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Heat transfer from convecting-radiating fin through optimized Chebyshev polynomials with interior point algorithm

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Abstract: In this paper, the problem of determining heat transfer from convecting-radiating fin of triangular and concave parabolic shapes is investigated. We consider one-dimensional, steady conduction in the fin and neglect radiative exchange between adjacent fins and between the fin and its primary surface. A novel intelligent computational approach is developed for searching the solution. In order to achieve this aim, the governing equation is transformed into an equivalent problem whose boundary conditions are such that they are convenient to apply reformed version of Chebyshev polynomials of the first kind. These Chebyshev polynomials based functions construct approximate series solution with unknown weights. The mathematical formulation of optimization problem consists of an unsupervised error which is minimized by tuning weights via interior point method. The trial approximate solution is validated by imposing tolerance constrained into optimization problem. Additionally, heat transfer rate and the fin efficiency are reported.

Keywords: Chebyshev polynomial of the first kind; Interior point method; Temperature distribution; Fin efficiency; Heat transfer rate

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1 Preliminaries and problem formulation

The heat dissipation mechanism considered in literature is either pure convection or pure radiation. In applications where fins operate in a free or natural convection environment, the contribution of radiation is equally significant, and therefore the design must allow for occurring both convection and radiation. As an application, it can be mentioned to stamped heat sink or extruded heat sink designed for cooling a transistor. Even if forced convection is employed for cooling, radiation is significant if the operating temperatures are high as is the case with a finned regenerator [1, 2].

Enhancement of heat transfer employing fins is important in a multitude of heat exchange equipment [3–16]. It is clear from the literature review that the research has been greatly focused on the theoretical and experimental thermal analysis of both solid fins and porous fins with different profiles and thermophysical properties due to wide range of applications [17–34], also see the refs [35–38, 38–44] to receive more information.

Furthermore, to see some very recent investigations on convective-radiative fin heat transfer, the interest readers are referred to [45–47]. Mosayebidorcheh et. al. have obtained an optimum design point for fin geometry so that heat transfer rate reaches to a maximum value in a constant fin volume [45]. In [46], authors have applied spectral collocation method for transient thermal analysis of coupled conductive, convective and radiative heat transfer in the moving plate with temperature dependent properties and heat generation. A spectral element method (SEM) has been developed in [47] in order to solve coupled conductive, convective and radiative heat transfer in moving porous fins of trapezoidal, convex parabolic and concave parabolic profiles.

We consider a longitudinal fin of arbitrary profile attached to a primary surface at temperature T_b. Let the fin length be L and its thicknesses at the base and at the tip be w_b and w_t, respectively. Further, let $w(x)$ represent the fin thickness at any distance x (measured from the base). Both top and bottom faces of the fin interact with the surroundings through convection and radiation. The convection process is characterized by the heat transfer coefficient h and the environment temperature T_∞. To describe the surface radiation loss, we assume an emissivity ε and an effective sink temperature T_S. Assuming one-dimensional conduction, constant thermal parameters, and neglecting fin-to-base and fin-to-fin radiation interaction, the governing equation for a unit depth of the fin is as follows [2]:

$$\frac{\mathrm{d}}{\mathrm{d}X}\left[w(X)\frac{\mathrm{d}T}{\mathrm{d}X}\right] = \frac{2h}{k}(T - T_\infty) + \frac{2\varepsilon\sigma}{k}\left(T^4 - T_s^4\right), \quad (1)$$

where h, k, ε, σ, T_∞ and T_S denote convective heat transfer coefficient, thermal conductivity of fin material, surface emissivity, Stefan-Boltzmann constant, environment temperature for convection and effective sink temperature for radiation, respectively. Introducing the dimensionless variables $\theta = \frac{T}{T_b}$, $\theta_\infty = \frac{T_\infty}{T_b}$, $\theta_S = \frac{T_s}{T_b}$, $x = \frac{X}{L}$, $\alpha = \frac{2L}{w_b}$, $Bi = \frac{hw_b}{2k}$, and $N_r = \frac{\varepsilon\sigma w_b T_b^3}{2k}$, Eq. (1) is converted to

$$w(x)\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \alpha^2 Bi(\theta - \theta_\infty) + \alpha^2 N_r\left(\theta^4 - \theta_s^4\right). \quad (2)$$

