On algorithmic complexity of double Roman domination

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Abstract

A double Roman dominating function (DRDF) on a graph \( G = (V, E) \) is a function \( f : V \rightarrow \{0, 1, 2, 3\} \) such that every vertex \( v \in V \) with \( f(v) = 0 \) is either adjacent to a vertex \( u \) with \( f(u) = 3 \) or two distinct vertices \( x \) and \( y \) with \( f(x) = f(y) = 2 \), and every vertex \( v \in V \) with \( f(v) = 1 \) is adjacent to a vertex \( u \) with \( f(u) \geq 2 \). The weight of \( f \) is the sum \( f(V) = \sum_{v \in V} f(v) \). The minimum weight of a DRDF on \( G \) is the double Roman domination number of \( G \), denoted by \( \gamma_{dr}(G) \). A graph \( G \) is a double Roman Graph if \( \gamma_{dr}(G) = 3\gamma(G) \), where \( \gamma(G) \) is the domination number of \( G \).

In this paper, we first show that the decision problem associated to double Roman domination is NP-complete even when restricted to planar graphs. Then, we study the complexity issue of a problem posed in [R.A. Beeler, T.W. Haynes and S.T. Hedetniemi, Double Roman domination, Discrete Appl. Math. 211 (2016), 23–29], and show that the problem of deciding whether a given graph is double Roman is NP-hard even when restricted to bipartite or chordal graphs. Then, we give linear algorithms that compute the domination number and the double Roman domination number of a given unicyclic graph. Finally, we give a linear algorithm that decides whether a given unicyclic graph is double Roman.

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1. Introduction

For notation and terminology not given here we refer to [10]. Let \( G = (V, E) \) be a graph with vertex set \( V \) of order \( n \) and edge set \( E \). The open neighborhood of a vertex \( v \in V \) is \( N(v) = \{ u \in V : uv \in E \} \) and the closed neighborhood of \( v \) is \( N[v] = N(v) \cup \{ v \} \). The degree of \( v \) is \( \deg(v) = |N(v)| \). A vertex of degree one is referred to as a leaf and its unique neighbor is called a support vertex. A tree \( T \) of order \( n \geq 2 \) is called a star if \( n = 2 \) or \( n \geq 3 \) and \( T \) contains exactly one vertex that is not leaf. A double star is a tree with precisely two vertices (as central vertices) that are not leaves. A path (respectively, cycle) of order \( n \) is denoted by \( P_n \) (respectively, \( C_n \)). A unicyclic graph is a graph obtained from a tree \( T \) of order at least three by joining precisely two non-adjacent vertices of \( T \). A planar graph is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints.

For a graph \( G = (V, E) \), a set \( S \subseteq V \) is called a dominating set (DS) of \( G \) if every \( v \in V - S \) is adjacent to at least one vertex \( u \in S \). Furthermore, if \( S \) induces a connected subgraph of \( G \), then \( S \) is a connected dominating set (CDS) of \( G \). The domination number (respectively, connected domination number) of \( G \), denoted by \( \gamma(G) \) (respectively, \( \gamma_c(G) \)), is the minimum cardinality of a dominating set (respectively, connected dominating set) of \( G \). A DS of \( G \) of minimum cardinality is referred as a \( \gamma(G) \)-set. A connected DS of \( G \) of minimum cardinality is referred as a \( \gamma_c(G) \)-set.

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A function \( f : V \rightarrow \{0, 1, 2\} \) is a Roman dominating function (RDF) of \( G \) if every vertex \( u \) with \( f(u) = 0 \) is adjacent to at least one vertex \( v \) with \( f(v) = 2 \). The weight of an RDF \( f \), denoted by \( w(f) \), is the sum \( w(f) = \sum_{v \in V} f(v) \). The mathematical concept of Roman domination, defined and discussed by Stewart [16], and ReVelle and Rosing [15], and subsequently developed by Cockayne et al. [8]. The results on Roman domination and its variations up to now, have recently been collected in two book chapters and three surveys, (See [3–7]). One of the new variants of Roman domination is introduced developed by Cockayne et al. [8]. The results on Roman domination and its variations up to now, have recently been collected in two book chapters and three surveys, (See [3–7]). One of the new variants of Roman domination is introduced.

Problem 1 (Beeler et al. [2]). Characterize the double Roman graphs.

For a DRDF \( f \) on \( G \), we denote by \( V_i \) (or \( V'_i \) to refer to \( f \)) the set of all the vertices of \( G \) with label \( i \) under \( f \). Thus a DRDF \( f \) can be represented by \((V_0, V_1, V_2, V_3)\), and we can use the notation \( f = (V_0, V_1, V_2, V_3) \).

Corollary 1 (Beeler et al. [2]). For any graph \( G \), there is a \( \gamma_{dr}(G) \)-function \( f = (V_0, V_1, V_2, V_3) \) with \( V_1 = \emptyset \).

Ahangar et al. [1] and Jafari Rad et al. [12] showed that the associated decision problem for double Roman domination is NP-complete, even for bipartite or chordal graphs. Zhang et al. [18] proposed a linear algorithm that computes the double Roman domination number of trees. Yue et al. [17] showed a linear algorithm that computes the double Roman domination number of cographs. Henning et al. [11] characterized the double Roman trees. In this paper we study the complexity issue of the double Roman domination and prove its NP-hardness in planar graphs. We also study the complexity issue of Problem 1 and study it in trees and unicyclic graphs.

The organization of the paper is as follows. We proceed as follows. In Section 3 we show that the associated decision problem for double Roman domination is NP-complete even when restricted to planar graphs. Then, we prove that the decision problem related to Problem 1 is NP-hard even when restricted to bipartite and chordal graphs. In Section 4, we give a linear algorithm that compute the double Roman domination number of a given unicyclic graph. In Section 5, we give a linear algorithm that compute the domination number of a given unicyclic graph. Finally, we give a linear algorithm that decides whether a given unicyclic graph is double Roman.

2. Preliminary

Consider the following family of graphs related to Problem 1:

- Family \( F_{dr3} \): the family of all graphs \( G \) with \( \gamma_{dr}(G) = 3\gamma(G) \).
- Family \( F_{dr33c} \): the family of all graphs \( G \) with \( \gamma_{dr}(G) = 3\gamma(G) = 3\gamma_{c}(G) \).

Note that \( F_{dr33c} \) is an infinite family even when restricted to chordal and bipartite graphs, since for any positive integer \( n \), if \( T_n \) is a tree obtained from \( P_n \) by adding five new leaves to any vertex of \( P_n \), then it can be seen that \( T_n \in F_{dr33c} \). The following is obvious.

Corollary 2. \( F_{dr33c} \subseteq F_{dr3} \).

Thus, to prove the NP-hardness of problem of whether a given graph belongs to \( F_{dr3} \) we only need to prove the NP-hardness of problem of whether a given graph belongs to \( F_{dr33c} \). To this end, we introduce a reduction from the 3-SAT problem. Recall that 3-SAT is the problem of deciding whether a given boolean formula in 3-conjunctive normal form is satisfiable. It is well-known that 3-SAT problem is NP-complete [9]. Let \( \Phi = \{C, x\} \) be an instance in 3-SAT Problem, that is, \( \Phi \) is a boolean formula in 3-conjunctive normal form. Let \( C = \{c_1, c_2, \ldots, c_l\} \) be a set of \( l \geq 1 \) clauses over a set \( X = \{x_1, \ldots, x_k\} \) of \( k \geq 3 \) variables. For each \( 1 \leq j \leq l \), the clause \( c_j \) (consisting of exactly three literals) is of the form \( c_j = \{y_{ij}, y_{kj}, y_{3j}\} \), where each of \( y_{ij}, y_{kj} \) and \( y_{3j} \) is either a variable or the negative of a variable in \( X \).

