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Mozhgan Rezakhanlou

Tarbiat Modares University

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A. Iranmanesh

Towards classifying finite groups whose non-linear irreducible characters are rational valued

M. Rezakhanlou* and H. Sharifi

Abstract

A finite group G is called a $\mathbb{Q}_1$-group if all of its non-linear irreducible characters are rational valued. In this paper we review some of the work done on the $\mathbb{Q}_1$-groups and we provide an approach to fully classify these types of groups.

Keywords and phrases: Frobenius group, Metabelian group, Irreducible character, $\mathbb{Q}$-group, $\mathbb{Q}_1$-group.
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1. Introduction

Suppose that G is a finite group and χ is a complex character of G. By definition, χ is called rational if $\chi(g) \in \mathbb{Q}$, $g \in G$. A finite group G is called rational group or $\mathbb{Q}$-group if every irreducible complex character is rational. We generalize $\mathbb{Q}$-group to $\mathbb{Q}_1$-group by imposing rationality merely on non-liner characters. It is clear that every $\mathbb{Q}$-group is $\mathbb{Q}_1$-group. For every natural number n , the symmetric group $\mathbb{S}_n$, is rational group [3]. The Alternating group on four letters and the dicyclic group of order 12, $T_{12} = \langle a, b | a^6 = 1, a^3 = b^2, b^{-1}ab = a^{-1} \rangle$, are the smallest examples of groups that are $\mathbb{Q}_1$-group but not $\mathbb{Q}$-group (we say $\mathbb{Q}_1'$-group).

The classification of $\mathbb{Q}_1$-groups has not been completed yet, as such, the complete classification of $\mathbb{Q}$-groups is not achieved. The purpose of this paper is to provide an approach to fully classify $\mathbb{Q}_1$-groups.

Throughout the paper we consider finite groups, and we employ the following notation and terminology. The symbol $\mathbb{Z}_n$ denotes a cyclic group of order n. For a prime p and a non-negative integer n, the symbol $E(p^n)$ denotes the elementary abelian p-group of order p^n. The semi-direct product of group K with group H is denoted by $K : H$. $A * B$ is the central product of groups A and B i.e., $A * B = AB$ and $[A, B] = 1$, where $[A, B]$ is the commutator of A and B. Also $\mathbb{Q}_8$ and D_8 are employed to denote the quaternion and dihedral group of order 8, respectively.

Let us close this introduction with a brief personal comment of second author. About 15 years ago , when I was still a Phd student and I was reading course of

* speaker

character theory with my best teacher Prof. Darafsheh, I asked him, what is benefit of character theory?, he laughed and said, Oh, that is a Love!. And, I am still looking for my love!. I have seen many applications of this theory so far, especially in the recent developments in wireless communication and fifth generation internet.

2. Preliminaries

It is a simple exercise, to check that finite group G is rational group if and only if every $g \in G$ conjugate in G to all elements of the form g^m for integers m such that $(m, |G|) = 1$, equivalently,

$$\frac{N_G(\langle x\rangle)}{C_G(\langle x\rangle)} \cong \mathrm{Aut}_G(\langle x\rangle)$$

for every element $g \in G$.

Now, we list some well-known properties of $\mathbb{Q}_1$-groups in the following Lemma. To see a proof of the following lemma, the reader can refer [2–4] and references therein.

Lemma 2.1. *Let G be a $\mathbb{Q}_1$-group. Then the following hold.*
(1) *$|G|$ is even.*
(2) *Every quotient of G is a $\mathbb{Q}_1$-group.*
(3) *If G be a p-group, then $p = 2$.*
(4) *$Z(G)$ is an elementary abelian 2-group.*
(5) *If $G = H \times K$ where H is abelian and K is non-abelian, then H is an elementary abelian 2-group.*
(6) *If G is a non-abelian and nilpotent then G is a 2-group.*

Lemma 2.2. *The direct product of a finite number of non-abelian groups $G_1, \ldots, G_n$, $(n \geq 2)$, is a $\mathbb{Q}_1$-group iff each G_i is a $\mathbb{Q}$-group.*

The vanishing-off subgroup of G is denoted as follows

$$V(G) = \langle g \in G : \exists x \in \mathrm{nl}(G) : \chi(g) \neq 0\rangle$$

$V(G)$ is an important tool for identifying structure $\mathbb{Q}_1$-groups. Here we review some theorems from [4].

Theorem 2.3. *Let G be a non-abelian finite group. Then G is a $\mathbb{Q}_1$-group if and only if every element of $V(G)$ is a rational element.*

Theorem 2.4. *Let G be a non-solvable group. Then G is a $\mathbb{Q}_1$-group iff G is a $\mathbb{Q}$-group.*

Corollary 2.5. *Let G be a non-cyclic simple group. Then G is a $\mathbb{Q}_1$-group if and only if $G \simeq \mathrm{Sp}_6(2)$ or $\mathbb{Q}_8^+(2)'$.*

Lemma 2.6. *The central product of a finite number of $\mathbb{Q}_1$-group is $\mathbb{Q}_1$-group.*

Definition 2.7. *A group G is said to be metabelian if its derived subgroup is abelian.*

Theorem 2.8. *[1] Let G be a finite metablian $\mathbb{Q}_1$-group and assume that G' is a cyclic subgroup. Then $|G'|$ is prime or $|G'|$ divides 12. Furthermore if $|G'| = p$, where p is an odd prime, then $|G| = 2^t p(p-1)$, if $|G'| \in \{2, 4\}$ then $|G| = 2^t$, and if $|G'| \in \{6, 12\}$ then $|G| = 2^t \cdot 3$.*

3. Some recent achievements

We mention some recent achievements.