To solve Eq. (2), the profile function $w(x)$ and the boundary conditions must be specified. The profile shapes chosen for the present work are triangular and concave parabolic. The fin base temperature in each case is assumed to be constant T_b. For the triangular and parabolic fins, the tip heat fluxes are obviously zero. Therefore

Triangular fin:

$$\theta\Big|_{x=0} = 1, \quad \frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{x=1} = 0, \quad w(x) = 1 - x, \quad (3)$$

Concave parabolic fin:

$$\theta\Big|_{x=0} = 1, \quad \frac{\mathrm{d}\theta}{\mathrm{d}x}\Big|_{x=1} = 0, \quad w(x) = 1 - x^2. \quad (4)$$

The problem formulated by Eq. (2) with boundary conditions (3) and (4) have been investigated numerically and semi-analytically by many researchers, see [2, 48–51] and references therein.

In this article, we propose a new intelligent computational approach to obtain solution for the non-linear second-order boundary value problem (2)-(4). First, we transform the governing equation into an equivalent problem whose boundary conditions are [−1, 1]. In this way, they are convenient to apply reformed version of Chebyshev polynomials of the first kind. Then we optimize Chebyshev polynomials of the first kind to construct approximate series solution with unknown weights. Furthermore, it is set up an optimization problem based on unsupervised error as objective function subject to a tolerance as constraint. This optimization problem is minimized by tuning weights via interior point method. This numerical based technique enables us to overcome the nonlinearity in the mentioned boundary value problem and then to obtain accurate solution. Moreover, in some exactly solvable cases, we compare the approximate solution with the exact one. Also, the fin efficiency and heat transfer rate with respect are reported.

2 High order derivatives of basis functions

Chebyshev polynomials [52] are very useful as orthogonal polynomials on the interval [−1, 1] of the real line. These polynomials have very good properties in the approximation of functions so that appear frequently in several fields of mathematics, physics and engineering.

2.1 Basic properties of Chebyshev polynomials

The Chebyshev polynomials of the first kind, known as $T_n(x) = \cos(n \arccos x)$, can be obtained by means of Rodrigue's formula [53]

$$T_n(x) = \frac{\Gamma(\frac{1}{2})}{(-2)^n\Gamma(n+\frac{1}{2})}\sqrt{1-x^2}\,\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(1-x^2\right)^{n-\frac{1}{2}}, \quad n = 0, 1, 2, \cdots. \quad (5)$$

The Chebyshev polynomials of the first kind can be developed by means of the generating function too, as follows:

$$\frac{1 - tx}{1 - 2tx + t^2} = \sum_{n=0}^{+\infty} T_n(x)t^n. \quad (6)$$

The first two Chebyshev polynomials $T_n(x) = 1$ and $T_n(x) = x$ are known from (5), all other polynomials $T_n(x)$, $n \geq 2$ can be obtained by means of the recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \quad (7)$$

The derivative of $T_n(x)$ with respect to x can be obtained from

$$\left(1 - x^2\right)T_n'(x) = -nxT_n(x) + nT_{n-1}(x), \ x \neq \pm 1, \quad (8)$$

$$T_n'(-1) = n^2(-1)^{n+1}, \quad T_n'(1) = n^2. \quad (9)$$

The following special values and properties of $T_n(x)$ are well established and will be useful:

$$T_n(-x) = (-1)^n T_n(x),\ T_n(1) = 1,\ T_n(-1) = (-1)^n,$$
$$T_{2n}(0) = (-1)^n,\ T_{2n+1}(0) = 0. \tag{10}$$

We can determine the orthogonality properties for the Chebyshev polynomials of the first kind from our knowledge of the orthogonality of the cosine functions, as

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\mathrm{d}x = \begin{cases} 0, & m \neq n; \\ \frac{\pi}{2}, & m = n \neq 0; \\ \pi, & m = n = 0. \end{cases} \tag{11}$$

We observe that the Chebyshev polynomials form an orthogonal set on the interval $[-1, 1]$ with the weighting function $\frac{1}{\sqrt{1-x^2}}$.