3. NP-hardness results

Consider the following decision problems.

Double Roman Domination (DRD) Problem:
Instance: A graph \( G \) and a positive integer \( m \).
Question: Does there exist a DRD \( f \) on \( G \) with \( w(f) \leq m? \)

Double Roman Graph (DRG) Problem:
Instance: A graph \( G \).
Claim 1. to see that $H$

Lemma 1.

Double Roman 3Connected-Domination (DR3CD) Problem:

Question: Is $G \in \mathcal{F}_{DR3}$?

Instance: A graph $G$.

Question: Is $G \in \mathcal{F}_{DR33c}$?

We first introduce a polynomial time reduction from PLANAR 3-SAT Problem to DRD Problem to show that DRD Problem is NP-complete even when restricted to planar graphs. A natural graph associated to 3-SAT Problem is the bipartite graph $G_{\{C, \lambda\}}$ that has $C \cup \lambda$ as its vertex set and has an edge between the vertices $x_i$ and $c_j$ if $c_j$ contains either $x_i$ or $\neg x_i$, where $\neg x_i$ refers to the negation of $x_i$. PLANAR 3-SAT is 3-SAT restricted to those instances $\{C, \lambda\}$ for which $G_{\{C, \lambda\}}$ is planar. It is well-known that PLANAR 3-SAT Problem is NP-complete [9, 13]. We introduce two polynomial time reductions from 3-SAT Problem to DR3CD Problem to show that the problem of deciding whether a given graph belongs to $\mathcal{F}_{DR33c}$ is NP-hard even when restricted to bipartite and chordal graphs. We construct graphs $H_\Phi$, $G_\Phi$ and $I_\Phi$ corresponding to $\Phi$ as follows.

3.1. The first reduction: $H_\Phi$

Let $\Phi = \{C, \lambda\}$ be an instance of 3-SAT Problem such that the associated graph $G_{\{C, \lambda\}}$ to $\Phi$ is planar. Assume that $H$ is a planar embedding of $G_{\{C, \lambda\}}$. We replace each variable-vertex $x_i$ of $H$, where $1 \leq i \leq k$, by a graph $H_i$ as variable gadget, where $H_i$ is obtained from a cycle of order 3 with vertices $u_i^1, \ldots, u_i^3$ such that each of vertices $u_i^j$, $u_i^{j-1}$ is adjacent to new vertices $v_{ij2-1}$, $v_{ij2}$, $v_{ij3-1}$ for each $1 \leq j \leq l$. We replace each clause-vertex $c_j$ of $H$, where $1 \leq j \leq l$, by a new vertex $z_j$. In the rest we fix indices $i$ and $j$, where $1 \leq i \leq k$ and $1 \leq j \leq l$. Assume that $c_{ij}, c_{ij+1}, \ldots, c_{im}$, where $1 \leq m \leq l$, is the sequence of all clause-vertices adjacent to $x_i$ in $H$ in clockwise direction starting from an arbitrary clause-vertex in $N_H(x_i)$, that is, $x_i c_j$ is an edge of $H$ for each $j \in \{i_1, i_2, \ldots, i_m\}$. If $x_i \in c_{ij}$ (respectively, $\neg x_i \in c_{ij}$), where $1 \leq r \leq m$, then we replace $x_i c_{ij}$ by new edge $u_i^{j-1} z_r$ (respectively, $u_i^{j-2} z_r$). Let $H_\Phi$ be the resulting graph. See Fig. 1. It is easy to see that $H_\Phi$ is a planar graph.

**Lemma 1.** The boolean formula $\Phi$ is satisfiable if and only if there is a DRDF $f$ on $H_\Phi$ with $w(f) \leq 3kl$.

**Proof.** Assume that $\Phi$ is satisfiable. Let $t$ be an assignment of truth values for the variables of $\lambda$ for which $\Phi$ evaluates to true. We construct a set $V_3$ on the vertex set of $H_\Phi$ as follows. If $t$ assigns the value true (respectively, the value false) to $x_i$, then we add all vertices $u_i^1, u_i^2, \ldots, u_i^{3l-1}$ (respectively, all vertices $u_i^2, u_i^3, \ldots, u_i^{3l-2}$) to $V_3$. It is easy to see that $f = (V(H_\Phi) - V_3, \emptyset, \emptyset)$ is a DRDF on $H_\Phi$ with $w(f) = 3kl$.

Assume that there is a DRDF $f = (V_0, V_1, V_2, V_3)$ on $H_\Phi$ with $w(f) \leq 3kl$. Consider values $f(v_{ij2-1}), f(v_{ij2}), f(u_i^j)$ for each $1 \leq i \leq k$ and $1 \leq j \leq l$. Since all vertices $v_{ij2-1}, v_{ij2}, v_{ij3-1}$ are only adjacent to vertices $u_i^j$ and $u_i^{j-1}$, we find that $S_{ij} = f(v_{ij2-1}) + f(v_{ij2}) + f(v_{ij3-1}) + f(u_i^j) + f(u_i^{j-1}) \geq 3$ for each $1 \leq i \leq l$ and $1 \leq j \leq l$. So, $w(f) \geq \sum_{i=1}^k \sum_{j=1}^l S_{ij} \geq 3kl$. Since $w(f) \leq 3kl$, we have $S_{ij} \geq 3$ and $f(z_j) = f(u_i^3) = 0$ for each $1 \leq i \leq k$ and $1 \leq j \leq l$. If $f(z_j) = 0$ for each $1 \leq j \leq l$, then $S_{ij} \geq 3$, a contradiction. So, $f(v_{ij2-1}) + f(v_{ij2}) + f(u_i^j) = 0$ for each $1 \leq i \leq l$ and $1 \leq j \leq l$. Thus, either both $f(u_i^j) = 0$ and $f(u_i^{j-1}) = 0$ or both $f(u_i^j) = 3$ and $f(u_i^{j-1}) = 0$ for each $1 \leq i \leq k$ and $1 \leq j \leq l$.

**Claim 1.** $|V_3 \cup \{u_i^{3j-2}, u_i^{3j+2}\}| \leq 1$, for each $1 \leq i \leq k$ and $1 \leq j \leq l$, where the addition is in modulo $3l$. 

**Fig. 1.** Illustration of replacing clause-vertex $c_j$ by $z_j$ for each $j \in \{1, 3, 4\}$, variable-vertex $x_i$ by $H_i$, clause-edge $x_i c_j$ by edges $u_i^j z_j$, clause-edge $x_i c_4$ by edge $u_i^j z_4$ and clause-edge $x_i c_1$ by edges $u_i^j z_1$ for which $l = 4$, $N_H(x_i) = \{c_1, c_4, c_1\}$, $x_i \in c_1$, $x_i \in c_4$ and $\neg x_i \in c_1$. 


