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A Nordhaus–Gaddum bound for Roman domination

Nader Jafari Rad*,‡ and Hadi Rahbani†

*Department of Mathematics
Shahed University, Tehran, Iran

†Department of Mathematics
Shahrood University of Technology, Shahrood, Iran
‡n.jafarirad@gmail.com

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A Roman dominating function of a graph G is a labeling $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The Roman domination number, $\gamma_R(G)$ of G, is the minimum of $\sum_{v\in V(G)} f(v)$ over such functions. Let G be an n-vertex graph. Chambers *et al.* [E. W. Chambers, B. Kinnersley, N. Prince and D. B. West, External Problems for Roman domination *Siam J. Discrete Math.* **23** (2009) 1575–1586.] proved that if G is a connected graph of order $n \geq 3$, then $\gamma_R(G)+\gamma_R(\overline{G}) \leq n+3$, with equality if and only if G or $\overline{G}$ is C_5 or $(n/2)K_2$. In this paper, we construct a specific family of graphs ξ, and prove that if $G \notin \xi$ and $\overline{G} \notin \xi$, then $\gamma_R(G) + \gamma_R(\bar{G}) \leq n + 1$, and this bound is sharp.

Keywords: Domination; Roman domination; Nordhaus–Gaddum.

Mathematics Subject Classification 2010: 05C69

1. Introduction

The original study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274–337 AD. He decreed that for all cities in the Roman Empire, at most two legions should be stationed. Further, if a location having no legions was attacked, then it must be within the vicinity of at least one location at which two legions were stationed, so that one of the two legions could be sent to defend the attacked city. The mathematical concept of Roman domination was defined and discussed by Stewart [11], and ReVelle and Rosing [10], and subsequently developed by Cockayne *et al.* [6]. Since then more than one hundred papers have been published. For more references on Roman domination, see for example [1–5, 7, 8].

Let $G = (V, E)$ be a graph having order $n = |V|$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$, and its *closed neighborhood* is

$N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of a vertex v is $|N(v)|$. The *maximum* and *minimum* degrees among the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. Any vertex $u \in N(v)$ is called a *neighbor* of v. A vertex with exactly one neighbor is called a *leaf* and its neighbor is a *support* vertex. A support vertex with two or more leaf neighbors is called a *strong support* vertex. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood of a set S is the set $N[S] = N(S) \cup S = \bigcup_{v \in S} N[v]$. The *disjoint union* of two graphs G and H is denoted by $G + H$. A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set in G, and a dominating set of G of cardinality $\gamma(G)$ is called a γ-set of G or just a $\gamma(G)$-set.

A function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted $\gamma_R(G)$. An RDF on G with weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function of G. For an RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus, an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$.

For a graph parameter ρ, bounds on $\rho(G) + \rho(\overline{G})$ and $\rho(G)\rho(\overline{G})$ in terms of the number of vertices are called results of "*Nordhaus–Gaddum*" type, honoring the paper of Nordhaus and Gaddum [9]. Chambers *et al.* [3] investigated Nordhaus–Gaddum type bounds for Roman domination.

Theorem 1 (Chambers *et al.* [3]). *If G is a connected graph of order $n \geq 3$, then $\gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3$, with equality if and only if G or $\overline{G}$ is C_5 or $n/2K_2$.*

By Theorem 1, if neither G nor $\overline{G}$ is C_5 or $n/2K_2$, then $\gamma_R(G) + \gamma_R(\overline{G}) \leq n + 2$. We characterize all graphs G with $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$ using a new family $\mathcal{G}$ of graphs, and thus we show that if none of G and $\overline{G}$ is C_5 or $n/2K_2$ or belongs to $\mathcal{G}$, then $\gamma_R(G) + \gamma_R(\bar{G}) \leq n + 1$. We make use of the following.

Proposition 2 (Chambers *et al.* [3]). *If G is a graph of order n and maximum degree $\Delta(G)$, then $\gamma_R(G) \leq n - \Delta(G) + 1$.*

Proposition 3 (Cockayne *et al.* [6]). *For paths and cycles of order $n \geq 3$, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil$.*

Theorem 4 (Chambers *et al.* [3]). *If G is a connected n-vertex graph with $\delta(G) \geq 2$ other than those shown Fig. 1, then $\gamma_R(G) \leq 8n/11$.*

Proposition 5 (Cockayne *et al.* [6]). *If G is a graph of order n with no isolated vertices, then $\gamma_R(G) = n$ if and only if n is even and $G = (n/2)K_2$.*

2. Main Result

According to Theorem 1, all graphs G with $\gamma_R(G)+\gamma_R(\overline{G}) = n+3$ are characterized. In this paper, we wish to characterize all graphs G of order $n \geq 2$ with $\gamma_R(G) + \gamma_R(\overline{G}) = n+2$. For this purpose we first introduce some families of graphs as follows. In the following, a cycle C_n is represented by $12\dots n1$, where $V(C_n) = \{1, 2, \dots, n\}$. Furthermore by $C_n + ij$, where $|i-j| > 1$, we mean a graph obtained from C_n by adding the chord ij. For two pairs of integers i, j and i', j' with $|i-j| > 1$ and $|i'-j'| > 1$, we mean by $C_n + ij + i'j'$ a graph obtained from C_n by adding chords ij and $i'j'$. The graph $C_n + ij + i'j' + i''j''$ is defined similarly.

- Family $\mathcal{G}_0$. The class of all graphs G of order $n \geq 2$ with $\Delta(G) = n-1$ and $\delta(G) \geq n-2$.
- Family $\mathcal{G}_1$. The class of graphs $P_i + sK_2$ $(3 \leq i \leq 5, s \geq 0), 2K_3, C_3 + K_2, C_3 + 2K_2, C_4 + K_2, C_4 + C_3, C_5 + K_2, C_5, C_6, C_6 + 35, C_6 + 36, C_6 + 36 + 14, C_7, C_7 + 15, C_7 + 15 + 26, C_7 + 46 + 37 + 35, C_8 + 15, C_8 + 15 + 26$.
- Family $\mathcal{G}_2$. The class of 7 specific graphs shown in Fig. 2.

