

## A numerical method for partial fractional Fredholm integro-differential equations

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**Abstract** In this paper, an expansion method based on orthonormal polynomials is presented to find the numerical solution of partial fractional Fredholm integro-differential equations (PFFIDEs). A PFFIDE is converted to a system of linear algebraic equations, which is solved for the expansion coefficients of approximate solution based on orthonormal polynomials. An estimation error is discussed and some illustrative examples are given to demonstrate the validity and applicability of the proposed method.

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**Keywords.** Partial integro-differential equations, Fractional operators, Expansion method.

**2010 Mathematics Subject Classification.** 65R20.

### 1. INTRODUCTION

Fractional ordinary and partial differential and integral equations have been applied for solving many practical problems which are modeled in various fields of science and engineering, such as, physics, electrochemistry, viscoelasticity, continuum and statistical mechanics [2, 16, 7, 10]. Since there are not simple analytical methods for solving these equations, many researchers proposed approximate methods for solving these kinds of equations, such as partial differential and integro-differential equations. For example, the Adomian decomposition, variational iteration, homotopy perturbation, fractional differential transform (FDTM), wavelet, block pulse operational matrix and Tau methods [22, 19, 15, 21, 26, 12, 27, 23, 8, 20, 24, 9, 11, 18, 25, 4, 5] are some instances of given methods.

More recently, a combination of the spectral method with Bernstein operational matrix in [13] and a combination of the Ritz method with fractional operational matrix method in [14] have been proposed to solve two-dimensional fractional optimal control problems.

But partial fractional integro-differential equations have not solved by any numerical

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Received: 23 September 2017 ; Accepted: 16 January 2018.

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methods. So in this paper, we propose a numerical method to solve these kinds of equations. To this end, we consider a PFFIDE of the form

$$D_t^\alpha u(x, t) - \lambda \int_0^1 \int_0^1 k(x, t, y, z) u(y, z) dy dz = f(x, t),$$

$$(x, t) \in I = [0, 1] \times [0, 1], \tag{1.1}$$

with initial conditions

$$\frac{\partial^k}{\partial t^k} u(x, 0) = h_k(x), \quad k = 0, 1, \dots, m - 1, \quad m - 1 < \alpha \leq m, \quad m \in \mathbf{N}, \tag{1.2}$$

where  $D_t^\alpha$  denotes the Caputo fractional operator of order  $\alpha$  with respect to  $t$ ,  $f(x, t)$  and  $k(x, t, y, z)$  are given continuous functions and  $u(x, t)$  is a solution to be determined. In this paper, we apply an expansion method based on some orthonormal polynomials for problem (1.1)- (1.2). Here, we replace the differential and integral parts of a PFFIDE by their operational matrix representations and then convert it to a corresponding system of linear algebraic equations. In a similar manner, we transform the initial conditions to a linear algebraic system of equations. Finally, by combining these two linear systems of algebraic equations, we obtain a system of linear algebraic equations and solve it to obtain an approximate solution of the problem.

## 2. BASIC DEFINITION OF FRACTIONAL CALCULUS

In this section, we briefly present some definitions and results in fractional calculus from [1, 6] which are used in this paper.

**Definition 2.1.** The Riemann-Liouville fractional integral operator  $J^\alpha$  of order  $\alpha$  is given by

$$J^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(\tau) d\tau, \quad \alpha > 0,$$

$$J^0 u(t) = u(t).$$

Some important properties of the fractional integral operator are as follows

$$1) J^\alpha J^\beta u(t) = J^{\alpha+\beta} u(t), \tag{2.1}$$

$$2) J^\alpha J^\beta u(t) = J^\beta J^\alpha u(t), \tag{2.2}$$

$$3) J^\alpha t^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(\alpha + \nu + 1)} t^{\alpha+\nu} \tag{2.3}$$

where  $\alpha, \beta > 0$  and  $\nu > -1$  are real numbers.