2.2 High order derivatives of Chebyshev polynomials

(The Leibniz Formula) For a function $f(x) = g(x)h(x)$, the derivatives of $f(x)$ can be represented as a sum of derivatives of $g(x)$ and $h(x)$ as:

$$f^{(k)}(x) = \sum_{n=0}^{k} \binom{k}{n} g^{(n)}(x)h^{(k-n)}(x), \tag{12}$$

where $\binom{k}{n}$ are the binomial coefficients.

Theorem 2.1. ***(Slevinsky-Safouhi)[54]*** *Let $G(x)$ be a function kth differentiable and with the term $\left(\frac{d}{xdx}\right)^k G(x)$ welldefined. The term $\frac{d^kG}{dx^k}$ is given by:*

$$\frac{d^kG}{dx^k} = \sum_{i=\lfloor \frac{k+1}{2} \rfloor}^{k} \hat{A}_k^i x^{2i-k} \left(\frac{d}{xdx}\right)^k G(x), \tag{13}$$

with coefficients:

$$\hat{A}_k^i = \begin{cases} 1, & i=k; \\ 2\hat{A}_{k-1}^i + \hat{A}_{k-1}^{i-1}, & i = \lfloor \frac{k+1}{2} \rfloor,\ k\ odd; \\ \hat{A}_{k-1}^i, & i = \lfloor \frac{k+1}{2} \rfloor,\ k\ even; \\ (2i-k+1)\hat{A}_{k-1}^i + \hat{A}_{k-1}^{i-1}, & \lfloor \frac{k+1}{2} \rfloor < i < k,\ k > 3. \end{cases} \tag{14}$$

where $\lfloor \alpha \rfloor$ is the integer floor function of argument α.

It is natural with the help of the Leibniz Formula as well as Rodrigue's formula to define the higher order derivatives of $T_n(x)$ as:

$$\frac{\mathrm{d}^i}{\mathrm{d}x^i} T_n(x) = \frac{\Gamma(\frac{1}{2})}{(-2)^n \Gamma(n+\frac{1}{2})} \sum_{l=0}^{i} \binom{i}{l} \frac{\mathrm{d}^l}{\mathrm{d}x^l}\sqrt{1-x^2}$$
$$\frac{\mathrm{d}^{n+i-l}}{\mathrm{d}x^{n+i-l}} \left(1-x^2\right)^{n-\frac{1}{2}}. \tag{15}$$

Without going into great detail, if we apply the result of Theorem 2.1 into the above equation then we develop a very effective formula as the final result [54]:

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} T_n(x) = \frac{\Gamma(\frac{1}{2})}{(-2)^n \Gamma(n+\frac{1}{2})}$$
$$\sum_{l=0}^{k} \left\{ \binom{k}{l} \left[\sum_{i=\lfloor \frac{l+1}{2} \rfloor}^{l} \hat{A}_l^i x^{2i-l} (-2)^i \left(1-x^2\right)^{\frac{1}{2}-i} \prod_{j=0}^{i-1} \left(\frac{1}{2}-j\right) \right] \right.$$
$$\left. \left[\sum_{i=\lfloor \frac{n+k-l+1}{2} \rfloor}^{n+k-l} \hat{A}_{n+k-l}^i x^{2i-n-k+l} (-2)^i \left(1-x^2\right)^{n-\frac{1}{2}-i} \prod_{j=0}^{i-1} \left(n-\frac{1}{2}-j\right) \right] \right\} \tag{16}$$

with coefficients $\hat{A}_k^i$ given by (14).