Finally, let $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \{H_3, H_4\}$, where H_3 and H_4 appeared in Fig. 1.

We will prove the following.

Theorem 6. *For a graph G of order $n \geq 2$, $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$ if and only if $G \in \mathcal{G}$ or $\overline{G} \in \mathcal{G}$.*

Let $\xi = \mathcal{G} \cup \{C_5\} \cup \{(n/2)K_2 : n \text{ is even}\}$.

Corollary 7. *If $G \notin \xi$ and $\overline{G} \notin \xi$, then $\gamma_R(G) + \gamma_R(\bar{G}) \leq n + 1$.*

We remark that the graph $P_3 + K_1$ shows that the bound of Corollary 7 is sharp.

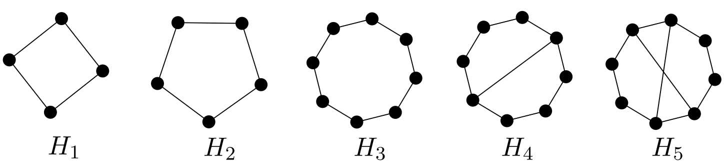

Fig. 1. Graphs corresponding to Theorem 4.

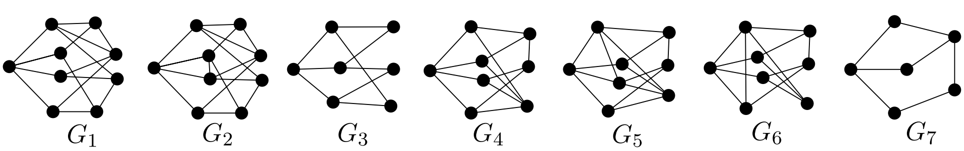

Fig. 2. The Family $\mathcal{G}_2$.

3. Proof of Theorem 6

The ($\Leftarrow$) part is straightforward. Thus we prove the ($\Rightarrow$) part. Let G be a graph of order n, and $\gamma_R(G)+\gamma_R(\overline{G})=n+2$. If $\gamma_R(G)\le n-\Delta(G)$ and $\gamma_R(\overline{G})\le n-\Delta(\overline{G})$, then $\gamma_R(G)+\gamma_R(\overline{G})\le n-\Delta(G)+n-\Delta(\overline{G})\le n+1$, a contradiction. Thus, without loss of generality and by Proposition 2, assume that $\gamma_R(G)=n-\Delta(G)+1$. Then $\gamma_R(\overline{G})=\Delta(G)+1$. Let $v\in V(G)$ be a vertex of maximum degree. If $V(G)-N[v]=\emptyset$, then $\deg(v)=n-1$ and $\gamma_R(G)=2$, since $n\ge 2$. From $\gamma_R(G)+\gamma_R(\overline{G})=n+2$ we obtain that $\gamma_R(\overline{G})=n$. By Proposition 5, $\Delta(\overline{G})\in\{0,1\}$, and thus $\delta(G)\in\{n-1,n-2\}$. Consequently, $G\in\mathcal{G}_0$. Thus assume that $V(G)-N[v]\ne\emptyset$.

Suppose that a component of $G[V(G)-N[v]]$ has at least three vertices. Let $xy,xz\in E(G[V(G)-N[v]])$. Then clearly the RDF, $f=(N(v)\cup\{y,z\},V(G)-(N[v]\cup N[x],\{v,x\})$ has weight at most $n-\Delta(G)$, a contradiction. Thus, each component of $G[V(G)-N[v]]$ has at most two vertices. If some vertex $u\in N(v)$ has at least three neighbors outside $N[v]$, then the RDF $f=(N(u)\cup N(v),V(G)-N[u]-N[v],\{u,v\})$ has weight at most $n-\Delta(G)$, a contradiction. Thus, every vertex of $N(v)$ has at most two neighbors outside $N[v]$. The following lemma plays an important role for the rest of the paper.

Lemma 8. $(n-\Delta(G)-1)\delta(G)-2|E(G[V(G)-N[v]])|\le 2\Delta(G)$.

Proof. Clearly, $\sum_{v\in V(G)-N[v]}\deg(v)\ge(n-\Delta(G)-1)\delta(G)$. Let $k=|E(G[V(G)-N[v]])|$. Observe that there are at least $(n-\Delta(G)-1)\delta(G)-2k$ edges joining $N(v)$ and $V(G)-N[v]$, and note that there are at most $2\Delta(G)$ edges joining $N(v)$ and $V(G)-N[v]$. Counting the edges joining $N(v)$ and $V(G)-N[v]$ from both sides yields $(n-\Delta(G)-1)\delta(G)-2k\le 2\Delta(G)$. $\square$

If $\Delta(G)>\delta(G)+1$, then $\Delta(G)+\Delta(\overline{G})>n$, and so $\gamma_R(G)+\gamma_R(\overline{G})\le(n-\Delta(G)+1)+(n-\Delta(\overline{G})+1)<n+2$, a contradiction. Thus $\delta(G)\le\Delta(G)\le\delta(G)+1$. We first consider the case $\Delta(G)=\delta(G)$.

Lemma 9. *If* $\Delta(G)=\delta(G)$, *then* G *or* $\overline{G}\in\{K_n,2K_3,C_3+C_4,C_5,C_6,C_7\}$.