Definition 3.1. *A finite group G is a Frobenius group if it contains a proper subgroup $H \neq \{1\}$, called a Frobenius complement, such that $H \cap H^x = \{1\}$.*

The following theorem is a classfication of Frobenius $\mathbb{Q}_1$-group, see [6].

Theorem 3.2. *If G is a Frobenius $\mathbb{Q}_1$-group, then one of the following occurs:*
(1) $G \cong E(p^n) : \mathbb{Z}_t$, *where p is an odd prime, $n \geq 1$, and $t \geq 1$ is even.*
(2) $G \cong G' : \mathbb{Z}_t$, *where the derived subgroup G' of G is a rational 2-group and $t \geq 1$ is odd.*
(3) $G \cong E(5^2) : \mathbb{Q}_8$ *or* $G \cong E(3^{2m}) : \mathbb{Q}_8$ *where $m \geq 1$.*
(4) $G \cong E(p^n) : H$, *where p is a Fermat prime, $n \geq 1$ and H is a metacyclic group of order $2^m q$, for some Fermat prime q and $m \geq 1$.*

Theorem 3.3. *[7]. Suppose G is a non-abelian solvable $\mathbb{Q}_1$-group with Sylow 2-subgroup P. Then one of the following occurs:*
(1) *If $P \subseteq V(G)$, then $G \cong V(G) : \mathbb{Z}_m$ or $G \cong V(G) : E(p^n)$. where m is an odd integer and p is coprime to $|V(G)|$.*
(2) *If P is non-abelian and $P \nsubseteq V(G)$ then $G \cong K : P$ where K is a $\{3, 5, 7\}$group*
(3) *If P is abelian and $P \nsubseteq V(G)$ then $G \cong G' : (\mathbb{Z}_m \times E(2^n))$ where the derived subgroup G' is a Hall subgroup of odd order and m, n are integers.*

Theorem 3.4. *[8] Suppose that G is a metabelian $\mathbb{Q}_1$-group and let $P \in Syl_2(G)$. Then one of the following occurs:*
(1) *G is a 2-group and $\exp(G')$ divides 16.*
(2) *$G \cong (E(3^n) : P) : \mathbb{Z}_m$ or $G \cong P : \mathbb{Z}_m$ where m is positive integer that is coprime to 6. Also P is a rational group, when $G \cong (E(3^n) : P) : \mathbb{Z}_m$ also $E(3^n) : P$ is a rational group, and $\exp(P')$ divides 8,*
(3) *$G \cong E(3^n) : P$ or $G \cong E(5^n) : P$, where P is a nonabelian $\mathbb{Q}_1$-group that is metabelian. Moreover, $\exp(P')$ divides 8.*
(4) *$G \cong E(p^n) : ((\mathbb{Z}_m) \times E(2^n))$, where p is an odd prime and m is an odd positive integer.*

Recently, $\mathbb{Q}_1$-groups with derived subgroup of prime order classified [5].

Theorem 3.5. *Suppose that G is a $\mathbb{Q}_1$-group and its derived subgroup G' is of order 2. Then $G \cong (A_1 * \cdots * A_n) \times E(2^m)$, where A_i, $1 \leq i \leq n$ is isomorphic to one of the following:*
(1) $\langle a, b : a^4 = b^4 = 1, a^b = a^3 \rangle$

(2) $\langle a, b, c : a^4 = b^4 = c^2 = 1, [a, c] = [b, c] = 1, [a, b] = c\rangle$
(3) $\langle a, b, c : a^4 = b^2 = c^2 = 1, [a, c] = [b, c] = 1, [a, b] = c\rangle$
(4) $\mathbb{Q}_8$
(5) D_8

Theorem 3.6. *Suppose that G is a group such that $|G'| = p$, where p is an odd prime. Then G is a $\mathbb{Q}_1$-group if and only if one of the following occurs*
(1) $G \cong (\mathbb{Z}_P : \mathbb{Z}_{p-1}) \times E(2^n)$
(2) $G \cong (\mathbb{Z}_p : \mathbb{Z}_{2(p-1)} \times E(2^n))$

4. An approach to classification of $\mathbb{Q}_1$-groups

We focus on vanishing-off subgroup $V(G)$ and Sylow 2-subgroup P of non-Abelian solvable $\mathbb{Q}_1$-group G, and consider Theorem 3.3. If we find the subgroup structure utilized in Theorem 3.3, then we are done. One of the problems is knowing the structure of $V(G)$. Firsit, in case $P \subseteq V(G)$, it can be proved that only $G \cong V(G) : \mathbb{Z}_m$ is established. Also, $V(G)$ is Hall $\mathbb{Q}$-group and m must be odd integer. Second, in case $P \nsubseteq V(G)$, if P is non-Abelian $\mathbb{Q}$-group then we can prove that G is $\mathbb{Q}$-group. And, if P is Abelian, then we can get results G' is either nilpotent or normal 2-complement.

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M. Rezakhanlou,
Department of Mathematics, Tarbiat Modares University, Tehran, Iran,

e-mail: rezakhanlou79@yahoo.com

H. Sharifi,
Faculty of Science, Department of Mathematics, Shahed University,Tehran, Iran,

e-mail: hsharifi@shahed.ac.ir