**Definition 2.2.** The partial Rimann-Liouville fractional integral operator with respect to  $t$  of order  $\alpha > 0$  is defined as

$$J_t^\alpha f(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(x, s) ds \tag{2.4}$$



**Definition 2.3.** The Caputo partial fractional derivative with respect to  $t$  of order  $\alpha > 0$  is defined by

$$\begin{aligned} D_t^\alpha f(x, t) &= J_t^{m-\alpha} D_t^m f(x, t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} \frac{\partial^m f(x, s)}{\partial s^m} ds, \quad m-1 < \alpha \leq m, \end{aligned} \quad (2.5)$$

where  $m$  is an integer.

### 3. MAIN RESULTS

In this section, we give some theorems and lemmas to convert the differential and integral parts of PFFIDE to their matrix representations. Let  $\{\phi_j(t)\}$  be a sequence of orthonormal polynomials. Then a function  $u(t) \in L^2[0, 1]$  can be expanded as

$$u(t) = \sum_{j=0}^{\infty} c_j \phi_j(t) = C\Phi X_t,$$

where  $C = (c_0, c_1, \dots, c_n, \dots)$  with

$$c_j = \int_0^1 u(t) \phi_j(t) dt$$

and  $X_t = (1, t, t^2, \dots, t^n, \dots)^T$  and  $\Phi$  is a lower triangular matrix such that  $\Phi X_t = (\phi_0(t), \phi_1(t), \dots, \phi_n(t), \dots)^T$ .

Now we recall the following lemma from [17].

**Lemma 3.1.** Let  $u(t) = CX_t$  be a polynomial. Then we have

$$D^r u(t) = \frac{d^r}{dt^r} u(t) = C\eta^r X_t, \quad r = 0, 1, 2, \dots, \quad (3.1)$$

where  $C = (c_0, c_1, \dots, c_n, 0, 0, \dots)$ ,  $X_t = (1, t, t^2, \dots, t^n, \dots)^T$  and

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

**Theorem 3.2.** For  $X_t = (1, t, t^2, \dots, t^n, \dots)^T$ , we have

$$J_t^{m-\alpha}(X_t) = GA\Phi X_t, \quad (3.2)$$

where  $G$  is a diagonal matrix with diagonal elements

$$G_{i,i} = \frac{\Gamma(i+1)}{\Gamma(m-\alpha+i+1)}, \quad i = 0, 1, 2, \dots, \quad (3.3)$$

and  $A = (a_{ij})_{i,j=0}^{\infty}$  with

$$a_{ij} = \int_0^1 t^{m-\alpha+i} \phi_j(t) dt, \quad i, j = 0, 1, 2, \dots \quad (3.4)$$



*Proof.* By (2.3), we have

$$\begin{aligned} J_t^{m-\alpha}(X_t) &= (J_t^{m-\alpha}(1), J_t^{m-\alpha}(t), \dots, J_t^{m-\alpha}(t^n), \dots)^T \\ &= \left( \frac{\Gamma(1)t^{m-\alpha}}{\Gamma(m-\alpha+1)}, \frac{\Gamma(2)t^{m-\alpha+1}}{\Gamma(m-\alpha+2)}, \dots, \frac{\Gamma(n+1)t^{m-\alpha+n}}{\Gamma(m-\alpha+n+1)}, \dots \right) \\ &= G\Pi, \end{aligned} \tag{3.5}$$

where  $G$  is a diagonal matrix with elements (3.3) and

$$\Pi = (t^{m-\alpha}, t^{m-\alpha+1}, \dots, t^{m-\alpha+n}, \dots)^T.$$

But  $t^{m-\alpha+i}$  can be written as:

$$t^{m-\alpha+i} = \sum_{j=0}^{\infty} a_{ij} \phi_j(t) = \mathbf{a}_i \Phi X_t, \quad \mathbf{a}_i = (a_{i0}, a_{i1}, \dots, a_{in}, \dots),$$

where  $a_{ij}$  obtained by (3.4). Therefore, we have

$$\Pi = (\mathbf{a}_0 \Phi X_t, \mathbf{a}_1 \Phi X_t, \dots, \mathbf{a}_n \Phi X_t, \dots)^T = A \Phi X_t, \tag{3.6}$$