Proof. Assume that $\Delta(G)=\delta(G)$. If $\Delta(G)=0$, then $G=\overline{K_n}$. Thus assume that $\Delta(G)\ge 1$, and so G is isolate-free, since $\Delta(G)=\delta(G)$. Assume that $\Delta(G)=1$. Then $\gamma_R(G)=n-\Delta(G)+1=n$ and $\gamma_R(\overline{G})=\Delta(G)+1=2$. From Proposition 5, we obtain $G=(n/2)K_2$. Now from $\gamma_R(\overline{G})=2$ we obtain $n=2$. Consequently, $G=K_2$. Assume next that $\Delta(G)=2$. Then $\gamma_R(G)=n-1$ and $\gamma_R(\overline{G})=3$. Since $\delta(G)=2$, G is a cycle. By Proposition 3, $n-1=\lceil 2n/3\rceil$ which implies that $3\le n\le 5$. Since $\gamma_R(\overline{G})=3$, we observe that $G=C_3$. Thus assume that $\Delta(G)\ge 3$. If $\Delta(G)\le n-5$, then $|E(G[V(G)-N[v]])|\le\frac{n-\Delta(G)-1}{2}$ and so by Lemma 8, we have $(n-\Delta(G)-1)(\delta(G)-1)\le 2\Delta(G)$ and so $4(\Delta(G)-1)\le 2\Delta(G)$, since $\delta(G)=\Delta(G)$ and $4\le n-\Delta(G)-1$. Hence, $\Delta(G)\le 2$, a contradiction. Thus

$\Delta(G) \geq n-4$. If $\Delta(G) = n-4$, then Lemma 8 implies that $\Delta(G) = 3$, and so $n = 7$, a contradiction, since there is no 3-regular graph of order 7. We deduce that $\Delta(G) \in \{n-3, n-2, n-1\}$.

If $\Delta(G) = n-2$, then $\gamma_R(G) = n - \Delta(G) + 1 = 3$, and $\Delta(\overline{G}) = 1$, (and also $\delta(\overline{G}) = 1$) since $\delta(G) = n-2$. By Proposition 5, $\gamma_R(\overline{G}) = n$ which implies that $\Delta(G) = \gamma_R(\overline{G}) - 1 = n-1$, a contradiction. If $\Delta(G) = n-1$, then $G = K_n$. Thus we assume that $\Delta(G) = n-3$. Then $|V(G) - N[v]| = 2$, $\gamma_R(G) = 4$, $\gamma_R(\overline{G}) = n-2$ and $\Delta(\overline{G}) = \delta(\overline{G}) = 2$. By applying Theorem 4 on $\overline{G}$, we obtain $n \leq 7$ and so $\Delta(G) \leq 4$. Let $V(G) - N[v] = \{x, y\}$, and $N(v) = \{v_1, v_2, \ldots, v_{\Delta(G)}\}$. If $\Delta(G) = 2$, then clearly $xy \in E(G)$, since $\delta(G) = 2$. Thus $G = C_5$. If $\Delta(G) = 3$, then $n = 6$ and $\gamma_R(\overline{G}) = 4$. If $xy \notin E(G)$, then $N(x) = N(y) = N(v)$, since $\Delta(G) = \delta(G) = 3$. Then we observe that $\overline{G} = 2K_3$. Assume next that $xy \in E(G)$. Without loss of generality, assume $N(x) = \{v_1, v_2, y\}$. If $N[x] = N[y]$, then $N(v_3) \cap \{v_1, v_2\} = \emptyset$, since $\Delta(G) = 3$. Then $\deg(v_3) = 1 < \delta(G)$, a contradiction. Thus $N[x] \neq N[y]$. We may assume that $N(y) = \{x, v_2, v_3\}$. Since $\Delta(G) = \delta(G) = 3$, we have $v_1v_3 \in E(G)$, and so $\overline{G} = C_6$. It remains to assume that $\Delta(G) = 4$. Then $n = 7$ and $\gamma_R(\overline{G}) = 5$. If $xy \notin E(G)$, then $N(x) = N(y) = N(v)$, since $\Delta(G) = \delta(G) = 4$. Hence, $\overline{G} = C_3 + C_4$. Assume next that $xy \in E(G)$. Without loss of generality, assume $N(x) = \{v_1, v_2, v_3, y\}$. If $N[x] = N[y]$, then $N(v_4) = \{v_1, v_2, v_3\}$ and the subgraph G induced by $\{v_1, v_2, v_3\}$ has no edge, since $\Delta(G) = \delta(G) = 4$, and thus $\overline{G} = C_3 + C_4$. Next assume that $N[x] \neq N[y]$. Without loss of generality assume that $N(y) = \{x, v_2, v_3, v_4\}$. If $v_2v_3 \in E(G)$, then $v_1 \notin N(v_2) \cup N(v_3)$, since $\Delta(G) = \delta(G) = 4$. Hence, $\deg(v_1) \leq 3$, a contradiction. So $v_2v_3 \notin E(G)$. If $v_1v_4 \notin E(G)$, then $\{v_2, v_3\} \subseteq N(v_1) \cap N(v_4)$, since $\delta(G) = 4$. Hence, $\deg(v_2) = \deg(v_3) = 5 > \Delta(G)$, a contradiction. Thus $v_1v_4 \in E(G)$, $|N(v_1) \cap \{v_2, v_3\}| = 1$, $|N(v_4) \cap \{v_2, v_3\}| = 1$ and $N(v_2) \cap N(v_3) \cap N(v) = \emptyset$. We may assume that $N(v_2) = \{v, v_1, x, y\}$ and $N(v_3) = \{v, v_4, x, y\}$. Hence, $\overline{G} = C_7$. $\square$

Henceforth, we assume that $\Delta(G) = \delta(G) + 1$. Suppose that $\Delta(\overline{G}) < n-5$ and $\Delta(G) < n-5$. Then $|V(G) - N(v)| \geq 5$. Let $x, y, z \in V(G) - N[v]$. Evidently, $N(x) \cap N(y) \cap N(z) = \emptyset$. Then the function $f = (V(G) - \{x, y, z\}, \emptyset, \{x, y, z\})$ is an RDF for $\overline{G}$ and so $\gamma_R(\overline{G}) \leq 6$. Then we have $n + 2 = \gamma_R(G) + \gamma_R(\overline{G} \leq (n - \Delta(G) + 1) + 6$ which implies that $\Delta(G) \leq 5$. Similarly, we can see that $\Delta(\overline{G}) \leq 5$. Then $\delta(G) \geq n-6$, and thus $\Delta(G) = \delta(G) + 1 \geq n-5$, a contradiction. We deduce that $\Delta(G) \geq n-5$ or $\Delta(\overline{G}) \geq n-5$. Without loss of generality, assume that $\Delta(G) \geq n-5$. Since $V(G) - N[v] \neq \emptyset$, we have $n - 5 \leq \Delta(G) \leq n-2$.