Proof. Suppose for a contradiction that $|V_3 \cup \{u_i^{3j-2}, u_i^{3j+2}\}| = 2$. So, both $u_i^{3j-1}, u_i^{3j+1} \in V_0$ and so $f(u_i^{3j}) \neq 0$, a contradiction. □

Claim 2. $|V_3 \cup \{u_i^{3j-1}, u_i^{3j+1}\}| \leq 1$, for each $1 \leq i \leq k$ and $1 \leq j \leq l$, where the addition is in modulo $3l$.

Proof. Suppose for a contradiction that $|V_2 \cup \{u_i^{3j-1}, u_i^{3j+1}\}| = 2$. Since $f(u_i^{3j+1}) = 3$, by Claim 1 we have $f(u_i^{3j+5}) = 0$ and so $f(u_i^{3j+4}) = 3$. Similarly, we find that $u_i^{3j+7}, u_i^{3j+10}, \ldots, u_i^{3j+3l-2} = u_i^{3j-2} \in V_3$, that is, both $u_i^{3j-1}, u_i^{3j-2} \in V_3$, a contradiction. □

By Claims 1 and 2 either $u_1^1, u_1^2, \ldots, u_1^{3l-2} \in V_0$ and $u_2^1, u_2^2, \ldots, u_2^{3l-2} \in V_3$ or $u_1^1, u_1^2, \ldots, u_1^{3l-2} \in V_3$ and $u_2^1, u_2^2, \ldots, u_2^{3l-1} \in V_0$, for each $1 \leq i \leq k$. We fix indices $i$ and $j$, where $1 \leq i \leq k$ and $1 \leq j \leq l$. If $u_1^1, u_1^2, \ldots, u_1^{3l-2} \in V_0$ and $u_1^2, u_2^2, \ldots, u_2^{3l-1} \in V_3$ (respectively, $u_1^1, u_1^2, \ldots, u_1^{3l-2} \in V_3$ and $u_2^1, u_2^2, \ldots, u_2^{3l-1} \in V_0$, then we assign the value true (respectively, the value false) to the variable $x_i$. We claim that $\Phi$ is satisfiable for this assignment.

Assume without loss of generality that $c_j = \{x_1, \neg x_2, x_6\}$. Since $f(z_1) = 0$, we have $f(u_1^{3j-1}) = 3, f(u_2^{3j-2}) = 3$ or $f(u_6^{3j-1}) = 1$ for some $j, j', j'' \in \{1, 2, \ldots, l\}$. Assume without loss of generality that $f(u_1^{3j-1}) = 3$. So, $x_1$ has the value true. It causes to satisfy the clause $c_j$, that is, the boolean formula $\Phi$ is satisfiable. This completes the proof. □

Clearly, we can compute $H_\Phi$ in polynomial time and the DRD Problem belongs to NP. By Lemma 1 we have the following.

Theorem 1. The DRD Problem is NP-complete even when restricted to planar graphs.

3.2. The second reduction: $G_\Phi$

Let $\Phi = \{C, X\}$ be an instance of 3-SAT Problem. For each variable $x_i$, where $1 \leq i \leq k$, we construct a graph $G_i$ as variable gadget, where $G_i$ is obtained from a path graph of order 2 with vertices $u_i^1, u_i^2$ such that each of vertices $u_i^1, u_i^2$ is adjacent to new vertex $v_i^3$ for each $s \in \{1, 2, 3\}$. For each clause $c_j = \{y_{j1}, y_{j2}, y_{j3}\}$, where $1 \leq j \leq l$, we add a new vertex $z_j$ such that $z_j$ is adjacent to five new leaves. For $s = 1, 2, 3$, if $y_{sj} = x_i$, for some $1 \leq i \leq k$, then we add edge $u_i^3 z_j$ and if $y_{sj} = \neg x_i$, for some $1 \leq i \leq k$, then we add edges $u_i^1 z_j$. We add a new vertex $o$ such that is adjacent to five new leaves and add edges $ou_1^1$ and $ou_2^2$ for each $1 \leq i \leq k$. Finally add all edges $ab$ for each $a \in \{u_1^1, u_2^2\}$ and $b \in \{u_1^1, u_2^2\}$ and for all $1 \leq i < j \leq k$. Let $G_\Phi$ be the resulting graph. See Fig. 2. It is easy to see that $G_\Phi$ is a chordal graph.

Lemma 2. $\gamma(G_\Phi) = k + l + 1$.

Proof. Let $S$ be a $\gamma(G_\Phi)$-set. Since each of vertices $o$ and $z_j$, where $1 \leq j \leq l$, is adjacent to five leaves, both $o, z_j \in S$. Since all vertices $u_1^1, u_2^2, v_i^3$, where $1 \leq i \leq k$, are only adjacent to vertices $u_1^1, u_2^2$, at least one of vertices $u_1^1, u_2^2$ belongs to $S$. So, $\gamma(G_\Phi) = |S| \geq k + l + 1$.

Let $S = \{o, z_j, u_i^1| 1 \leq i \leq k, 1 \leq j \leq l\}$. Clearly, $S$ is a DS on $G_\Phi$ with $|S| = k + l + 1$. So, $\gamma(G_\Phi) \leq k + l + 1$. This completes the proof. □

Lemma 3. $\gamma_{ab}(G_\Phi) = 3(k + l + 1)$.
Proof. Let $f$ be a $\gamma_{AB}(G_\Phi)$-function. Since each of vertices $o$ and $z_j$, where $1 \leq j \leq l$, is adjacent to five leaves, we have $f(o) = f(z_j) = 3$. Since all vertices $u_1, u_2, u_3$, where $1 \leq i \leq k$, are only adjacent to vertices $u_1, u_2$, we find that $\sum_{i=1}^{k} f(u_i) + f(u_3) \geq 3$. So, $\gamma_{AB}(G_\Phi) = w(f) \geq 3(k + l + 1)$.

Let $V_3 = \{o, z_j, u_i | 1 \leq i \leq k, 1 \leq j \leq l\}$. Clearly, $f = (V(G_\Phi) - V_3, \emptyset, V_3)$ is a DRDF on $G_\Phi$ with $w(f) = 3(k + l + 1)$. So, $\gamma_{AB}(G_\Phi) \leq 3(k + l + 1)$. This completes the proof. □

**Lemma 4.** The boolean formula $\Phi$ is satisfiable if and only if $G_\Phi \in F_{DR33c}$.

**Proof.** Assume that $\Phi$ is satisfiable. Let $t$ be an assignment of truth values for the variables of $\lambda$ for which $\Phi$ evaluates to true. We construct a set $S$ on the vertex set of $G_\Phi$ as follows. Initialize $S$ to be $\{o, z_j : 1 \leq j \leq l\}$. If $t$ assigns the value true (respectively, the value false) to $x_i$, then we add vertex $u_i$ (respectively, vertex $u_3$) to $S$. It is easy to see that $S$ is a $\lambda$-CDS on $G_\Phi$ with $|S| = k + l + 1$. So, $\gamma_{C}(G_\Phi) \leq k + l + 1$. By Lemma 2 we have $\gamma(G_\Phi) = k + l + 1$. By the fact $\gamma(G) \leq \gamma_{C}(G)$ for any graph $G$, it obtains that $\gamma_{C}(G_\Phi) = k + l + 1$. By Lemma 3 we have $\gamma_{AB}(G_\Phi) = 3(k + l + 1)$. So, $\gamma_{AB}(G_\Phi) = 3\gamma_{C}(G_\Phi)$, that is, $G_\Phi \in F_{DR33c}$.

