper, we consider the system of nonlinear Volterra integral equations (NVIEs) of the first kind of the form:

$$\int_0^t K_i(t, s)G_i(u_1(s), \dots, u_n(s))ds = f_i(t), \quad i = 1, \dots, n, \quad t \in [0, T] \quad (1)$$

where for  $i = 1, \dots, n$ ,  $G_i$  is a nonlinear and continuous function,  $f_i$  is a continuous function on  $[0, T]$  and the kernel  $k_i(t, s)$  is continuous on  $[0, T] \times [0, T]$ .

It is clear that for solvability of the equation (1.1), it is necessary that  $f_i(0) = 0$ ,  $i = 1, \dots, n$ . For more details about the solvability of the volterra integral equations of the first kind see [12].

We first set  $g_i(s) := G_i(u_1(s), \dots, u_n(s))$ ,  $i = 1, \dots, n$ , to convert the system (1.1) to a linear system of the form:

$$\int_0^t K_i(t, s)g_i(s)ds = f_i(t), \quad i = 1, \dots, n. \quad (2)$$

To discuss the existence and uniqueness of the solution of (1.2), by differentiating of (1.2) with respect to  $t$ , we obtain

$$K_i(t, t)g_i(t) + \int_0^t \frac{\partial K_i(t, s)}{\partial t}g_i(s)ds = f_i'(t), \quad i = 1, \dots, n. \quad (3)$$

If  $K_i(t, t) \neq 0$  for every  $t \in [0, T]$ , then (1.2) is equivalent to the Volterra integral equation of the second kind:

$$g_i(t) + \int_0^t \frac{\partial K_i(t, s)/\partial t}{K_i(t, t)}g_i(s)ds = \frac{f_i'(t)}{K_i(t, t)}, \quad i = 1, \dots, n. \quad (4)$$

We know that this system has a unique solution under the suitable conditions, see [13, 14]. Thus, the system (1.2) has a unique solution. For more study about existence and uniqueness of

the solution of integral equations see [12].

## 2 description of the method

In this section, we present the method of this paper. To this end, consider the  $i$ th equation of the system (1.2) as:

$$\int_0^t K(t, s)y(s)ds = f(t) \quad (5)$$

where the subscribe  $i$  is suppressed for simplicity.

For a given positive integer  $N$ , let  $t_i = ih$ ,  $i = 0, 1, \dots, N$  with  $h = T/N$ . Setting  $t = t_i$  in (2.1) implies

$$\int_0^{t_i} K(t_i, s)y(s)ds = f(t_i), \quad i = 1, \dots, n \quad (6)$$

Now, we use a quadrature rule to approximate the integral part of this equation. At each step we determine the approximate value of  $y_i$ , and at the next step, we determine the next value of the unknown function, namely  $y_{i+1}$ , using previous values  $y_j$ ,  $j = 0, \dots, i$ . Note that, in the initial steps, choosing of the quadrature rules is restricted by the number of mesh points.

Assume that we are given  $r$  starting values which may be obtained from a multistep method or we know them.

So we define [9]:

$$[\varphi_h y]_i = \begin{cases} h(y_i - \widehat{y}_i), & i = 0, 1, \dots, r - 1 \\ h \sum_{j=0}^i w_{ij} k_{ij} y_j - f_i, & i = r, \dots, N. \end{cases} \quad (7)$$

where, for  $i = 0, 1, \dots, r - 1$ ,  $\widehat{y}_i$ s are the starting values and for  $i = r, \dots, N$ ,  $f_i$ ,  $k_{ij}$  denote  $f(t_i)$ ,  $k(t_i, t_j)$  respectively. Also  $w_{ij}$ s are the weights of the quadrature rules and for  $i = 0, \dots, N$ ,  $y_i$  denotes the approximate of  $y(t_i)$ .

In matrix notation we have

$$\varphi_h y = hU_h y - g$$



Simpson's rule and etc from the main integral equation.

The following algorithm can be given to the presented method.