Assume that $\Delta(G) = n-2$. Then $\gamma_R(G) = 3$, $\delta(G) = n-3$, $\gamma_R(\overline{G}) = n-1$, $\Delta(\overline{G}) = 2$ and $\delta(\overline{G}) = 1$. Hence, any component of $\overline{G}$ is a path. Also there exist exactly one component P_t of G with $\gamma_R(P_t) = t-1$ and other components of G are P_2. By Proposition 3, $t - 1 = \gamma_R(\overline{P_t}) = \lceil 2t/3 \rceil$, which implies that $t \leq 5$. Hence, $\overline{G} \in \{P_3 + sK_2, P_4 + sK_2, P_5 + sK_2\}$, where $s \geq 0$ is an integer. Thus

$n-5 \le \Delta(G) \le n-3$. We proceed with three lemmas namely Lemmas 10, 11 and 12 according to the value $\Delta(G) = n-5$, $n-4$ or $n-3$.

Lemma 10. *If $\Delta(G) = n-5$, then G or $\overline{G} \in \{2K_2 + \overline{K_2}, K_2 + \overline{K_4}, P_5 + K_2, P_3 + 2K_2, C_3 + 2K_2, P_5 + K_2, C_5 + K_2, C_8 + 15, C_8 + 15 + 26\} \cup \{G_1, G_2\}$.*

Proof. Assume that $\Delta(G) = n-5$. Then $\gamma_R(G) = 6$, $\gamma_R(\overline{G}) = n-4$ and $\delta(G) = n-6$. Let $V(G) - N[v] = \{x, y, z, w\}$ and $k = |E(G[G - N[v]])|$. It can be seen that $k \le 2$ and so by Lemma 8 we have $\Delta(G) \le 4$.

If $\Delta(G) = 1$, then clearly $G = 2K_2 + \overline{K_2}$ or $G = K_2 + \overline{K_4}$. If $\Delta(G) = 2$, then $\gamma_R(\overline{G}) = 3$ and $\delta(G) = 1$. Let $N(v) = \{v_1, v_2\}$. Since $\Delta(G) = 2$, every vertex $N(v)$ must have at most one neighbor outside $N[v]$. Similarly $\delta(G) = 1$ implies that every vertex outside $N[v]$ must have at most one neighbor in $N(v)$. Hence, $|N(v_1) \cap (V(G) - N[v])| \le 1$, $|N(v_2) \cap (V(G) - N[v])| \le 1$ and $N(v_1) \cap N(v_2) = \emptyset$. If $k = 0$, then we may assume $x \notin N(v_1) \cap N(v_2)$, since $\Delta(G) = 2$. Then $\deg(x) = 0 < \delta(G) = 1$, a contradiction. If $k = 1$, then we may assume that $xy \in E(G)$ and $zw \notin E(G)$. Since $\delta(G) = 1$, we have $N(z) \cap N(v) \neq \emptyset$ and similarly $N(w) \cap N(v) \neq \emptyset$. On the other hand the subgraph induced by $N(v)$ has no edge. Hence, $G = P_5 + K_2$. If $k = 2$, then we may assume that $x \in N(y)$ and $z \in N(w)$. If there is no edge joining $N(v)$ and $V(G) - N(v)$, then $G = P_3 + 2K_2$ or $G = C_3 + 2K_2$. Otherwise we may assume $|(N(x) \cup N(y)) \cap N(v)| \ge 1$. If $|(N(x) \cup N(y)) \cap N(v)| = 1$, then $G \in \{P_5 + K_2, P_7\}$. Since $\gamma_R(P_7) = 5$, we obtain that $G = P_5 + K_2$. Thus assume that $|(N(x) \cup N(y)) \cap N(v)| = 2$. Then, clearly $G = C_5 + K_2$.

If $\Delta(G) = 3$, then $\gamma_R(\overline{G}) = 4$ and $\delta(G) = 2$. Let $N(v) = \{v_1, v_2, v_3\}$. It is evident that $k \ge 1$. If $k = 1$, then we may assume that $xy \in E(G)$. Then $N(x) \cap N(v) \neq \emptyset$ and $N(y) \cap N(v) \neq \emptyset$. If $u \in N(x) \cap N(y)$, then $N(z) = N(w) = N(v) - \{u\}$, since $\delta(G) = 2$. Then $\{u, z\}$ is a dominating set for $G - w$, a contradiction. Thus $N(x) \cap N(y) = \emptyset$, and so $|N(z) \cap N(w)| = 1$, since $\delta(G) = 2$. We may assume that $N(z) \cap N(w) = \{v_3\}$. Then $\{v_3\} \notin N(x) \cup N(y)$, and we may assume that $N(x) = \{y, v_1\}$ and $N(y) = \{x, v_2\}$. Since $\Delta(G) = \delta(G) + 1 = 3$, we have $\deg(z) = \deg(w) = 2$. Thus, we may assume that $N(z) = \{v_2, v_3\}$ and $N(w) = \{v_1, v_3\}$. Then $\{z, v_1\}$ is a dominating set for $G - y$, a contradiction. Hence, $k = 2$, and we may assume that $x \in N(y)$ and $z \in N(w)$. If $u \in N(x) \cap N(y)$, then the RDF $f = (N(w) \cup N(u), V(G) - N[w] - N[u], \{w, u\})$ has weight at most 5, a contradiction. Thus $N(x) \cap N(y) = \emptyset$, and similarly, $N(z) \cap N(w) = \emptyset$. Clearly, there exist at least one vertex of $N(v)$ with exactly two neighbors in $V(G) - N[v]$. Without loss of generality, assume that $N(v_2) \cap (V(G) - N[v]) = \{y, z\}$. If $N(x) \cap N(w) \neq \emptyset$, then without loss of generality, assume that $v_3 \in N(x) \cap N(w)$. Then $\{v_2, v_3\}$ is a dominating set for $G - v_1$, and so $\gamma_R(G) \le 5$, a contradiction. Hence, $N(x) \cap N(w) = \emptyset$, and we may assume that $N(x) = \{v_1, y\}$ and $N(w) = \{v_3, z\}$. If $v_1 v_3 \in E(G)$, then $\{v_1, v_2\}$ is a dominating set for G, a contradiction. Thus, the subgraph induced by $N(v)$ has no edge. If $\{v_1 z, v_3 y\} \subseteq E(G)$, then $\{z, y\}$ is a dominating set for