Let $G_\Phi \in F_{DR33c}$. By Lemma 2 we have $\gamma(G_\Phi) = k + l + 1$. Let $S$ be a $\lambda$-CDS on $G_\Phi$. So, $|S| = k + l + 1$. Clearly, both $o$ and $z_j$, where $1 \leq j \leq l$, belong to $S$. Since $S$ is a connected dominating set and $o$ belongs to $S$, at least one of vertices $u_1$ and $u_2$ belongs to $S$ for each $1 \leq i \leq k$. If both $u_1, u_2 \in S$ for some $1 \leq i \leq k$, then $|S| > k + l + 1$, a contradiction. So, either both $u_1 \in S$ and $u_2 \notin S$ or both $u_1 \notin S$ and $u_2 \in S$ for each $1 \leq i \leq k$.

We fix indices $i$ and $j$, where $1 \leq i < k$ and $1 \leq j \leq l$. Recall that either both $u_1 \in S$ and $u_2 \notin S$ or both $u_1 \notin S$ and $u_2 \in S$ (respectively, $u_1 \in S$ and $u_2 \notin S$), then we assign the value true (respectively, the value false) to the variable $x_i$. We claim that $\Phi$ is satisfiable for this assignment.

Assume without loss of generality that $c_j = \{x_1, x_2, x_6\}$. Since $z_j \in S$, we have $u_2^j \in S$, $u_2^j \in S$ or $u_6^j \in S$. Assume without loss of generality that $u_2^j \in S$. So, $x_1$ has the value true. It causes to satisfy the clause $c_j$, that is, the boolean formula $\Phi$ is satisfiable. This completes the proof. □

Note that we can compute $G_\Phi$ in polynomial time. Thus by Lemma 4 and the fact that $G_\Phi$ is a chordal graph we have the following.

**Theorem 2.** The DR3CD Problem is NP-hard even when restricted to chordal graphs.

By Corollary 2 and Theorem 2 we have the following.

**Corollary 3.** The DRCD Problem is NP-hard even when restricted to chordal graphs.

3.3. The third reduction: $I_\phi$

Let $\Phi = \{C, \lambda\}$ be an instance of 3-SAT Problem. For each variable $x_i$, where $1 \leq i \leq k$, we construct a graph $l_i$ as variable gadget, where $l_i$ is obtained from a path graph of order 3 with vertices $u_1^i, u_2^i, u_3^i$ such that $u_2^i$ is not a leaf, $u_3^i$ is adjacent to new five leaves and both vertices $u_1^i$ and $u_3^i$ are adjacent to $v_s^i$ for each $s \in \{1, 2, 3\}$. For each clause $c_j = \{y_{i_1}, y_{i_2}, y_{i_3}\}$, where $1 \leq j \leq l$, we add a new vertex $z_j$ such that $z_j$ is adjacent to five new leaves. For $s = 1, 2, 3$, if $y_{i_s} = x_i$, for some $1 \leq i \leq k$, then we add edge $u_2^j z_j$ and if $y_{i_s} = \neg x_i$, for some $1 \leq i \leq k$, then we add edges $u_1^j z_j$. We add a new vertex $x_1$ such that is adjacent to five new leaves and add edges $ou_1^i$ and $ou_2^i$ for each $1 \leq i \leq k$. Let $I_\phi$ be the resulting graph. See Fig. 3. It is easy to see that $I_\phi$ is a bipartite graph.

**Lemma 5.** $\gamma(I_\phi) = 2k + l + 1$.

**Proof.** Let $S$ be a $\gamma(I_\phi)$-set. Since each of vertices $o, u_2^i$ and $z_j$, where $1 \leq i \leq k$ and $1 \leq j \leq l$, is adjacent to five leaves, we have $o, u_2^i, z_j \in S$. Since all vertices $v_1^i, v_2^i, v_3^i$, where $1 \leq i \leq k$, are only adjacent to vertices $u_1^i, u_3^i$, at least one of vertices $u_1^i, u_3^i$ belongs to $S$. So, $\gamma(I_\phi) = |S| \geq 2k + l + 1$.

Let $S = \{o, z_j, u_1^i, u_3^i | 1 \leq i \leq k, 1 \leq j \leq l\}$. Clearly, $S$ is a DS on $I_\phi$ with $|S| = 2k + l + 1$. So, $\gamma(I_\phi) \leq 2k + l + 1$. This completes the proof. □

**Lemma 6.** $\gamma_{AB}(I_\phi) = 6k + 3l + 3$.

**Proof.** Let $f$ be a $\gamma_{AB}(I_\phi)$-function. Since each of vertices $o, u_2^i$ and $z_j$, where $1 \leq i \leq k$ and $1 \leq j \leq l$, is adjacent to five leaves, we have $f(o) = f(u_2^i) = f(z_j) = 3$. Since all vertices $v_1^i, v_2^i, v_3^i$, where $1 \leq i \leq k$, are only adjacent to vertices $u_1^i, u_3^i$, we find that $f(u_1^i) + f(u_2^i) + f(u_3^i) \geq 3$. So, $\gamma_{AB}(I_\phi) = w(f) \geq 6k + 3l + 3$.

Let $V_3 = \{o, z_j, u_1^i, u_3^i | 1 \leq i \leq k, 1 \leq j \leq l\}$. Clearly, $f = (V(I_\phi) - V_3, \emptyset, V_3)$ is a DRDF on $I_\phi$ with $w(f) = 6k + 3l + 3$. So, $\gamma_{AB}(I_\phi) \leq 6k + 3l + 3$. This completes the proof. □
Corollary 4. The DRG Problem is NP-hard even when restricted to bipartite graphs.

Lemma 7. The boolean formula \( \Phi \) is satisfiable if and only if \( I_\Phi \in \mathcal{F}_{dR3C} \).

Proof. Assume that \( \Phi \) is satisfiable. Let \( t \) be an assignment of truth values for the variables of \( \mathcal{X} \) for which \( \Phi \) evaluates to true. We construct set \( S \) on the vertex set of \( I_\Phi \) as follows. Initialize \( S \) to be \( \{ a, u_i^2 : 1 \leq i \leq k, 1 \leq j \leq l \} \). Let \( t \) assigns the value \( \text{true} \) (respectively, the value \( \text{false} \)) to \( x_i \), then we add vertex \( u_i^1 \) (respectively, vertex \( u_i^3 \)) to \( S \). It is easy to see that \( S \) is a CDS on \( I_\Phi \) with \( |S| = 2k + l + 1 \). By Lemma 5 we have \( \gamma(I_\Phi) = 2k + l + 1 \). By the fact \( \gamma(G) \leq \gamma_c(G) \) for any graph \( G \), it obtains that \( \gamma_c(I_\Phi) = 2k + l + 1 \). By Lemma 6 we have \( \gamma_{dr}(I_\Phi) = 3(k + l + 1) \). So, \( \gamma_{dr}(I_\Phi) = 3\gamma(I_\Phi) = 3\gamma_C(I_\Phi) \), that is, \( I_\Phi \in \mathcal{F}_{dR3C} \).