where

$$A = (\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n, \dots)^T.$$

Hence, by substituting (3.6) into (3.5), (3.2) is obtained. □

Consider the approximate solution of the Eq.(1.1) in the form

$$u_n(x, t) = \sum_{i=0}^n \sum_{j=0}^n u_{ij} \phi_i(x) \phi_j(t) = X_x^T \Phi^T U \Phi X_t, \tag{3.7}$$

where  $X_x^T = (1, x, x^2, \dots, x^n, \dots)$  and

$$U = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0n} & 0 & \cdots \\ u_{10} & u_{11} & \cdots & u_{1n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

with the unknown elements  $u_{ij}, i, j = 0, \dots, n$ . By using (2.5), (3.1), (3.2) and (3.7), we obtain the matrix form

$$\begin{aligned} D_t^\alpha u_n(x, t) &= J_t^{m-\alpha} D_t^m (X_x^T \Phi^T U \Phi X_t) = J_t^{m-\alpha} (X_x^T \Phi^T U \Phi \eta^m X_t) \\ &= X_x^T \Phi^T U \Phi \eta^m J_t^{m-\alpha} (X_t) = X_x^T \Phi^T U \Phi \eta^m G A \Phi X_t, \end{aligned} \tag{3.8}$$

for differential operator  $D_t^\alpha$ . On the other hand, the function  $f(x, t)$  can be expanded as

$$f(x, t) = \sum_{i=0}^n \sum_{j=0}^n f_{ij} \phi_i(x) \phi_j(t) = X_x^T \Phi^T F \Phi X_t, \tag{3.9}$$



where  $F = (f_{ij})_{i,j=0}^n$  with

$$f_{ij} = \int_0^1 \int_0^1 f(x, t) \phi_i(x) \phi_j(t) dx dt. \quad (3.10)$$

**Theorem 3.3.** *If  $k(x, t, y, z) = \sum_{s=0}^n \sum_{r=0}^n \sum_{j=0}^n \sum_{i=0}^n k_{ijrs} \phi_i(x) \phi_j(t) \phi_r(y) \phi_s(z)$ , then*

$$\int_0^1 \int_0^1 k(x, t, y, z) u_n(y, z) dy dz = X_x^T \Phi^T \Lambda \Phi X_t \quad (3.11)$$

where  $\Lambda$  is an  $(N+1) \times (N+1)$  matrix with the elements

$$\Lambda_{ij} = \sum_{s=0}^n \sum_{r=0}^n k_{ijrs} u_{rs}, \quad i, j = 0, 1, \dots, n.$$

*Proof.* Since  $u_n(x, t) = \sum_{i=0}^n \sum_{j=0}^n u_{ij} \phi_i(x) \phi_j(t)$ , then we have

$$\begin{aligned} & \int_0^1 \int_0^1 k(x, t, y, z) u_n(y, z) dy dz \\ &= \int_0^1 \int_0^1 \left( \sum_{s=0}^n \sum_{r=0}^n \sum_{j=0}^n \sum_{i=0}^n k_{ijrs} \phi_i(x) \phi_j(t) \phi_r(y) \phi_s(z) \right) \\ & \times \left( \sum_{l=0}^n \sum_{m=0}^n u_{lm} \phi_l(y) \phi_m(z) \right) dy dz \\ &= \sum_{s=0}^n \sum_{r=0}^n \sum_{j=0}^n \sum_{i=0}^n \sum_{l=0}^n \sum_{m=0}^n k_{ijrs} u_{lm} \phi_i(x) \phi_j(t) \int_0^1 \phi_r(y) \phi_l(y) dy \int_0^1 \phi_s(z) \phi_m(z) dz \\ &= \sum_{s=0}^n \sum_{r=0}^n \sum_{j=0}^n \sum_{i=0}^n \sum_{l=0}^n \sum_{m=0}^n k_{ijrs} u_{lm} \phi_i(x) \phi_j(t) \delta_{rl} \delta_{sm} \\ &= \sum_{s=0}^n \sum_{r=0}^n \sum_{j=0}^n \sum_{i=0}^n k_{ijrs} u_{rs} \phi_i(x) \phi_j(t) = X_x^T \Phi^T \Lambda \Phi X_t. \end{aligned}$$

□

By substituting from (3.8), (3.9) and (3.11) into Eq.(1.1) and noting that  $X_x^T \Phi^T$  and  $\Phi X_t$  are basis vectors, we obtain the operational matrix representation

$$U \Phi \eta^m G A - \Lambda = F, \quad (3.12)$$

which is a system of linear algebraic equations.