$G - v$, a contradiction. Hence, $|\{v_1z, v_3y\} \cap E(G)| \le 1$ and so $G = C_8 + 15$ or $G = C_8 + 15 + 26$.

Now assume that $\Delta(G) = 4$. Then $\gamma_R(\overline{G}) = 5$ and $\delta(G) = 3$. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. Lemma 8 implies that $k = 2$. We may assume $x \in N(y)$ and $z \in N(w)$. If $u \in V(G) - N[v]$, then $|N(u) \cap N(v)| \ge 2$, since $\delta(G) = 3$. Then $|N(x) \cap N(y)| \le 2$. If $|N(x) \cap N(y)| \ge 1$, then we may assume that $v_1 \in N(x) \cap N(y)$. If $\deg(v_1) = 3$, then $\{v_1, v\}$ is a dominating set for $\overline{G}$, and so $\gamma_R(G) \le 4$, a contradiction. Thus assume that $\deg(v_1) = 4$. Then $N(v_1) \cap \{z, w\} = \emptyset$. Hence, we may assume that $N(v_1) \cap N(v) = \{v_2\}$. Since $\delta(G) = 3$, we have $|N(z) \cap \{v_2, v_3, v_4\}| \ge 2$. Then the RDF $f = (N(v_1) \cup N(z), V(G) - N(v_1) - N(z), \{v_1, z\})$ has weight at most 5, a contradiction. Hence, $N(x) \cap N(y) = \emptyset$, and similarly $N(z) \cap N(w) = \emptyset$. If $N(x) \cap N(z) = \emptyset$, then we may assume that $N(x) = \{y, v_1, v_2\}$ and $N(z) = \{w, v_3, v_4\}$. Then $\{x, z\}$ is a dominating set for $G - v$, a contradiction. Thus $N(x) \cap N(z) \ne \emptyset$, and similarly $N(x) \cap N(w) \ne \emptyset$, $N(y) \cap N(z) \ne \emptyset$ and $N(y) \cap N(w) \ne \emptyset$. Then we may assume that $N(x) = \{y, v_1, v_2\}, N(y) = \{x, v_3, v_4\}, N(z) = \{w, v_1, v_3\}$ and $N(w) = \{z, v_2, v_4\}$. If $v_1v_2 \in E(G)$, then $\{v_1, y\}$ is a dominating set for $G - w$ and so $\gamma_R(G) \le 5$. Similarly, we can see that $\{v_3v_4, v_1v_3, v_2v_4\} \cap E(G) = \emptyset$. If $\{v_1v_4, v_2v_3\} \subseteq E(G)$, then $\{v_1, v_2\}$ is a dominating set for $G - y$, and so $\gamma_R(G) \le 5$, a contradiction. Therefore $|\{v_1v_4, v_2v_3\} \cap E(G)| \le 1$. Then $G = G_1$ or $G = G_2$. $\square$

Lemma 11. *If $\Delta(G) = n - 4$, then G or $\overline{G} \in \{2K_2 + K_1,\ K_2 + \overline{K}_3, P_4 + K_2, C_4 + K_2, C_7 + 15 + 26, C_7 + 15\} \cup \{G_3, G_4, G_5, G_6\}$.*

Proof. Assume that $\Delta(G) = n - 4$. Then $\delta(G) = n - 5$, $\gamma_R(G) = 5$ and $\gamma_R(\overline{G}) = n - 3$. Let $V(G) - N(v) = \{x, y, z\}$. Observe that $G - N[v]$ has at most one edge and so by Lemma 8, we have $\Delta \le 5$. If $\Delta(G) = 5$, then $\gamma_R(\overline{G}) = 6$ and $k = 1$. We may assume that $xy \in E(G)$. Since $\delta(G) = 4$, we have $N(x) \cap N(y) \ne \emptyset$. Let $\{u\} \subseteq N(x) \cap N(y)$. By Lemma 8, $u \notin N(z)$, and so $N(v) - \{u\} \subseteq N(z)$. Thus $\{u, z\}$ is a dominating set for G, a contradiction. Hence $\Delta(G) \le 4$.

If $\Delta(G) = 1$, then clearly $G = 2K_2 + K_1$ or $G = K_2 + \overline{K_3}$. If $\Delta(G) = 2$, then clearly $k = 1$. Without loss of generality, assume that $xy \in E(G)$. If $(N(x) \cup N(y)) \cap N(v) \ne \emptyset$, then $G = P_6$, a contradiction since $\gamma_R(P_6) = 4$. Hence, $(N(x) \cup N(y)) \cap N(v) = \emptyset$. Since $\delta(G) = 1$, we have $|N(z) \cap N(v)| \ge 1$. Then $G = P_4 + K_2$ if $|N(z) \cap N(v)| = 1$, and $G = C_4 + K_2$ otherwise.

