maximizing numerical radius of matrices with fixed spectral norm and spectral radius

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Abstract. Let $0 \leq p \leq 1$ and M_n be the algebra of n-by-n matrices with complex entries. In this paper we discuss about the following optimization problem

$$M(p) := \max\{r(A) : A \in M_n,\ \rho(A) = p, \|A\| = 1\},$$

Where $\|A\|$, $\rho(A)$ and $r(A)$, are spectral norm, spectral radius and numerical radius of A respectively.

A well known inequality of Haagerup and Harpe [2] states that $M(0) = \cos\left(\frac{\pi}{n+1}\right)$ and by Wintner theorem [3] $M(p) = 1$ if and only if $p = 1$. If we take the maximum over all upper triangular non-negative matrices, then by [1] the following inequality holds

$$M(p) \leq p + (1 - p^2)\cos\left(\frac{\pi}{n+1}\right).$$

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1. Introduction

For $1 \leq p \leq 1$, calculating

$$M(p) := \max\{r(A) : A \in M_n,\ \rho(A) = p, \|A\| = 1\},$$

is a challenging problem. The problem states that how much we can increase the numerical radius of matrices with fixed spectral radius and spectral norm? There are some partial answers to this question [**1**, **2**, **3**].

All of our simulations approves that the maximum in this optimization problem is attained at the matrix in the following form

$$A(n,p,q) = \begin{pmatrix} p & q & 0 & \cdots & \cdots & 0 \\ 0 & p & q & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & p & q \\ 0 & \cdots & \cdots & \cdots & 0 & p \end{pmatrix}_{n\times n}. \tag{1}$$

We will prove that if this guess is true, then

$$r(A) \leq p + [\sqrt{p^2cos^2(\frac{\pi}{n}) + 1 - p^2} - pcos(\frac{\pi}{n})]cos(\frac{\pi}{n+1}).$$

Note that for $n \geq 3$ we have

$$\sqrt{p^2cos^2(\frac{\pi}{n}) + 1 - p^2} - pcos(\frac{\pi}{n}) \leq 1 - p^2.$$

Therefore the last inequality improves the main inequality in [1], for $n \geq 3$.
In addition, we present a new proof of Wintner theorem.

2. Main results

Through this section we assume that the guess (1) is true.

Lemma 2.1. *Let $0 \leq k, x \leq 1$ and $\|A(n,k,x)\| = 1$. Then*

$$x \leq \sqrt{k^2 cos^2(\frac{\pi}{n}) + 1 - k^2} - k cos(\frac{\pi}{n}).$$

Proof. Clearly if $k = 0$ then $x = 1$ and we have desired equality. Suppose that $0 < k \leq 1$ and set $\tilde{A} = \frac{1}{k}A$. We have

$$\frac{1}{k^2} = \|\tilde{A}\|^2 = \|\tilde{A}\tilde{A}^t\| = \rho(\tilde{A}\tilde{A}^t).$$

But $\tilde{A}\tilde{A}^t$ is the following tridiagonal matrix

$$\tilde{A}\tilde{A}^t = \begin{pmatrix} 1+\frac{x^2}{k^2} & \frac{x}{k} & 0 & \cdots & \cdots & 0 \\ \frac{x}{k} & 1+\frac{x^2}{k^2} & \frac{x}{k} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \frac{x}{k} & 1+\frac{x^2}{k^2} & \frac{x}{k} \\ 0 & \cdots & \cdots & \cdots & \frac{x}{k} & 1 \end{pmatrix}.$$

Now consider the following matrix

$$H = \begin{pmatrix} 1+\frac{x^2}{k^2} & \frac{x}{k} & 0 & \cdots & \cdots & 0 \\ \frac{x}{k} & 1+\frac{x^2}{k^2} & \frac{x}{k} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \frac{x}{k} & 1+\frac{x^2}{k^2} & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

Since $0 \leq H \leq \tilde{A}\tilde{A}^t$, we have $\rho(H) \leq \rho(\tilde{A}\tilde{A}^t)$. Setting $a = c = \frac{x}{k}$, $b = 1 + \frac{x^2}{k^2}$, by [4] we have

$$\rho(H) = 1 + \frac{x^2}{k^2} + \frac{2x}{k} cos(\frac{\pi}{n}) \leq \frac{1}{k^2}.$$