3 Nonlinear optimization model

By the change of variable $x \mapsto \frac{1}{2}(x+1)$, the boundary value problems (2)-(3) and (2)-(4) can be rewritten as

$$(1-x)\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{2}\left[\alpha^2 Bi\,(\theta - \theta_\infty) + \alpha^2 N_r \left(\theta^4 - \theta_s^4\right)\right], \tag{17}$$

$$\theta(-1) = 1, \theta'(1) = 0, \tag{18}$$

and

$$(3 - x^2 - 2x)\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - 2(1+x)\frac{\mathrm{d}\theta}{\mathrm{d}x} = \alpha^2 Bi\,(\theta - \theta_\infty) + \alpha^2 N_r \left(\theta^4 - \theta_s^4\right), \tag{19}$$

$$\theta(-1) = 1, \theta'(1) = 0, \tag{20}$$

respectively. Now, it is convenient to treat them by Chebyshev polynomials of the first kind. Moreover, the change of function $\theta \mapsto \theta + 1$ transforms the problems into

$$(1-x)\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{2}\left[\alpha^2 Bi\,(1 + \theta - \theta_\infty) + \alpha^2 N_r \left((1+\theta)^4 - \theta_s^4\right)\right], \tag{21}$$

$$\theta(-1) = 0, \theta'(1) = 0, \tag{22}$$

and

$$(3 - x^2 - 2x)\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - 2(1+x)\frac{\mathrm{d}\theta}{\mathrm{d}x} = \alpha^2 Bi\,(1 + \theta - \theta_\infty) + \alpha^2 N_r \left((1+\theta)^4 - \theta_s^4\right), \tag{23}$$

$$\theta(-1) = 0, \theta'(1) = 0, \tag{24}$$

such that the boundary conditions become homogenous.

3.1 Reformed version of Chebyshev polynomials

Define $\hat{T}_n,\ n \geq 1$ as

$$\hat{T}_n(x) = T_n(x) - n^2 x - n^2 - (-1)^n, \quad n \geq 1, \tag{25}$$

then obviously, from (10), we have

$$\hat{T}_n(-1) = 0, \quad n \geq 1. \tag{26}$$

Eq. (9) implies

$$\hat{T}'_n(1) = T'_n(1) - n^2 = 0 \quad n \geq 1. \tag{27}$$

Therefore, from Eqs. (26)-(27), we conclude that the boundary conditions (22) and (24) hold.

Furthermore, the second derivative of the reformed version of Chebyshev polynomials of the first kind are given by

$$\hat{T}'_n(x) = T'_n(x) - n^2, \quad n \geq 1, \tag{28}$$

$$\hat{T}''_n(x) = T''_n(x), \quad n \geq 1, \tag{29}$$

where the right hand side can be obtained by the formula (16) when $k = 1, 2$.

3.2 Corresponding optimization problem

Define a approximate series solution of order M as

$$\Theta_M(x) = \sum_{n=1}^{M} \alpha_n \hat{T}_n(x), \tag{30}$$

and consider the number of N regularly distributed nodal points in interval $[-1, 1]$, namely x_i, $i = 1, 2, \ldots, N$, then we define the unsupervised errors as the sum of mean squared errors:

$$\epsilon_1(N, \alpha) = \frac{1}{N} \sum_{i=1}^{N} \Bigg\{ (1 - x_i) \sum_{n=1}^{M} \alpha_n \hat{T}''_n(x_i) - \sum_{n=1}^{M} \alpha_n \hat{T}'_n(x_i) - \frac{1}{2} \Bigg[\alpha^2 Bi \left(1 + \sum_{n=1}^{M} \alpha_n \hat{T}_n(x_i) - \theta_\infty \right) + \alpha^2 N_r \left(\left(1 + \sum_{n=1}^{M} \alpha_n \hat{T}_n(x_i) \right)^4 - \theta_s^4 \right) \Bigg] \Bigg\}^2, \tag{31}$$

for triangular fin, and

$$\epsilon_2(N, \alpha) = \frac{1}{N} \sum_{i=1}^{N} \Bigg\{ (3 - x_i^2 - 2x_i) \sum_{n=1}^{M} \alpha_n \hat{T}''_n(x_i) - 2(1 + x_i) \sum_{n=1}^{M} \alpha_n \hat{T}'_n(x_i) - \Bigg[\alpha^2 Bi \left(1 + \sum_{n=1}^{M} \alpha_n \hat{T}_n(x_i) - \theta_\infty \right) + \alpha^2 N_r \left(\left(1 + \sum_{n=1}^{M} \alpha_n \hat{T}_n(x_i) \right)^4 - \theta_s^4 \right) \Bigg] \Bigg\}^2, \tag{32}$$