If $\Delta(G) = 3$, then $\delta(G) = 2$ and $n = 7$. Let $N(v) = \{v_1, v_2, v_3\}$. Lemma 8 implies that $k \le 2$. If $k = 1$, then we may assume that $xy \in E(G)$. If $u \in N(x) \cap N(y)$, then Lemma 8 implies that $N(z) = N(v) - \{u\}$. Then $\{z, u\}$ is dominating set for G, a contradiction. Thus $N(x) \cap N(y) = \emptyset$ and $|N(x)| = 2$ or $|N(y)| = 2$. If $|N(x)| = |N(y)| = 2$, then we may assume that $N(x) = \{v_1, y\}, N(y) = \{v_2, x\}$. If $\deg(z) = 3$, then $N(v_3) \cap N(v) = \emptyset$, since $\Delta(G) = 3$. Thus $G = C_7 + 15 + 26$. If $\deg(z) = 2$, then $v_3 \in N(z)$, since $\delta(G) = \Delta(G) - 1 = 2$. We may assume that $N(z) = \{v_2, v_3\}$. If $v_1v_3 \in E(G)$, then $\{v_1, v_2\}$ is a dominating set for G, a contradiction. On the other hand, since $\Delta(G) = 3$, we have $\{v_1, v_3\} \cap N(v_2) = \emptyset$. Thus $G = C_7 + 15$. Assume next that $|N(x)| = 2$ and $|N(y)| = 3$. Without loss of generality, we may assume that $N(x) = \{v_1, y\}, N(y) = \{v_2, v_3, x\}$. If $v_1 \in N(z)$, then $\{v_1, y\}$ is a dominating set for G, a contradiction. Thus $N(z) = \{v_2, v_3\}$. Since $\Delta(G) = 3$, the subgraph induced by $N(v)$ has no edges, and thus $G = C_7 + 15 + 26$. Now assume that the subgraph induced by $V(G) - N(v)$ has no edges. Lemma 8 implies that $\deg(x) = \deg(y) = \deg(z) = 2$. Without loss of generality, assume that $N(x) = \{v_1, v_2\}, N(y) = \{v_2, v_3\}$ and $N(z) = \{v_1, v_3\}$. Since $\Delta(G) = 3$, the subgraph induced by $N(v)$ has no edges, and thus $G = G_3$.

Now assume that $\Delta(G) = 4$. Then Lemma 8 implies that $k = 1$. We may assume, without loss of generality, that $xy \in E(G)$. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. If $\deg(z) = 4$, then by Lemma 8, $N(x) \cap N(y) = \emptyset$. We may assume that $N(x) = \{y, v_1, v_2\}$ and $N(y) = \{x, v_3, v_4\}$. If $v_1v_2 \in E(G)$, then $\{v_2, y\}$ is a dominating set for G, and so $\gamma_R(G) \leq 4$, a contradiction. Thus $v_1v_2 \notin E(G)$. Similarly, $v_3v_4 \notin E(G)$. If the subgraph induced by $N(v)$ has no edges, then $G = G_4$. Thus the subgraph induced by $N(v)$ has some edges. Assume next that $v_1v_3 \in E(G)$. Then $\{v_2, v_4\} \cap (N(v_1) \cup N(v_3)) = \emptyset$, since $\Delta(G) = 4$. Also if $v_2v_4 \in E(G)$, then $\{v_2, v_3\}$ is a dominating set for G, and so $\gamma_R(G) \leq 4$, a contradiction. Thus $G = G_5$. Similarly, if $v_2v_4 \in E(G)$, then $G = G_5$. Assume next that $\deg(z) = 3$, then Lemma 8 implies that $|N(x) \cap N(y)| \leq 1$. Assume that $|N(x) \cap N(y)| = 1$. If $\deg(x) = 4$, then, $|N(x) \cap N(y)| \geq 2$, a contradiction since $\deg(y) \geq \delta(G) = 3$. Thus $\deg(x) = 3$, and similarly $\deg(y) = 3$. We can assume that $N(x) = \{y, v_1, v_2\}$, $N(y) = \{x, v_1, v_3\}$ and $N(z) = \{v_2, v_3, v_4\}$. If $v_2 \in N(v_4)$, then $\{v_2, y\}$ is a dominating set for G, a contradiction. Thus $v_2v_4 \notin E(G)$ and similarly, $v_3v_4 \notin E(G)$. If $v_1 \in N(v_4)$, then $\{v_1, z\}$ is a dominating set for G, a contradiction. Thus $v_1 \notin N(v_4)$ and $\deg(v_4) = 2 < \delta(G)$, a contradiction. Thus $N(x) \cap N(y) = \emptyset$. Then $\deg(x) = \deg(y) = 3$. We can assume that $N(x) = \{y, v_1, v_2\}$, $N(y) = \{x, v_3, v_4\}$ and $N(z) = \{v_1, v_2, v_3\}$. Then $N(v_4) \cap N(v) \neq \emptyset$, since $\delta(G) = 3$. If $v_3v_4 \in E(G)$, then $\{x, v_3\}$ is a dominating set for G, a contradiction. Thus $v_3v_4 \notin E(G)$. Without loss of generality, we assume that $v_1v_4 \in E(G)$, since $\delta(G) = 3$. If $v_2v_4 \notin E(G)$ then $v_1v_2, v_1v_3 \notin E(G)$, since $\Delta(G) = 4$. If $v_2v_3 \in E(G)$, then $\{v_1, v_3\}$ is a dominating set for G, a contradiction. Thus $v_2v_3 \notin E(G)$. Consequently, $G = G_6$. It remains to assume that $v_2v_4 \in E(G)$. Then the assumption $\Delta(G) = 4$ implies that $G = G_5$. $\square$

Lemma 12. *If* $\Delta(G) = n-3$, *then* G *or* $\overline{G} \in \{K_2+\overline{K_2}, P_3+K_2, P_5, C_3+K_2, C_5, C_6+35, C_6+36, C_6+36+14, C_7+15, C_7+15+26, C_7+46+37+35\} \cup \{H_3, H_4, G_7\}$.