Similarly, we convert the conditions (1.2) to a system of linear algebraic equations. To this end, let us assume that

$$h_k(x) = \sum_{j=0}^n \bar{h}_{kj} \phi_j(x) = X_x^T \Phi^T \mathbf{h}_k, \quad k = 0, 1, \dots, m-1, \quad (3.13)$$



where  $\mathbf{h}_k = (\bar{h}_{k0}, \bar{h}_{k1}, \dots, \bar{h}_{kn}, 0, 0, \dots)^T$ . Then by using (3.8), we have

$$\frac{\partial^k}{\partial t^k} u(x, t)|_{t=0} = X_x^T \Phi^T \eta^k U \Phi X_t|_{t=0}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (1.2) yields

$$\eta^k U \Phi X_t|_{t=0} = \mathbf{h}_k, \quad k = 0, 1, \dots, m - 1. \tag{3.15}$$

By solving the system of equations (3.12) and (3.15) simultaneously and determining the unknown coefficients  $u_{ij}$ , the approximate solution  $u_n(x, t)$  can be determined from (3.7).

**Remark 3.4.** By applying the integral operator (2.4) to both sides of (1.1), using the conditions (1.2) and some classical results from fractional calculus, we get:

$$\begin{aligned} u(x, t) &= \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x, s) ds \\ &+ \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^1 \int_0^1 k(x, s, y, z) u(y, z) dy dz ds, \end{aligned} \tag{3.16}$$

and by changing the order of integration in (3.16), we can derive

$$u(x, t) = g(x, t) + \lambda \int_0^1 \int_0^1 K(x, t, y, z) u(y, z) dy dz \tag{3.17}$$

where

$$g(x, t) = \sum_{k=0}^{m-1} h_k(x) \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x, s) ds$$

and

$$K(x, t, y, z) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} k(x, s, y, z) ds.$$

Therefore, problem (1.1)-(1.2) is equivalent to the Fredholm integral equation (3.17). Since  $f$  and  $k$  are continuous functions on their corresponding domains, it follows that  $g(x, t)$  and  $K(x, t, y, z)$  are continuous. The convergence of the proposed method with Legendre polynomial bases for (3.17) was proved in [3], so the proposed method is converge.

#### 4. ERROR ESTIMATION

In this section, we present an estimation error for the approximate solution. To do this, we define the error function as

$$e_n(x, t) := u(x, t) - u_n(x, t), \tag{4.1}$$

where  $u(x, t)$  and  $u_n(x, t)$  are the exact and the approximate solutions, respectively. By replacing  $u(x, t)$  by  $u_n(x, t)$  in (1.1)-(1.2), a perturbed problem is obtained as

$$D_t^\alpha u_n(x, t) - \int_0^1 \int_0^1 k(x, t, y, z) u_n(y, z) dy dz = f(x, t) + p(x, t), \tag{4.2}$$



with the initial conditions

$$\frac{\partial^k}{\partial t^k} u_n(x, 0) = h_k(x), \quad k = 0, 1, \dots, m-1, \quad (4.3)$$

where the perturbation term  $p$  is computed from

$$p(x, t) = D_t^\alpha u_n(x, t) - \int_0^1 \int_0^1 k(x, t, y, z) u_n(y, z) dy dz - f(x, t), \quad (4.4)$$

By subtracting (4.2) from (1.1) and (4.3) from (1.2) and using (4.1), we obtain the following initial value problem for the error function  $e_n(x, t)$

$$D_t^\alpha e_n(x, t) - \int_0^1 \int_0^1 k(x, t, y, z) e_n(y, z) dy dz = -p(x, t), \quad (4.5)$$

$$\frac{\partial^k}{\partial t^k} e_n(x, 0) = 0, \quad i = 0, 1, \dots, m-1. \quad (4.6)$$

Solving problem (4.5)-(4.6) by the same way as we did before for the solution of (1.1)-(1.2), an approximation  $e_{nl}(x, t)$  of degree  $l$  is obtained for the exact error function  $e_n(x, t)$ . We should notice that, only the right-hand sides of equations (4.5)-(4.6) differ from (1.1) and (1.2).