**Algorithm of the method:**

**Step 1:** For a given positive integer  $N$ , set  $h = T/N$  and  $t_i = ih$  for  $i = 0, \dots, N$ .

**Step 2:** Select main repeated formula and end formula(e) and form the matrix  $U_h$  from (2.4).

**Step 3:** Obtain  $y_0 = \hat{y}_0$  exactly from (11).

**Step 4:** Obtain the approximate starting values  $\hat{y}_1, \dots, \hat{y}_{r-1}$  by a suitable method (of order  $p$ ).

**Step 5:** Form the vector  $g$  from (2.5).

**Step 6:** Solve the lower triangular system  $hU_h y = g$ .

### 3 Numerical examples

In this section, we give some numerical examples to clarify the efficiency of the presented method.

As mentioned previously, we shall use the 3/8th-rule as the main repeated formula combined with a 4-step and a 5-step end formula over even and odd subintervals, respectively.

We show the coefficients of 4-step and 5-step rules with  $\alpha_i$  and  $\beta_i$ , respectively. Choosing the  $\alpha_i$  and  $\beta_i$  such that the quadrature rules are exact for polynomials of degree three, we will have:

$$\left\{ \begin{array}{l} \alpha_0 = 0.39740084, \\ \alpha_1 = 1.07706330, \\ \alpha_2 = 1.05107170, \\ \alpha_3 = 1.07706330, \\ \alpha_4 = 0.39740084. \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \beta_0 = 0.43284615, \\ \beta_1 = 1.02180185, \\ \beta_2 = 0.95933115, \\ \beta_3 = 1.24606743, \\ \beta_4 = 0.87843368, \\ \beta_5 = 0.46151978. \end{array} \right.$$

**Example 3.1.** *As the first example, consider the system of nonlinear VIEs:*

$$\begin{cases} \int_0^t \cos(t-s)u(s)v(s)ds = \sin(t) \\ \int_0^t (s^3+1)(u(s)+v(s))^3 ds = 2t^4+8t, \end{cases}$$

on  $[0, 1]$ , with exact solutions  $u(t) = v(t) = 1$ .

First, we set  $g_1(s) = u(s)v(s)$  and  $g_2(s) = (u(s) + v(s))^3$ . Then by solving the linearised integral equations by the presented method, we obtain the values of  $g_i(s)$ ,  $i = 1, 2$  at mesh points. Finally, by solving the nonlinear algebraic system

$$\begin{cases} u(s_j)v(s_j) = g_1(s_j) \\ (u(s_j) + v(s_j))^3 = g_2(s_j), \end{cases}$$

for  $j = 1, \dots, n$ , by Newton method we determine the approximate values of unknown functions at mesh points.

In this example, we use three exact starting values, that is  $u(t_j) = v(t_j) = 1$  for  $j = 0, 1, 2$ .

The numerical results are reported in Table 1, which includes the errors  $e_j = |u(t_j) - u_j|$  and  $\varepsilon_j = |v(t_j) - v_j|$  for some mesh points  $t_j$ .

$t$	$e_j$	$\varepsilon_j$
0.2	$3.259e-10$	$3.259e-10$
0.4	$1.910e-8$	$1.910e-8$
0.6	$1.386e-8$	$1.386e-8$
1.0	$1.966e-8$	$1.966e-8$

Table 1: Computational results of Example 3.1 for  $n = 20$  at some nodes.

**Example 3.2.** *Consider the system of nonlinear VIEs:*

$$\begin{cases} \int_0^t (3u^2(s) + v^2(s))ds = t^3 + t \\ \int_0^t \exp(t-s)(u(s)v(s))ds = \exp(t) - t - 1 \end{cases}$$

on  $[0, 1]$ , with exact solutions  $u(t) = t, v(t) = 1$ .

The Table 2 shows the approximate values at some mesh points, with three exact starting values.

$t$	$e_j$	$\varepsilon_j$
0.2	1.913e-6	1.145e-6
0.4	6.770e-5	8.116e-5
0.6	2.923e-4	5.265e-4
1.0	1.564e-5	4.697e-5

Table 2: Computational results of Example 3.2 for  $n = 20$  at some nodes.