for concave parabolic fin. It is worth to mention here that $\Theta_M(x)$ automatically satisfy boundary conditions (22) and (24). Now, define the following optimization problems

$$\min_{\alpha} \epsilon_1(N, \alpha) \quad \text{subject to} \quad \epsilon_1(N, \alpha) - \varepsilon \leq 0, \tag{33}$$

$$\min_{\alpha} \epsilon_2(N, \alpha) \quad \text{subject to} \quad \epsilon_2(N, \alpha) - \varepsilon \leq 0, \tag{34}$$

for triangular and concave parabolic fins, respectively, where ε is a given tolerance. In our approach, the interior point method (IPM) is used for tuning of weights of the approximate series solution (30). IPM belongs to a class of algorithms which are used for treating constrained optimization problems. The technique is based on Karmarkar's algorithm which has been developed by Narendra Karmarkar in 1984 for linear programming resolution [55]. Detailed in formation about the algorithm is available in references [56, 57]. IPMs have been applied to many optimization problems in engineering and applied science such as multi-area optimal reactive power flow [58] and economic dispatch problem [59]. The fundamental trait of interior point methods are based on self-concordant barrier functions which play important role in encoding the convex set. In contrast to the classical simplex method, search for an optimal solution is made by traversing the interior of the feasible region and solving a sequence of subproblems [60].

4 Numerical experiments and comparison

In this section, we show the results obtained for some case studies which have been adopted from Refs. [2, 48–51] using proposed method described in the previous sections. In these examples, $N = 30$, the number of total nodal points covering $[-1, 1]$, is regularly distributed. Moreover, the number of basis function in approximate series solution in Eq. (30) is $M = 10$. The obtained solutions can be compared to those of Refs. [2, 48–51] and references therein. All approximate solutions reported here obtained in seconds by MATLAB softwares programm, therefore the method is highly robust.

MATLAB provides an efficient optimization toolbox that contains functions for finding minimum of a multivariable function while satisfying constraints. The toolbox includes solvers that perform optimization on the various types of linear or nonlinear problems. The function,

$fmincon(\cdot)$, of this toolbox is a general, multipurpose optimizer that well tested and frequently used to solve nonlinear programming problems with general equality, inequality, and bound constraints of small, medium, and large scale. To handle optimization problem (33)-(34), we use $fmincon(\cdot)$ augmented to the interior point method (IPM) as described in the previous section.

Figures 1 – 3 show temperature distribution versus x for different values of Biot number $Bi = 0.01, 0.05, 0.1, 0.5, 1, 5, 10$ and $\alpha = 1, 4, 10$ and $N_r = 0, 0.1, 1$ when $\theta_s = \theta_\infty = 0.2$ in the case of triangular fin. The same graphs for the case of concave parabolic fin are plotted in Figures 4 – 6.

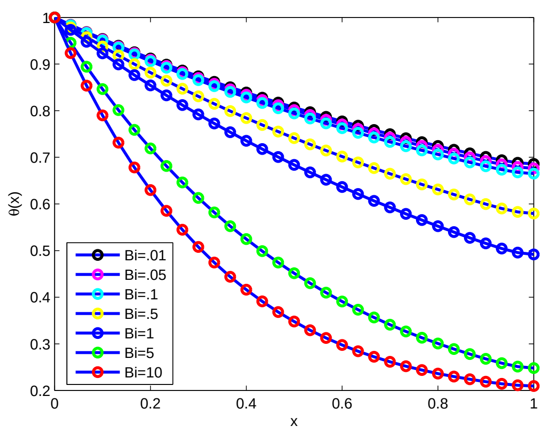

Figure 1: Diagram of temperature distribution versus x for the triangular fin with $\alpha = 1$ and $N_r = 1$.