## 5. NUMERICAL EXAMPLES

In this section, we use normalized Legendre polynomials as basis functions and solve some examples to illustrate efficiency and applicability of the proposed method.

**Example 5.1.** Consider the equation

$$D_t^{0.5} u(x, t) - \int_0^1 \int_0^1 (y^2 + tz) u(y, z) dy dz = \frac{2x\sqrt{t}}{\Gamma(0.5)} - \frac{1}{3}t - \frac{13}{40}, \quad (x, t) \in I,$$

with the initial condition  $u(x, 0) = x^2$  and exact solution  $u(x, t) = x^2 + xt$ . By applying the presented method, we obtain  $u_2(x, t) = x^2 + xt$  which is the exact solution.

**Example 5.2.** As the second example, consider the equation

$$\begin{aligned} D_t^{0.75} u(x, t) - \int_0^1 \int_0^1 (xz + t \sin(y)) u(y, z) dy dz \\ = \frac{4 t^{1/4} \cos(x)}{\Gamma(0.25)} - \frac{1}{3} \sin(1) x - \frac{1}{4} \sin(1)^2 t, \quad (x, t) \in I, \end{aligned}$$

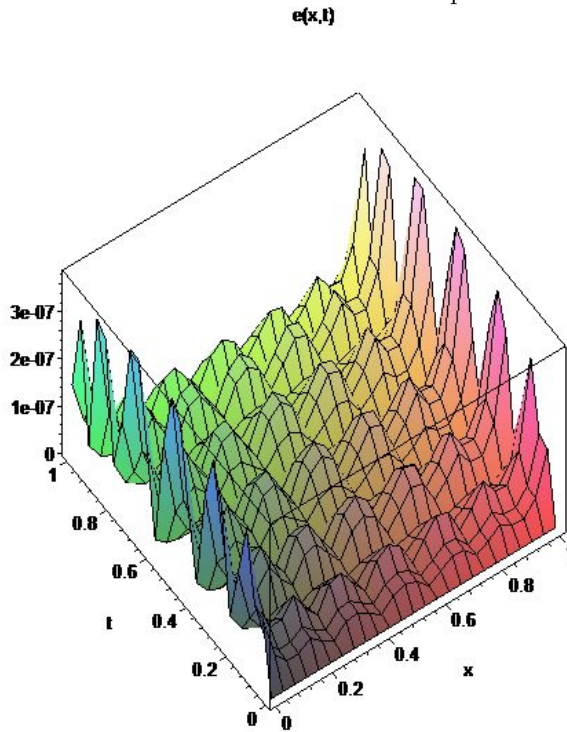
with the initial condition  $u(x, 0) = 0$ . The exact solution of this problem is  $u(x, t) = t \cos(x)$ . Table 1 shows the absolute errors ( $e(x, t)$ ) and the corresponding estimations ( $e_n(x, t)$ ) at some selected points for  $n = 8$ . The absolute errors plotted in in Figure 1.



TABLE 1. Numerical results of Example (5.2).

$(x, t)$	$e(x, t)$	$e_n(x, t)$
(0,0.6)	0.6651e -7	0.1734e -7
(0.1,0.1)	0.4075e -8	0.4508e -8
(0.2,0.7)	0.3101e -7	0.1054e -7
(0.3,0.8)	0.3899e -7	0.1717e -7
(0.4,0.4)	0.3746e -8	0.3568e -8
(0.6,0.9)	0.8366e -8	0.3533e -8
(0.8,0.5)	0.8094e -7	0.1248e -7
(0.9,0.1)	0.4074e -8	0.4508e -8

FIGURE 1. The absolute errors of Example 5.2 for  $n = 8$ .