Let \( I_\Phi \in \mathcal{F}_{dR3C} \). By Lemma 5 we have \( \gamma(I_\Phi) = 2k + l + 1 \). Let \( S \) be a CDS on \( I_\Phi \). So, \( |S| = 2k + l + 1 \). Clearly, all vertices \( o, u_i^2 \) and \( z_j \), where \( 1 \leq i \leq k \) and \( 1 \leq j \leq l \), belong to \( S \). Since \( S \) is a connected dominating set and \( o \) belongs to \( S \), at least one of vertices \( u_i^1 \) and \( u_i^3 \) belongs to \( S \) for each \( 1 \leq i \leq k \). If both \( u_i^1, u_i^3 \in S \) for some \( 1 \leq i \leq k \), then \( |S| > 2k + l + 1 \), a contradiction. So, either both \( u_i^1 \in S \) and \( u_i^3 \not\in S \) or both \( u_i^1 \not\in S \) and \( u_i^3 \in S \) for each \( 1 \leq i \leq k \).

We fix indices \( i \) and \( j \), where \( 1 \leq i \leq k \) and \( 1 \leq j \leq l \). Recall that either both \( u_i^1 \in S \) and \( u_i^3 \not\in S \) or both \( u_i^1 \not\in S \) and \( u_i^3 \in S \). If \( u_i^1 \not\in S \) and \( u_i^3 \in S \) (respectively, \( u_i^1 \in S \) and \( u_i^3 \not\in S \)), then we assign the value \( \text{true} \) (respectively, the value \( \text{false} \)) to the variable \( x_i \). We claim that \( \Phi \) is satisfiable for this assignment.

Assume without loss of generality that \( c_j = \{ x_1, \neg x_2, x_3 \} \). Since \( z_j \in S \), we have \( u_3^1 \in S, u_1^2 \in S \) or \( u_2^3 \in S \). Assume without loss of generality that \( u_1^3 \in S \). So, \( x_1 \) has the value \( \text{true} \). It causes to satisfy the clause \( c_j \) that is, the boolean formula \( \Phi \) is satisfiable. This completes the proof. \( \square \)

We can compute \( I_\Phi \) in polynomial time. Thus by Lemma 7 and the fact that \( I_\Phi \) is a bipartite graph we have the following.

Theorem 3. The DRG Problem is NP-hard even when restricted to bipartite graphs.

By Corollary 2 and Theorem 3 we have the following.

Corollary 4. The DRG Problem is NP-hard even when restricted to bipartite graphs.

4. Computing double Roman domination number of unicyclic graphs

In this section, we give a linear algorithm that computes the double Roman domination number of unicyclic graphs. Let \( G = (V, E) \) be a graph, let \( u \in V \), let \( v \) be a vertex not in \( V \) and let \( a \in \{ 0, 1, 2, 3 \} \). We define the following.

- \( \gamma_{dr}(G, u = a) = \min \{ w(f) | f \text{ is a DRDF on } G \text{ with } f(u) = a \} \)
- \( \gamma_{dr}(G, u, v) = \min \{ w(f) | f \text{ is a DRDf on } G + uv \text{ with } f(u) = 0 \text{ and } f(v) = 2 \} \)

A \( \gamma_{dr}(G, u = a) \)-function (respectively, \( \gamma_{dr}(G, u, v) \)-function) is a DRDF \( f \) on \( G \) with \( w(f) = \gamma_{dr}(G, u = a) \) and \( f(u) = a \) (respectively, \( w(f) = \gamma_{dr}(G, u, v) \), \( f(u) = 0 \) and \( f(v) = 2 \)).

Lemma 8. Let \( H_1 = (V_1, E_1) \) and \( H_2 = (V_2, E_2) \) be two graphs with \( V_1 \cap V_2 = \emptyset \) such that \( u \in V_1, v \in V_2 \) and a vertex \( w \in V_1 \cup V_2 \). Let \( G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{ uv \}) \). Then, we have the following.

\((i) \) \( \gamma_{dr}(G, u = 0) = \min \{ \gamma_{dr}(H_1, u = 0) + \gamma_{dr}(H_2, u = 0), \gamma_{dr}(H_1, u, w) + \gamma_{dr}(H_2, v = 2) - 2, \gamma_{dr}(H_1 - u) + \gamma_{dr}(H_2, v = 3) \} \),

\((ii) \) \( \gamma_{dr}(G, u = 2) = \gamma_{dr}(H_1, u = 2) + \min \{ \gamma_{dr}(H_2, u, w) - 2, \gamma_{dr}(H_2, v = 2), \gamma_{dr}(H_2, v = 3) \} \),
\(\gamma_{G}(u, v) = \min\{\gamma_{H}(u, v) + \gamma_{H}(v, u)\}\) for the DRDF on \(G + uv\) with both \(u, v \in V\). We define the following.

- \(\gamma_{G}(u, v) = \min\{\gamma_{H}(u, v) + \gamma_{H}(v, u)\}\) is a DRDF on \(G\) with \(f(u) = a\) and \(f(v) = b\).
- \(\gamma_{G}'(u, v) = \min\{\gamma_{H}(u, v) + \gamma_{H}(v, u)\}\) is a DRDF on \(G + uv\) with \(f(u) = 0, f(v) = a\) and \(f(w) = 2\).
- \(\gamma_{G}''(u, v) = \min\{\gamma_{H}(u, v) + \gamma_{H}(v, u)\}\) is a DRDF on \(G + uv\) with \(f(u) = 0, f(v) = a\) and \(f(w) = 3\).

Let \(U\) be a connected unicyclic graph with the unique cycle \(C = v_{0}, \ldots, v_{k-1}, v_{0}\), where \(k \geq 3\). Let \(T(v_{0}, R) = U - v_{0}v_{1}\). Clearly, \(T(v_{0}, R)\) is a tree with the vertex set \(V(U)\).

**Lemma 10.** Let \(U\) be a connected unicyclic graph with the unique cycle \(v_{0}, \ldots, v_{k-1}, v_{0}\) \((k > 2)\) and let \(a, b \in [2, 3]\). Then, \(\gamma_{U}(U) = \min\{\gamma_{G}(T(v_{0}, R), v_{0}) = 0, v_{1} = 0\}, \gamma_{G}(T(v_{0}, R), v_{0}, v_{1} = 2, w) - 1, \gamma_{G}(T(v_{0}, R), v_{1}, v_{2} = 2, w) - 1, \gamma_{G}(T(v_{0}, R), v_{0} = a, v_{1} = b), \gamma_{G}(T(v_{0}, R) - v_{0}, v_{1} = 3), \gamma_{G}(T(v_{0}, R) - v_{1}, v_{0} = 3)\}. \)
Algorithm 4.1: DRD(T)

Input: A rooted tree $T = (V, E)$ with $V = \{v_1, \ldots, v_n\}$, Property 1 and a vertex $w \notin V$.