Proof. Assume that $\Delta(G) = n-3$. Then $\gamma_R(G) = 4$, $\gamma_R(\overline{G}) = n-2$, $\delta(G) = n-4$ and $\delta(\overline{G}) = 2$. Let $V(G) - N[v] = \{x, y\}$. Without loss of generality, we may assume $\deg(x) \geq \deg(y)$. If $\Delta(G) \geq 5$, then $n \geq 8$. If $\overline{G}$ is none of the graphs shown Fig. 1, then by Theorem 4, we obtain $n \leq 7$, a contradiction. Thus $\overline{G}$ is one of the graphs shown Fig. 1. Now the assumption on G and $\overline{G}$ leads to $\overline{G} \in \{H_3, H_4\}$. Now assume that $\Delta(G) \leq 4$. We proceed according to the value of $\Delta(G)$.

If $\Delta(G) = 1$, then $n = 4$, $\delta(G) = 0$ and so $xy \notin E(G)$. Hence, $G = K_2 + \overline{K_2}$.

If $\Delta(G) = 2$, then $\delta(G) = 1$. Let $N(v) = \{v_1, v_2\}$. If $xy \notin E(G)$, then $N(x) \cap N(v) \neq \emptyset$ and $N(y) \cap N(v) \neq \emptyset$. Since $\Delta(G) = 2$, $v_1v_2 \notin E(G)$ and $N(x) \cap N(y) = \emptyset$, we may assume, without loss of generality, that $N(x) \cap N(v) = \{v_1\}$ and $N(y) \cap N(v) = \{v_2\}$. Hence, $G = P_5$. Assume next that $xy \in E(G)$. Clearly $N(x) \cap N(y) = \emptyset$, since $\Delta(G) = 2$. If there exist two edges joining $N(v)$ and $V(G) - N[v]$, then $G = C_5$. If there exist one edges joining $N(v)$ and $V(G) - N[v]$, then $G = P_5$. Otherwise $G = P_3 + K_2$ or $G = C_3 + K_2$.

If $\Delta(G) = 3$, then $\gamma_R(\overline{G}) = 4$ and $\delta(G) = 2$. Let $N(v) = \{v_1, v_2, v_3\}$. First assume that $xy \in E(G)$. Clearly $2 \leq \deg(x) \leq \deg(y)) \leq 3$. Assume that $\deg(x) = 3$. Let $N[x] = \{x, y, v_1, v_2\}$. Assume that $\deg(y) = 3$. If $N[x] = N[y]$, then $N(v_3) \cap N(v) = \emptyset$, since $\Delta(G) = 3$. So $\deg(v_3) = 1$, a contradiction, since $\delta(G) = 2$. Hence, $N[x] \neq N[y]$. We may assume that $N[y] = \{y, x, v_2, v_3\}$. If $v_1v_3 \in E(G)$, then $\delta(G) \geq 3$, a contradiction. Hence, $v_1v_3 \notin E(G)$. On the other hand since $\Delta(G) = 3$, $N(v_2) \cap N(v) = \emptyset$. We conclude that $\overline{G} = C_6 + 35$. Assume next that $\deg(y) = 2$. If $N[y] \subset N[x]$, we may assume that $N[y] = \{y, x, v_2\}$. Then since $\delta(G) = 2$ and $\Delta(G) = 3$, we obtain that $v_1v_3 \in E(G)$ and $N(v_2) \cap N(v) = \emptyset$. Hence, $\overline{G} = C_6 + 36$. Now assume that $N[y] \not\subseteq N[x]$. We may assume that $N[y] = \{y, x, v_3\}$. Since $\Delta(G) = 3$, the subgraph induced by $N(v)$ has at most one edge. Hence, $\overline{G} \in \{G_7, \overline{G_7}, C_6 + 35\}$. Now assume that $\deg(x) = 2$. Then $\deg(y) = 2$. Let $N[x] = \{x, y, v_1\}$. If $N[x] = N[y]$, then $N(v_1) \cap N(v) = \emptyset$ and $v_2v_3 \in E(G)$, since $\delta(G) = 2$ and $\Delta(G) = 3$. Hence, $\overline{G} = C_6 + 36 + 14$. Next we assume that $N[x] \neq N[y]$. Without loss of generality, assume that $N[y] = \{y, x, v_2\}$. If $v_1v_2 \in E(G)$, then $N(v_3) \cap N(v) = \emptyset$, since $\Delta(G) = 3$, and so $\deg(v_3) = 1 < \delta(G) = 2$, a contradiction. Hence, $v_1v_2 \notin E(G)$. Moreover, $N(v_3) \cap \{v_1, v_2\} \neq \emptyset$, since $\delta(G) = 2$. If $v_1v_3 \notin E(G)$, then $G = C_6 + 35$. Otherwise $G = \overline{G_7}$.

Next, assume that $xy \notin E(G)$ and that $\deg(x) = 3$. If $\deg(y) = 3$, then $\delta(G) \geq 3$, a contradiction. Hence, $\deg(y) = 2$. Without loss of generality, $N(y) = \{v_2, v_3\}$. Since $\Delta(G) = 3$, the subgraph induced by $N(v)$ has no edge, and so $G = C_6 + 36 + 14$. Next assume next that $\deg(x) = 2$. Assume that $N(x) = \{v_1, v_2\}$. If $N(x) = N(y)$, then $N(v_3) \cap \{v_1, v_2\} = \emptyset$, since $\Delta(G) = 3$. Thus $\deg(v_3) = 1 < \delta(G)$, a contradiction. Hence, $N(x) \neq N(y)$. We may assume $N(y) = \{v_2, v_3\}$. Since $\Delta(G) = 3$, we have $N(v_2) \cap N(v) = \emptyset$. On the other hand if $v_1v_3 \in E(G)$, then $\Delta(\overline{G}) = 2$, a contradiction. Hence, $v_1v_3 \notin E(G)$. Therefore, $G = C_6 + 36$.