**Example 5.3.** As the final example, we consider the equation

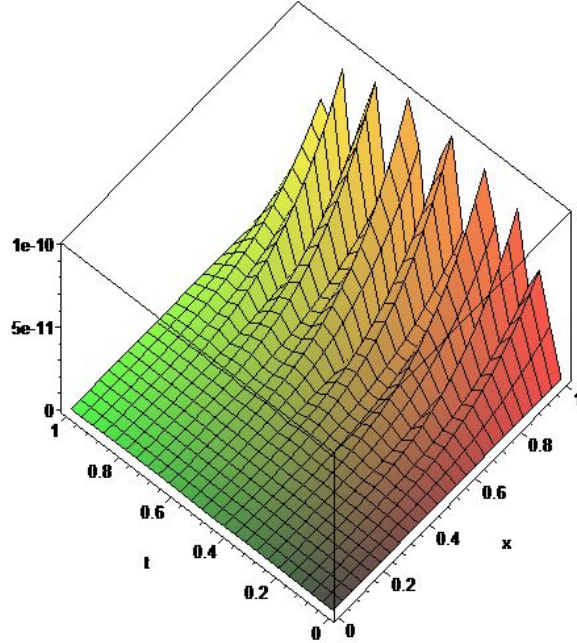
$$D_t^\alpha u(x, t) - \int_0^1 \int_0^1 u(y, z) dy dz = 2x^4 e^{2t} + \frac{1}{10} - \frac{1}{10} e^2, \quad (x, t) \in I,$$

$$0 < \alpha \leq 1,$$





FIGURE 2. The absolute error for  $n = 10$  and  $\alpha = 1$  of Example 3.



subject to the initial condition  $u(x, 0) = x^4$ . For  $\alpha = 1$ , the exact solution is  $u(x, t) = x^4 e^{2t}$ . For  $n = 10$  and different values of  $\alpha$  the approximate values of the solution are reported in Table 2. As shown in this table, the results of our method for  $\alpha = 1$  (the only case that we know the exact solution) is in the best agreement with the exact values. For  $\alpha = 1$  the plot of the absolute error is shown in Figure 2.

## 6. CONCLUSION

In this paper, we presented an expansion method, based on orthonormal polynomials to solve PFFIDEs. We applied the presented method on three test problems and showed that the numerical results have high accuracy comparing with exact values. We concluded that the presented method is efficient and it can be developed for similar partial fractional Volterra integro-differential equations.

## ACKNOWLEDGEMENTS

The authors would like to thank the editors and anonymous referees for their valuable comments and suggestions.



TABLE 2. The numerical results of Example (5.3) for various  $\alpha$ .

$(x, t)$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution for $\alpha = 1$
(0.1,0.1)	0.111603	0.048368	0.015931	0.000122	0.000122
(0.2,0.2)	0.187484	0.087515	0.032154	0.002387	0.002387
(0.3,0.3)	0.263031	0.134484	0.058627	0.014759	0.014759
(0.4,0.4)	0.440334	0.214756	0.117241	0.056974	0.056974
(0.5,0.5)	0.571382	0.378956	0.252958	0.169893	0.169893
(0.6,0.6)	0.955369	0.716930	0.549075	0.430287	0.430287
(0.7,0.7)	1.697018	1.386118	1.152511	0.973653	0.973653
(0.8,0.8)	3.095794	2.658581	2.311338	2.028762	2.028762
(0.9,0.9)	5.599890	4.963044	4.427511	3.969174	3.969174
(1,1)	9.967113	8.993351	8.142245	7.389056	7.389056