Output: $(\gamma_{dr}(T, v_1 = 0), \gamma_{dr}(T, v_1 = 2), \gamma_{dr}(T, v_1 = 3), \gamma_{dr}(T, v_1, w), \gamma_{dr}(T - v_1)).$

1 for $i = 1$ to $n$ do
  2 \quad $\gamma_{dr}(v_i) = 0 = \infty$;
  3 \quad $\gamma_{dr}(v_i) = 2$;
  4 \quad $\gamma_{dr}(v_i) = 3$;
  5 \quad $\gamma_{dr}(v_i, w) = \infty$;
  6 \quad $\gamma_{dr}(v_i) = 0$;

7 for $i = n$ to 2 do
  8 \quad Let $v_{j}$ be the parent of $v_i$;
  9 \quad $\gamma_{dr}(v_i) = \min\{\gamma_{dr}(v_{j} - v_{i} = 0) + \gamma_{dr}(v_{i} = 0), \gamma_{dr}(v_{j}, w) + \gamma_{dr}(v_{i} = 2) - \gamma_{dr}(v_{j}) + \gamma_{dr}(v_{i} = 3)\}$;
  10 \quad $\gamma_{dr}(v_i) = 2 = \min\{\gamma_{dr}(v_{j} = 2) + \min\{\gamma_{dr}(v_{j}, w) - 2, \gamma_{dr}(v_{i} = 2), \gamma_{dr}(v_{j} = 3)\}\}$;
  11 \quad $\gamma_{dr}(v_i) = 3 = \min\{\gamma_{dr}(v_{j} = 3) + \min\{\gamma_{dr}(v_{j}, w) - 2, \gamma_{dr}(v_{i} = 2), \gamma_{dr}(v_{j} = 3)\}\}$;
  12 \quad $\gamma_{dr}(v_i, w) = \min\{\gamma_{dr}(v_{j}, w) + \gamma_{dr}(v_{i} = 0), \gamma_{dr}(v_{j}) + \gamma_{dr}(v_{i} = 2) + 2, \gamma_{dr}(v_{j}) + \gamma_{dr}(v_{i} = 3) + 2\}$;
  13 \quad $\gamma_{dr}(v_i) = 0 = \min\{\gamma_{dr}(v_{j} = 0) + \gamma_{dr}(v_{i} = 2) + \gamma_{dr}(v_{j} = 3)\}$;
  14 return $(\gamma_{dr}(v_1 = 0), \gamma_{dr}(v_1 = 2), \gamma_{dr}(v_1 = 3), \gamma_{dr}(v_1, w), \gamma_{dr}(v_1))$;

Proof. Assume that $\gamma = \min\{\gamma_{dr}(T(v_0, R), v_0 = 0, v_1 = 0), \gamma_{dr}(T(v_0, R), v_0, v_1 = 2, w) - 1, \gamma_{dr}(T(v_0, R), v_0 = 2, w) - 1, \gamma_{dr}(T(v_0, R), v_0, v_1 = 3), \gamma_{dr}(T(v_0, R), v_0, v_1 = 0)\}$. Let $f$ be a DRDF on $T(v_0, R)$ with $f(v_0) = f(v_1) = 0$ and $w(f) = \gamma_{dr}(T(v_0, R), v_0, v_1 = 0)$. Then, $f$ is a DRDF on $U$ and so $\gamma_{dr}(U) \leq \gamma_{dr}(T(v_0, R), v_0 = 0, v_1 = 0)$.

Let $f$ be a DRDF on $T(v_0, R) + v_0w$ with $f(v_0) = 0, f(v_1) = f(w) = 2$ and $w(f) = \gamma_{dr}(T(v_0, R), v_0, v_1 = 2, w)$. Then, the restriction of $f$ to $V(T)$ is a DRDF on $U$ and so $\gamma_{dr}(U) \leq \gamma_{dr}(T(v_0, R), v_0 = 1, v_1 = 2)$.

Now we consider the following cases.

- If $f(v_0) = f(v_1) = 0$, then $f$ is a DRDF on $T(v_0, R)$ with $f(v_0) = f(v_1) = 0$ and so $\gamma_{dr}(T(v_0, R), v_0 = 0, v_1 = 0) \leq \gamma_{dr}(U)$.

- Let $f(v_0) = 0$ and $f(v_1) = 1$ and let $g(w) = 1$. Then, $h = f \cup g$ is a DRDF on $T(v_0, R) + v_0w$ with $h(v_0) = 0$ and $h(v_1) = h(w) = 1$ and so $\gamma_{dr}(T(v_0, R), v_0 = 1, v_1 = 3) \leq \gamma_{dr}(U)$.

- Let $f(v_0) = 0$ and $f(v_1) = 2$. Then, the restriction of $f$ to $U(v_0, R) = \gamma_{dr}(T(v_0, R), v_0, v_1 = 3)$.

- Let $f(v_0) = 0$ and $f(v_1) = 2$. Then, the restriction of $f$ to $U(v_0, R) = \gamma_{dr}(T(v_0, R), v_0, v_1 = 2) \leq \gamma_{dr}(U)$.

By Lemma 10, we have $f(v_0), f(v_1) \in \{0, 2, 3\}$. In the following we consider the above cases.

Lemma 11. Let $T$ be a rooted tree with the root $u$, $v \in V(T)$ and a vertex $w \notin V(T)$ and let $(\gamma_{dr}, \gamma_{dr}, \gamma_{dr}, \gamma_{dr}, \gamma_{dr})$ be the output of Algorithm DRDF(T, u, v). Then,

- $\gamma_{dr}(T, u = 0, v = 0) = \gamma_{dr}(T, u = 0, v = 0),$
- $\gamma_{dr}(T, u = 0, v = 0) = \gamma_{dr}(T, u = 0, v = 0),$
- $\gamma_{dr}(T, u = 0, v = 0) = \gamma_{dr}(T, u = 0, v = 0),$
- $\gamma_{dr}(T, u = 0, v = 0) = \gamma_{dr}(T, u = 0, v = 0),$
- $\gamma_{dr}(T, u = 0, v = 0) = \gamma_{dr}(T, u = 0, v = 0).$

Proof. Let $P(T, v, u) = u_0(= v), \ldots, u_k(= u) (k > 0)$ be the shortest path between $v$ and $u$ in $T$. The proof is by induction on $k = |P(T, v, u)|$. Let $k = 1$. So, $u$ is the parent of $v$. Let $T' = T_u - T_v$. So,

- $\gamma_{dr}(T, u = 0, v = 0) = \gamma_{dr}(T, v = 0) + \gamma_{dr}(T, u = 0),$
- $\gamma_{dr}(T, u = 0, v = 0) = \gamma_{dr}(T, v = 0) + \gamma_{dr}(T', u, w).$
Algorithm 4.2: DRD0(T, u, v)

Input: A rooted tree T with root u, a vertex v ∈ V(T) and a vertex w ∈ V(T).

Output: (γdr(T, u = 0, v = 0), γdr(T, u, v = 0, w), γdr(T, u, v = 0, w), γdr(T, u = 2, v = 0), γdr(T, u = 3, v = 0)).

1. Let P(T, u, v) = w_u(= v), . . . , w_k(= u) (k > 0) be the shortest path between u and v in T.
2. T′ = T_u1 − 1
3. γ00 = γdr(T_w0, w0 = 0) + γdr(T′, w1 = 0);
4. γ02 = γdr(Tw0, w0 = 0) + γdr(T′, w1, w);
5. γ03 = γdr(Tw0, w0 = 0) + γdr(T′ - w0 - 1).
6. γ02 = γdr(Tw0, w0 = 0) + γdr(T′, w1 = 2) - 2;
7. γ03 = γdr(Tw0 - w0) + γdr(T′, w1 = 3);
8. for i = 2 to k do
9. α0 = min{γdr(T′, wi + 1) + γ00, γdr(T′, wi + 1) + γ02, γdr(T′ - wi) + γ03};
10. α1 = min{γdr(T′, wi, w1) + γ00, γdr(T′ - wi) + γ02 + 2, γdr(T′ - wi) + γ03 + 2};
11. α2 = γdr(T′ - wi) + min{γ00, γ02, γ03 + 3};
12. α3 = γdr(T′, w1 = 3) + min{γ00 - 2, γ02, γ03};
13. γ00 = α0;
14. γ02 = α1;
15. γ03 = α2;
16. γ03 = α3;
17. return (γ00, γ02, γ03);

Algorithm 4.3: DRD2(T, u, v)

Input: A rooted tree T with root u, a vertex v ∈ V(T) and a vertex w ∈ V(T).