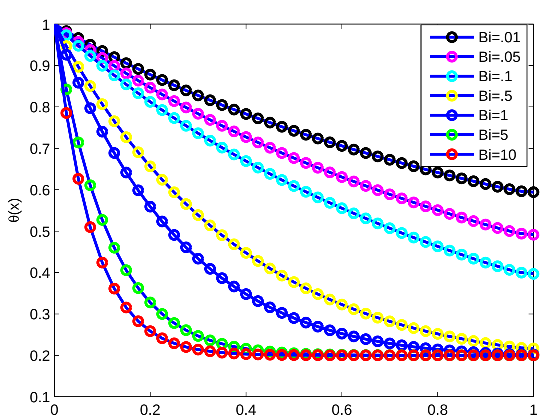

Figure 2: Diagram of temperature distribution versus x for the triangular fin with $\alpha = 4$ and $N_r = 0.1$.

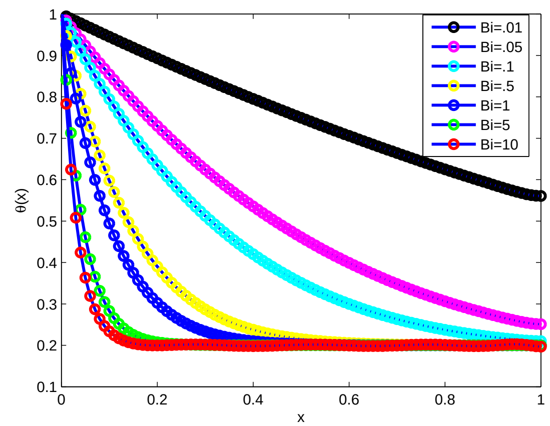

Figure 3: Diagram of temperature distribution versus x for the triangular fin with $\alpha = 10$ and $N_r = 0$.

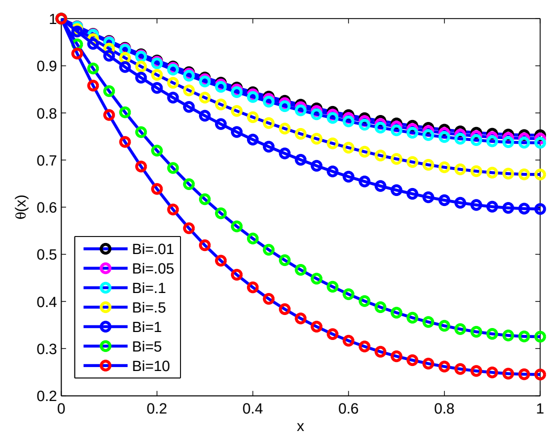

Figure 4: Diagram of temperature distribution versus x for the concave parabolic fin with $\alpha = 1$ and $N_r = 1$.

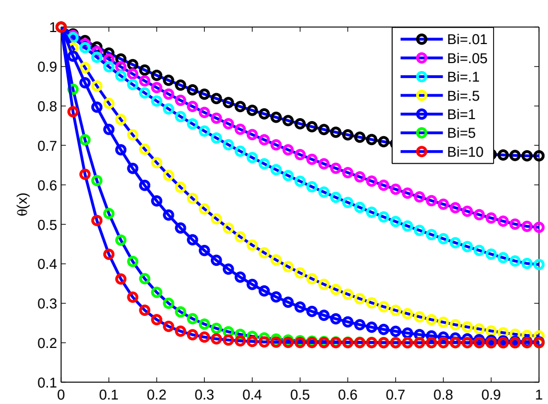

Figure 5: Diagram of temperature distribution versus x for the concave parabolic fin with $\alpha = 4$ and $N_r = 0.1$.

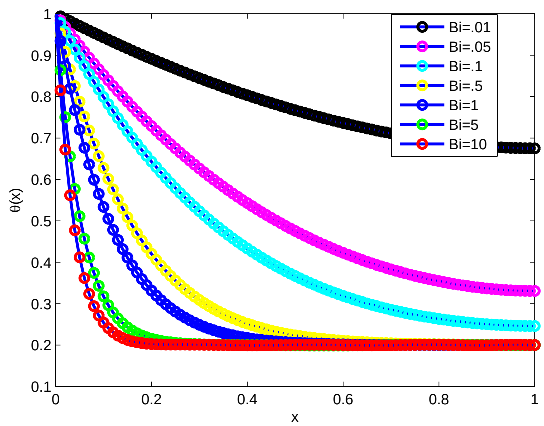

Figure 6: Diagram of temperature distribution versus x for the concave parabolic fin with $\alpha = 10$ and $N_r = 0$.