## REFERENCES

- [1] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, 2004.
- [2] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore 2000.
- [3] S. A. Hosseini, S. Shahmorad, F. Talati, *A matrix based method for two dimensional nonlinear volterra-fredholm integral equations*, Numer. Algor., 68(3) (2015), 511–529.
- [4] S. Karimi Vanani and A. Aminataei, *Operational Tau approximation for a general class of fractional integro-differential equations*, Comp. Appl. Math., 30(3) (2011), 655–674.
- [5] S. Karimi Vanani and A. Aminataei, *Tau approximate solution of fractional partial differential equations*, Computers and Mathematics with Applications, 62 (2011), 1075–1083.
- [6] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam, Netherlands: Elsevier, 2006.
- [7] R. C. Koeller, *Application of fractional calculus to the theory of viscoelasticity*, J. Appl. Mech., 51 (1984), 299–307.
- [8] Y. Li, *Solving a nonlinear fractional differential equation using Chebyshev wavelets*, Commun. Nonlinear Sci. Numer. Simulat., 15(9) (2010), 22842292.
- [9] Y. Li and N. Sun, *Numerical solution of fractional differential equations using the generalized block pulse operational matrix*, Comput. Math. Appl., 62 (2011), 1046–1054.
- [10] F. Mainardi, *Fractional calculus: some basic problems in continuum and statistical mechanics*, Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien, 1997, 291–348.
- [11] P. Mokhtary, F. Ghoreishi, and H.M. Srivastava, *The Muntz-Legendre Tau method for fractional differential equations*, Appl. Math. Modelling, 40 (2015), 671–684.
- [12] D. Nazari and S. Shahmorad, *Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions*, J. Comput. Appl. Math., 234 (2010), 883–891.
- [13] A. Nemati, *Numerical solution of 2D fractional optimal control problems by the spectral method combined with Bernstein operational matrix*, International Journal of Control, doi:10.1080/00207179.2017.1334267.
- [14] A. Nemati and S.A. Yousefi, *A numerical scheme for solving two-dimensional fractional optimal control problems by the Ritz method combined with fractional operational matrix*, IMA Journal of Mathematical Control and Information, doi: 10.1093/imamci/dnw009.
- [15] Z. Odibat and S. Momani, *Application of variational iteration method to nonlinear differential equations of fractional order*, Int. J. Nonlinear Sci. Numer. Simul., 7 (2006), 271–279.



- [16] K. Oldham, *Fractional differential equations in electrochemistry*, Adv. Eng. Softw., 41 (2010) , 9–17
- [17] E. L. Ortiz and L. Samara , *An operational approach to the Tau method for the numerical solution of nonlinear differential equations*, Computing, 27 (1981), 15–25.
- [18] J.A. Rad, S. Kazem, M. Shaban, K. Parand, and A. Yildirim, *Numerical solution of fractional differential equations with a Tau method based on Legendre and Bernstein polynomials*, Math. Meth. Appl. Sci., 37(3) (2014), 329–342.
- [19] S. S. Ray, *Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method*, Commun. Nonlinear Sci. Numer. Simulat., 14 (2009) , 129–306.
- [20] M. Tahami, A. Askari Hemmat, S.A. Yousefi, *Numerical solution of two-dimensional first kind Fredholm integral equations by using linear Legendre wavelet*, International Journal of Wavelets, Multiresolution and Information Processing, 14(1) (2016).
- [21] V. Turut and N. Güzel, *On solving partial differential equations of fractional order by using the variational iteration method and multivariate pade Approximations*, Eur. J. Pure App. Math., 6(2), (2013), 147–171.
- [22] Q. Wang, *Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method*, Appl. Math. Comput., 182 (2006), 1048–1055.
- [23] J. L. Wu, *A wavelet operational method for solving fractional partial differential equations numerically*, Appl. Math. Comput., 214(1) (2009), 31–40.
- [24] M. Yi , J. Huang, and J. Wei, *Block pulse operational matrix method for solving fractional partial differential equation*, Appl.Math.Comput., 221 (2013), 121–131.
- [25] A. Yousefi , T. Mahdavi-Rad, and S.G. Shafiei, *A quadrature Tau method for solving fractional integro-differential equations in the Caputo sense*, J. Math. Computer Sci., 15 (2015), 97–107.
- [26] X. Zhang, J. Liu, J. Zhao, and B. Tang , *Homotopy perturbation method for two dimensional time-fractional wave equation*, Appl. Math. Model., 38 (2014), 5545–5552.
- [27] L. Zhu and Q. Fan, *Numerical solution of nonlinear fractional-order Volterra integro-differential equations by SCW*, Commun. Nonlinear Sci. Numer. Simulat., 18 (2013), 1203–1213.