Output: (γdr(T, u = 0, v = 2), γdr(T, u, v = 2, w), γdr(T, u, v = 2, w), γdr(T, u = 2, v = 2), γdr(T, u = 3, v = 2)).

1. Let P(T, u, v) = w_u(= v), . . . , w_k(= u) (k > 0) be the shortest path between u and v in T.
2. T′ = T_u1 − 1
3. γ20 = γdr(T_w0, w0 = 2) + γdr(T′, w1, w) - 2;
4. γ20 = γdr(T_w0, w0 = 2) + γdr(T′ - w1) + 2;
5. γ20 = γdr(T_w0, w0 = 2) + γdr(T′ - w1) + 3;
6. γ22 = γdr(T_w0, w0 = 2) + γdr(T′, w1 = 2);
7. γ22 = γdr(T_w0, w0 = 2) + γdr(T′, w1 = 3);
8. for i = 2 to k do
9. α0 = min{γdr(T′, wi) + γ20, γdr(T′, wi, w) + γ22 - 2, γdr(T′ - wi) + γ23};
10. α2 = min{γdr(T′, wi, w) + γ20, γdr(T′ - wi) + γ22 + 2, γdr(T′ - wi) + γ23 + 2};
11. α3 = γdr(T′ - wi) + min(γ20, γ22, γ23 + 3);
12. α3 = γdr(T′, w1 = 3) + min{γ20 - 2, γ22, γ23};
13. γ20 = α0;
14. γ22 = α1;
15. γ22 = α2;
16. γ22 = α3;
17. return (γ20, γ20, γ20, γ22, γ23);

Algorithm 4.4: DRD0(T, u, v)

Input: A rooted tree T with root u, a vertex v ∈ V(T) and a vertex w ∈ V(T).

Output: (γdr(T, u = 0, v = 0), γdr(T, u, v = 0, w), γdr(T, u, v = 0, w), γdr(T, u = 2, v = 0), γdr(T, u = 3, v = 0)).

1. Let P(T, u, v) = w_u(= v), . . . , w_k(= u) (k > 0) be the shortest path between u and v in T.
2. T′ = T_u1 − 1
3. γ00 = γdr(T_w0, w0 = 0) + γdr(T′, w1 = 0);
4. γ02 = γdr(T_w0, w0 = 0) + γdr(T′, w1, w);
5. γ03 = γdr(Tw0, w0 = 0) + γdr(T′ - w0 - 1) + 3;
6. γ02 = γdr(Tw0, w0 = 0) + γdr(T′, w1 = 2) - 2;
7. γ03 = γdr(Tw0 - w0) + γdr(T′, w1 = 3);
8. for i = 2 to k do
9. α0 = min{γdr(T′, wi) + γ00, γdr(T′, wi) + γ02, γdr(T′ - wi) + γ03};
10. α2 = min{γdr(T′, wi, w) + γ00, γdr(T′ - wi) + γ02 + 2, γdr(T′ - wi) + γ03 + 2};
11. α3 = γdr(T′ - wi) + min{γ00, γ02, γ03 + 3};
12. α3 = γdr(T′, w1 = 3) + min{γ00 - 2, γ02, γ03};
13. γ00 = α0;
14. γ02 = α1;
15. γ03 = α2;
16. γ03 = α3;
17. return (γ00, γ00, γ00, γ00, γ00);

Since k = 1, the for loop of Algorithm DRD0(T, u, v) does not execute. This proves the base case of the induction. Assume that the result is true for any rooted tree T with the root u, v ∈ V(T) and |P(T, u, v)| ≤ m, where m ≥ 1. Let T be a rooted tree with the root u, v ∈ V(T), a vertex w ∈ V(T) and P(T, u, v) = w_u(= v), ..., w_m, w_m+1(= u). Let (γ00, γ02, γ03, γ03) be values of variables (γ00, γ02, γ02, γ03) of Algorithm DRD(T, u, v), respectively, after the iteration of the for loop for each 2 ≤ i ≤ m + 1. Let T_u1 be the rooted subtree of T with the root w_m. Let (α00, α02, α03, α03) be outputs of Algorithms DRD(T, u, v) and DRD(T, u, v), respectively. Clearly, (α00, α00, α00, α02, α03) = (γ00, γ02, γ03, γ03) and (β00, β00, β00, β02, β03) = (γ00, γ02, γ03, γ03).
Algorithm 4.4: DRD3(T, u, v)

Input: A rooted tree T with root u, a vertex v ∈ V(T) and a vertex w ∈ V(T).
Output: γ_{dr}(T, u = 0, v = 3), γ′_{dr}(T, u, v = 3, w), γ''_{dr}(T, u, v = 3, w), γ_{dr}(T, u = 2, v = 3), γ_{dr}(T, u = 3, v = 3).

1. Let P(T, u, v) = w_0(= u), ..., w_k(= u) (K > 0) be the shortest path between u and v in T.
2. T' = T_{w_0} - T_{w_0};
3. γ_{30} = γ_{dr}(T_{w_0}, w_0 = 3) + γ_{dr}(T' - w_1);
4. γ'_{30} = γ_{dr}(T_{w_0}, w_0 = 3) + γ_{dr}(T' - w_1) + 3;
5. γ''_{30} = γ_{dr}(T_{w_0}, w_0 = 3) + γ_{dr}(T' - w_1) + 3;
6. γ_{32} = γ_{dr}(T_{w_0}, w_0 = 3) + γ_{dr}(T', w_1 = 2);
7. γ'_{32} = γ_{dr}(T_{w_0}, w_0 = 3) + γ_{dr}(T', w_1 = 2);
8. for i = 2 to k do
   9.   T' = T_{v_i} - T_{v_{i+1}};
   10.  α_0 = min{γ_{dr}(T', w_i = 0) + γ_{30}, γ_{dr}(T', w_i, w) + γ''_{30} - 2, γ_{dr}(T' - w_i) + γ_{33}};
   11.  α_i = min{γ_{dr}(T', w_i, w) + γ_{30}, γ_{dr}(T' - w_i) + γ_{32} + 2, γ_{dr}(T' - w_i) + γ_{33} + 2};
   12.  γ_{33} = γ_{dr}(T', w_i = 3) + min{γ''_{30} - 3, γ_{32}, γ_{33}};
   13.  γ_{30} = α_0;
   14.  γ'_{30} = α_1;
   15.  γ''_{30} = α_3;
   16.  γ_{32} = α_2;
   17.  γ'_{32} = α_3;
   18.  γ''_{32} = α_3;

return (γ_{30}, γ'_{30}, γ''_{30}, γ_{32}, γ'_{32});