The heat transfer rate q (per unit depth) is given by

$$q = -kw_b \frac{\mathrm{d}T(0)}{\mathrm{d}X}, \quad (35)$$

which is in dimensionless form as

$$Q = \frac{q}{kT_b} = -\frac{1}{\alpha}\frac{\mathrm{d}\theta(0)}{\mathrm{d}x}. \quad (36)$$

Fin efficiency is the ratio of the real heat transfer rate to the ideal heat transfer rate for a fin of infinite thermal conductivity

$$\eta = \frac{q}{h\left(2L + w_t\right)\left(T_b - T_\infty\right) + \left(2L + w_t\right)\varepsilon\sigma\left(T_b^4 - T_s^4\right)}, \quad (37)$$

which can be rewritten in dimensionless form as

$$\eta = \frac{Q}{2\left(\alpha + 1\right)\left[Bi\left(1 - \theta_\infty\right) + N_r\left(1 - \theta_s^4\right)\right]}, \quad (38)$$

We have reported dimensionless heat transfer rate and the fin efficiency in the cases of the triangular and concave parabolic fin for different Biot number, radiation-conduction number and α in Tables 1-4 when $\theta_s = \theta_\infty = 0.2$.

To validate our results, consider the boundary value problem (2)-(4) in the case $Bi(1 - \theta_\infty) = N_r(\theta_s^4 - 1)$, then it is easy to see that the unique solution to the above problem is $\theta(x) \equiv 1$ in both cases, triangular and concave parabolic fins. This point is in full agreement with our approximation results shown in Figs. 7 and 8 for any values satisfying $Bi(1 - \theta_\infty) = N_r(\theta_s^4 - 1)$.

5 Conclusions

In this article, the problem of the evaluation of heat transfer rate from convecting-radiating fin in the cases of triangular and concave parabolic shapes has been investigated.

Table 1: Dimensionless heat transfer rate for the triangular fin with $\theta_s = \theta_\infty = 0.2$.

Bi	$\alpha = 1, N_r = 1$	$\alpha = 4, N_r = 0.1$	$\alpha = 10, N_r = 0$
0.01	0.48333	0.17135	0.05572
0.05	0.49763	0.21644	0.15614
0.1	0.51538	0.26672	0.22919
0.5	0.65161	0.53104	0.52836
1	0.80703	0.73927	0.74467
5	1.65183	1.57822	1.59101
10	2.31777	2.14760	2.16689

Table 2: Fin efficiency for the triangular fin with $\theta_s = \theta_\infty = 0.2$.

Bi	$\alpha = 1, N_r = 1$	$\alpha = 4, N_r = 0.1$	$\alpha = 10, N_r = 0$
0.01	0.12006	0.15889	0.31659
0.05	0.11981	0.15478	0.17743
0.1	0.11948	0.14831	0.13022
0.5	0.11649	0.10624	0.06004
1	0.11219	0.08156	0.04231
5	0.08262	0.03849	0.01808
10	0.06439	0.02651	0.01231

Table 3: Dimensionless heat transfer rate for the concave parabolic fin with $\theta_s = \theta_\infty = 0.2$.

Bi	$\alpha = 1, N_r = 1$	$\alpha = 4, N_r = 0.1$	$\alpha = 10, N_r = 0$
0.01	0.50120	0.17525	0.05939
0.05	0.51591	0.22097	0.15904
0.1	0.53416	0.27154	0.22319
0.5	0.67405	0.51838	0.47271
1	0.83310	0.69683	0.65199
5	1.65707	1.40174	1.36063
10	2.24615	1.88566	1.84815

Table 4: Fin efficiency for the concave parabolic fin with $\theta_s = \theta_\infty = 0.2$.