Now assume that $\Delta(G) = 4$. Then $\delta(G) = 3$ and $\gamma_R(\overline{G}) = 5$. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. We show that $xy \in E(G)$. Suppose that $xy \notin E(G)$. Assume that $\deg(x) = 4$. If $\deg(y) = 4$, then there exist vertex $u \in N(v)$ such that $\deg(u) = 3$, since $\Delta(\overline{G}) = 3$. Therefore, $\{u, v\}$ is a dominating set for $\overline{G}$, a contradiction. Thus $\deg(y) = 3$, since $\delta(G) = 3$. Without loss of generality, we may assume that $N(y) = \{v_1, v_2, v_3\}$. Since $\delta(G) = 3$, we have $N(v_4) \cap N(v) \neq \emptyset$. Without loss of generality, assume that $v_3 \in N(v_4)$. Since $\Delta(G) = 4$, we have $N(v_3) \cap \{v_1, v_2\} = \emptyset$. Then $\{y, v_3\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $xy \in E(G)$. We consider the following cases.

Case 1. $\deg(x) = 4$. Assume that $N[x] = \{x, y, v_1, v_2, v_3\}$. We consider the following subcases:

Subcase 1.1. $\deg(y) = 4$. We show that $N[x] \neq N[y]$. Suppose that $N[x] = N[y]$. Since $\delta(G) = 3$, we may assume that $|N(v_4) \cap \{v_1, v_2, v_3\}| \geq 2$. If $\deg(v_4) = 4$, then G is a 4-regular graph, a contradiction, since $\Delta(G) = \delta(G) + 1$. Hence, $\deg(v_4) = 3$ and we may assume without loss of generality that $N(v_4) = \{v, v_2, v_3\}$. Since $\Delta(G) = 4$, the subgraph induced by $\{v_1, v_2, v_3\}$ has no edges. Then $\{v_1, v\}$ is a dominating set for $\overline{G}$, a contradiction. Thus $N[x] \neq N[y]$. We thus may assume that $N[y] = \{x, v_2, v_3, v_4\}$. If $v_2v_3 \in E(G)$, then, $N(v_4) \cap \{v_2, v_3\} = \emptyset$, since $\Delta(G) = 4$. On other hand since $\delta(G) = 3$, we have $v_1v_4 \in E(G)$. Then $\{y, v_4\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $v_2v_3 \notin E(G)$. Assume next that $v_1v_4 \in E(G)$. If $\deg(v_1) = 3$, then $\{v_1, x\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $\deg(v_1) = 4$, and similarly $\deg(v_4) = 4$. Hence, G is a 4-regular graph, a contradiction. Hence, $v_1v_4 \notin E(G)$. Since $\delta(G) = 3$ and $\Delta(G) = 4$, we may assume $N(v_1) = \{v, v_2, x\}$ and $N(v_4) = \{v, v_3, y\}$. Hence, $G = C_7 + 15$.

Subcase 1.2. $\deg(y) = 3$. If $N[y] \subset N[x]$, then we may assume $N(y) = \{x, v_1, v_2\}$. If $v_1v_2 \in E(G)$, then $N(v_4) \cap \{v_1, v_2\} = \emptyset$, and so $\deg(v_4) \leq 2$, a contradiction. Hence, $v_1v_2 \notin E(G)$. If $\deg(v_4) = 4$, then $N(v_3) \cap \{v_1, v_2\} = \emptyset$, since $\Delta(G) = 4$. So $\{v_3, x\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $\deg(v_4) = 3$. If $\{v_1, v_2\} \subseteq N(v_4)$, then $\deg(v_3) = 2$, a contradiction. Hence, $|N(v_4) \cap \{v_1, v_2\}| = 1$, and we may assume that $N(v_4) = \{v, v_2, v_3\}$. If $v_1v_3 \in E(G)$, then $\{v_2, y\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $G = C_7 + 46 + 37 + 35$. If $N[y] \not\subseteq N[x]$. We may assume that $N(y) = \{x, v_3, v_4\}$. If $v_3v_4 \notin E(G)$, then $\{v_4, y\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $v_3v_4 \in E(G)$. Since $\Delta(G) = 4$, we have $N(v_3) \cap \{v_1, v_2\} = \emptyset$. If $v_1v_2 \notin E(G)$, then $\{v_1, x\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $v_1v_2 \in E(G)$. On the other hand $|N(v_4) \cap \{v_1, v_2\}| \leq 1$, since $\Delta(G) = 4$. If $v_4 \notin E(G)$, then $\overline{G} = C_7 + 15 + 26$. Otherwise $G = C_7 + 15$.

Case 2. $\deg(x) = 3$. Clearly $\deg(y) = 3$. Without loss of generality, assume that $N(x) = \{y, v_1, v_2\}$. We show that $N[x] = N[y]$. Suppose that $N[x] \neq N[y]$. If $N(x) \cap N(y) \neq \emptyset$, then we may assume that $N(y) = \{x, v_2, v_3\}$. Since $\Delta(G) = 4$, we have $|N(v_2) \cap N(v)| \leq 1$. Without loss of generality, we may assume that $v_1 \notin N(v_2)$. Then $\{v_1, x\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $N(x) \cap N(y) = \emptyset$. Then $\{x, y\}$ is a dominating set for $\overline{G}$, a contradiction. Hence, $N[x] = N[y]$. Then $N[y] = \{x, y, v_1, v_2\}$. Since $\Delta(G) = 4$ and $\delta(G) = 3$, we have $v_3v_4 \in E(G)$. If $\deg(v_3) = 4$, then $N(v_4) \cap \{v_1, v_2\} = \emptyset$, since $\Delta(G) = 4$. Then $\deg(v_4) = 2 < \delta(G) = 3$, a contradiction. Hence, $\deg(v_3) = 3$, and similarly $\deg(v_4) = 3$. Since $\Delta(G) = 4$, $N(v_4) \cap N(v_3) = \{v\}$ and $v_1v_2 \notin E(G)$, we may assume that $N(v_3) = \{v, v_4, v_1\}$ and $N(v_4) = \{v, v_3, v_2\}$. Hence, $\overline{G} = C_7 + 15 + 26$. $\square$

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