γ′′_{4m}. By the induction hypothesis, we have (β_{00}, β'_{00}, β''_{00}, β_{02}, β_{03}) = (γ_{dr}(T_{w_m}, w_m = 0, v = 0), γ'_{dr}(T_{w_m}, w_m, v = 0, w), γ_{dr}(T_{w_m}, w_m, v = 0, w), γ_{dr}(T_{w_m}, w_m = 2, v = 0), γ_{dr}(T_{w_m}, w_m = 3, v = 0)). Let T' = T - T_{w_m}. Since u is the parent of w_m (≠ v) (i.e., u is adjacent to w_m) in T, we have

• γ_{dr}(T, u = 0, v = 0) = min{γ_{dr}(T', u = 0) + β_{00}, γ_{dr}(T', u, w) + β_{02} - 2, γ_{dr}(T' - u) + β_{03}};
• γ'_{dr}(T, u, v = 0) = min{γ_{dr}(T', u, w) + β_{00}, γ_{dr}(T' - u) + β_{02} - 2, γ_{dr}(T' - u) + β_{03}};
• γ''_{dr}(T, u, v = 0) = min{γ_{dr}(T' - u) + β_{02} - 2, γ_{dr}(T' - u) + β_{03}};
• γ_{dr}(T, u = 2, v = 0) = min{γ_{dr}(T', u = 2) + β'_{00} - 2, γ_{dr}(T', u = 2) + β_{02}, γ_{dr}(T', u = 2) + β_{03}};
• γ_{dr}(T, u = 3, v = 0) = min{γ_{dr}(T', u = 3) + β'_{00} - 3, γ_{dr}(T', u = 3) + β_{02}, γ_{dr}(T', u = 3) + β_{03}}.

This completes the proof. □

Similar to Lemma 11 we have the following results.

Lemma 12. Let T be a rooted tree with the root u, v ∈ V(T) and a vertex w ∉ V(T) and let (γ_{20}, γ'_{20}, γ''_{20}, γ_{22}, γ''_{22}) be the output of Algorithm DRD2(T, u, v). Then,

• γ_{20} = γ_{dr}(T, u = 0, v = 2);
• γ'_{20} = γ_{dr}(T, u, v = 2, w);
• γ''_{20} = γ_{dr}(T, u, v = 2, w);
• γ_{22} = γ_{dr}(T, u = 2, v = 2);
• γ''_{22} = γ_{dr}(T, u = 2, v = 2).

Lemma 13. Let T be a rooted tree with the root u, v ∈ V(T) and a vertex w ∉ V(T) and let (γ_{30}, γ'_{30}, γ''_{30}, γ_{32}, γ''_{32}) be the output of Algorithm DRD3(T, u, v). Then,

• γ_{30} = γ_{dr}(T, u = 0, v = 3);
• γ'_{30} = γ_{dr}(T, u, v = 3, w);
• γ''_{30} = γ_{dr}(T, u, v = 3, w);
• γ_{32} = γ_{dr}(T, u = 2, v = 3);
• γ'_{32} = γ_{dr}(T, u = 2, v = 3);
• γ''_{32} = γ_{dr}(T, u = 2, v = 3).

Theorem 4. There is a linear algorithm that computes the double Roman domination number of a given unicyclic graph.

Proof. Let U be a connected unicyclic graph with the unique cycle v_0, ..., v_{k-1}, v_0 such that a vertex w ∉ V(U). By Lemma 10, γ_{dr}(U) = min{γ_{dr}(T(v_0, R), v_0 = 0, v_1 = 0), γ'_{dr}(T(v_0, R), v_0, v_1 = 2, w) - 2, γ''_{dr}(T(v_0, R), v_1, v_0 =
follows from Lemmas 9, 11, 12 and 13 we can compute \( \gamma \).

5. Computing domination number of unicyclic graphs

Let \( H \) be a graph such that

\[
\alpha(H) = \min\{\alpha(S) : S \text{ is a DS on } G \text{ such that } u \notin S\},
\]

\[
\gamma(H, u = 0) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \notin S\};
\]

\[
\gamma(H, u = 1) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \in S\}.
\]

Similar to Lemma 8 we have the following.

**Lemma 14.** Let \( H_1 = (V_1, E_1) \) and \( H_2 = (V_2, E_2) \) be two graphs with \( V_1 \cap V_2 = \emptyset \) such that \( u \in V_1 \) and \( v \in V_2 \). Let \( G = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv\}) \). Then,

\[
\gamma(G, u = 0) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \notin S\},
\]

\[
\gamma(G, u = 1) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \in S\}.
\]
Lemma 16. Let $T$ be a connected unicyclic graph with the unique cycle $C = v_0, \ldots, v_{k-1}, v_0$ where $k \geq 3$. Recall that $T(v_0, R) = U - v_0 v_1$. Let $G = (V, E)$ be a graph with $u, v \in V$ and a vertex $w \notin V$. We define the following.

\begin{itemize}
  \item $\gamma(G, u = 0, v = 0) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \notin S \text{ and } v \notin S\}$,
  \item $\gamma(G, u = 0, v = 1) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \notin S \text{ and } v \in S\}$,
  \item $\gamma(G, u = 1, v = 0) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \in S \text{ and } v \notin S\}$,
  \item $\gamma(G, u = 1, v = 1) = \min\{|S| : S \text{ is a DS on } G \text{ such that } u \in S \text{ and } v \in S\}$,
  \item $\gamma'(G, u, v = 0, w) = \min\{|S| : S \text{ is a DS on } G + uw \text{ such that } w \in S \text{ and } u, v \notin S\}$,
  \item $\gamma'(G, u, v = 1, w) = \min\{|S| : S \text{ is a DS on } G + uw \text{ such that } v, w \in S \text{ and } u \notin S\}$.
\end{itemize}

Similar to Lemma 10 we have the following.

Lemma 17. Let $T$ be a rooted tree with the root $u, v \in V(T)$ and $w \notin V(T)$ and let $(\gamma_{00}, \gamma_{01}, \gamma_{01})$ be the output of Algorithm D0$(T, u, v)$. Then,

\begin{itemize}
  \item $\gamma_{00} = \gamma(T, u = 0, v = 0)$,
  \item $\gamma'_{00} = \gamma'(T, u, v = 0, w)$,
  \item $\gamma_{01} = \gamma(T, u = 1, v = 0)$.
\end{itemize}

Lemma 18. Let $T$ be a rooted tree with the root $u, v \in V(T)$ and $w \notin V(T)$ and let $(\gamma_{10}, \gamma'_{10}, \gamma_{10})$ be the output of Algorithm D1$(T, u, v)$. Then,

\begin{itemize}
  \item $\gamma_{10} = \gamma(T, u = 0, v = 1)$,
  \item $\gamma'_{10} = \gamma'(T, u, v = 1, w)$,
  \item $\gamma_{10} = \gamma(T, u = 1, v = 1)$.
\end{itemize}

Similar to Theorem 4 we have the following.

Theorem 5. There is a linear algorithm that computes the domination number of a given unicyclic graph.
By Theorems 4 and 5 we obtain the following.

**Theorem 6.** There is a linear algorithm that decides whether a given unicyclic graph is double Roman.

**CRediT authorship contribution statement**

**Abolfazl Poureidi:** Conceptualization, Methodology, Writing - original draft preparation, Visualization, Investigation, Writing - reviewing and editing. **Nader Jafari Rad:** Conceptualization, Methodology, Writing - original draft preparation, Visualization, Investigation, Writing - reviewing and editing.

**References**