Bi	$\alpha = 1, N_r = 1$	$\alpha = 4, N_r = 0.1$	$\alpha = 10, N_r = 0$
0.01	0.12450	0.16251	0.33744
0.05	0.12421	0.15802	0.18073
0.1	0.12383	0.15099	0.12681
0.5	0.12050	0.10371	0.05372
1	0.11581	0.07744	0.03704
5	0.08288	0.03419	0.01546
10	0.06240	0.02328	0.01050

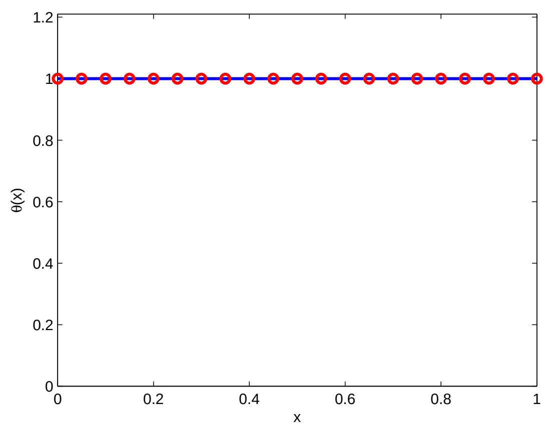

Figure 7: Diagram of temperature distribution versus x for the triangular fin when $Bi(1-\theta_\infty) = N_r(\theta_s^4 - 1)$.

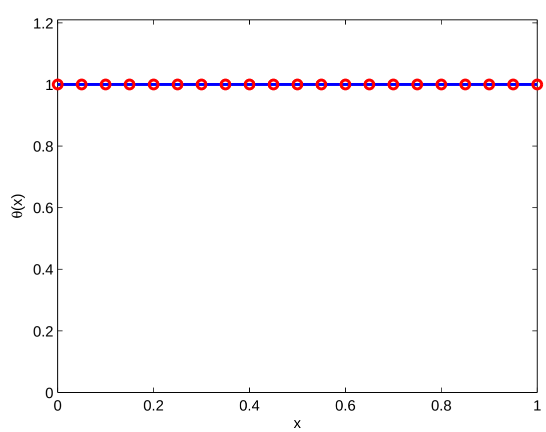

Figure 8: Diagram of temperature distribution versus x for the concave parabolic fin when $Bi(1-\theta_\infty) = N_r(\theta_s^4 - 1)$.

We have considered one-dimensional, steady conduction in the fin and neglected radiative exchange between adjacent fins and its primary surface.

It has been proposed a new intelligent computational technique to obtain approximate solution for the mentioned problem. First, the governing equation is transformed into an equivalent problem whose boundary conditions are homogeneous in interval $[-1, 1]$. Then, it is optimized Chebyshev polynomials of the first kind to construct approximate series solution with unknown weights. Furthermore, by defining an optimization problem and minimizing it, all weights are obtained via interior point method. As a result, we have reported heat transfer rate and the fin efficiency in the cases of the triangular and concave parabolic fin for different Biot number and radiation-conduction number with desired order of accuracy.

The method includes three steps: The first and most important step is to find Reformed Version of Chebyshev Polynomials i.e. Eq. (25) so that they satisfy the boundary conditions. The second step is to construct the optimizations problems (33) or (34) and the final step is to demand $fmincon(\cdot)$ augmented to interior point algorithm using MATLAB. It has been revealed through test studies that the method is highly robust and reliable.

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Nomenclature	
T	temperature
T_b	fin base temperature
T_s	effective sink temperature for radiation
T_∞	environment temperature for convection
L	fin length
h	convective heat transfer coefficient
k	thermal conductivity
N_r	radiation-conduction number
X	dimensional space coordinate
q	heat transfer rate
Q	dimensionless heat transfer rate
x	non-dimensional space coordinate
w_b	fin thickness at the base
w_t	fin thickness at the tip
Bi	Biot number
$w(x)$	Profile function
$T_n(x)$	n'th Chebyshev polynomials of the first kind
$\epsilon_i(N, \alpha)$	Objective function in optimization model
Greeks symbols	
σ	Stefan-Boltzmann constant
η	fin efficiency
α	ratio of length to one-half base thickness
ε	surface emissivity
θ	dimensionless temperature
θ_s	dimensionless effective sink temperature for radiation
θ_∞	dimensionless environment temperature for convection

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