Therefore

$$x \leq \sqrt{k^2 cos^2(\frac{\pi}{n}) + 1 - k^2} - k cos(\frac{\pi}{n}).$$

□

Theorem 2.2. *For every $A \in M_n$, the following inequality holds*

$$r(A) \leq \rho(A) + [\sqrt{\rho(A)^2 cos^2(\frac{\pi}{n}) + \|A\|^2 - \rho(A)^2} - \rho(A) cos(\frac{\pi}{n})] cos(\frac{\pi}{n+1}).$$

Proof. By Lemma 2.1, we have

$$\begin{aligned} r(A) &\leq \rho(A) + q cos(\frac{\pi}{n+1}) \\ &\leq \rho(A) + [\sqrt{\rho(A)^2 cos^2(\frac{\pi}{n}) + \|A\|^2 - \rho(A)^2} - \rho(A) cos(\frac{\pi}{n})] cos(\frac{\pi}{n+1}). \end{aligned}$$

□

Corollary 2.3. *(Wintner Theorem) If $B \in M_n$ and $r(B) = \|B\|$, then $\rho(B) = \|B\|$.*

Proof. Let $n \geq 2$, $\rho(B) = p$ and without loss of generality suppose that $\|B\| = 1$. By Haagerup-Harpe inequality, we have

$$1 = r(B) \leq p + qcos(\frac{\pi}{n+1}) = r(A(n,p,q)) \leq \|A(n,p,q)\| = 1.$$

Hence $1 \geq p > 1 - cos(\frac{\pi}{n+1})$, $q = \frac{1-p}{cos(\frac{\pi}{n+1})}$ and $\|A(n,p,\frac{1-p}{cos(\frac{\pi}{n+1})})\| = 1$. Now consider the following unit vector

$$\begin{aligned} x &= \frac{(sin\frac{\pi}{n+1}, sin\frac{2\pi}{n+1}, \ldots, sin\frac{n\pi}{n+1})}{\sqrt{\sum_{k=1}^{n} sin^2\frac{k\pi}{n+1}}} \\ &= \frac{(sin\frac{\pi}{n+1}, sin\frac{2\pi}{n+1}, \ldots, sin\frac{n\pi}{n+1})}{\sqrt{\frac{n+1}{2}}}. \end{aligned}$$

Setting $A = A(n,p,q)$, if $p < 1$ then we will have

$$\begin{aligned} 1 \geq\ & \|Ax\|^2 = \frac{2}{n+1}\|(psin\frac{\pi}{n+1} + \frac{(1-p)sin\frac{2\pi}{n+1}}{cos\frac{\pi}{n+1}}, \ldots, psin\frac{(n-1)\pi}{n+1} + \frac{(1-p)sin\frac{n\pi}{n+1}}{cos\frac{2\pi}{n+1}}, psin\frac{n\pi}{n+1})\|^2 \\ =\ & \frac{2}{n+1}[p^2\sum_{k=1}^{n} sin^2\frac{k\pi}{n+1} + \frac{(1-p)^2}{cos^2\frac{\pi}{n+1}}\sum_{k=2}^{n} sin^2\frac{k\pi}{n+1} + \frac{2p(1-p)}{cos\frac{\pi}{n+1}}\sum_{k=1}^{n-1} sin\frac{k\pi}{n+1}sin\frac{(k+1)\pi}{n+1}] \\ =\ & \frac{2}{n+1}[\frac{p^2(n+1)}{2} + \frac{(1-p)^2}{cos^2\frac{\pi}{n+1}}(\frac{n-1}{2} + cos^2\frac{\pi}{n+1}) + \frac{p(1-p)}{cos\frac{\pi}{n+1}}\sum_{k=1}^{n-1}(cos\frac{\pi}{n+1} - cos\frac{(2k+1)\pi}{n+1}] \\ >\ & \frac{2}{n+1}[\frac{p^2(n+1)}{2} + \frac{(1-p)^2(n-1)}{2} + (1-p)^2 + p(1-p)(n-1) \\ -\ & \frac{2p(1-p)}{sin\frac{2\pi}{n+1}}\sum_{k=1}^{n-1}(sin\frac{\pi}{n+1}cos\frac{(2k+1)\pi}{n+1}] \\ =\ & \frac{2}{n+1}[\frac{n+1}{2} - p(1-p) - p(1-p)\underbrace{\frac{sin\frac{2n\pi}{n+1}}{sin\frac{2\pi}{n+1}}}_{-1}] = 1, \end{aligned}$$

Which is a contradiction. Hence $p = 1$ and the proof is complete. □

References

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