**Example 3.3.** This example is proposed in [15]:

$$\begin{cases} \int_0^t (1 - t^2 + s^2)(u(s) + v^3(s))ds = -1/12t^6 - 2/15t^5 + 1/4t^4 + 1/3t^3 \\ \int_0^t (5 + t - s)(u^3(s) - v(s))ds = -5/2t^2 - 1/6t^3 + 5/7t^7 + 1/56t^8 \end{cases}$$

on  $[0, 1]$ , with exact solutions  $u(t) = t^2, v(t) = t$ .

We proceed similar to previous examples and obtain the numerical results reported in Tables 3 and 4.

$t$	$e_j$	$\varepsilon_j$
0.2	5.87e-6	1.625e-6
0.4	2.11e-4	1.169e-6
0.6	5.33e-4	1.359e-4
1.0	3.65e-4	2.633e-5

Table 3: Computational results of Example 3.3 for  $n = 20$  at some nodes.

$t$	$e_j$	$\varepsilon_j$
0.2	1.81e-6	3.23e-6
0.4	2.05e-6	8.14e-7
0.6	2.96e-6	9.58e-7
1.0	2.26e-6	3.15e-7

Table 4: Computational results of Example 3.3 for  $n = 80$  at some nodes.

**Example 3.4.** As the last example, consider the system of nonlinear VIEs:

$$\begin{cases} \int_0^t 2t \sin(2s)(u^2(s) - v^2(s))ds = t(1 - \cos 4t)/4 \\ \int_0^t 2(t + s)u(s)v(s)ds = t/2 + (\sin 2t)/4 - t \cos 2t \end{cases}$$

on  $[0, 1]$ , with exact solutions  $u(t) = \cos t, v(t) = \sin t$ .

We show the numerical results of this example in the figure 1 at some nodes.

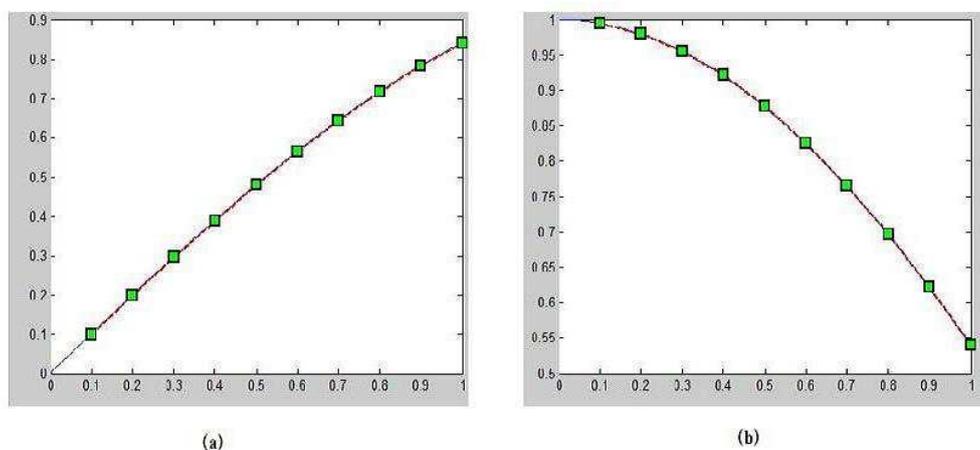


Figure 1: Numerical results of Example 4.4:  $v(t) = \sin t$  (a) and  $u(t) = \cos t$  (b).

The above results show the efficiency of the presented method. Also, by comparing two last tables we see that we can obtain higher accuracy of the solution by increasing number of the mesh points.

## 4 Conclusion

In this paper, we extended the multistep methods to a class of nonlinear systems of the Volterra integral equations of the first kind. We converted the system of NVIE of the first kind to LVIE of the first kind and then by using multistep methods we solved the resulted system. Comparison with available papers shows that the presented method gives results with high accuracy. We hope that the suggested method in this paper can be applied to the two-dimensional VIEs of the first kind.

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