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Stability and stabilisation of switched time-varying delay systems: a multiple discontinuous Lyapunov function approach

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ABSTRACT
In this paper, stability and stabilisation of switched time-varying delay systems are investigated. By developing a multiple discontinuous Lyapunov function approach and using the mode-dependent average dwell time (MDADT) switching signal, sufficient conditions are first proposed to guarantee the exponential stability of the switched system with stable and unstable subsystems where the time-varying delay is considered in the states. Relaxed stability conditions are also obtained by using a form of the Bessel–Legendre inequality. Then stabilisation of the proposed system is presented by designing a sliding mode controller. Sufficient conditions and sufficient relaxed conditions of a reduced-order sliding mode dynamics are derived, and desired sliding surfaces in the form of linear matrix inequalities (LMIs) are also given. The obtained conditions are checked by using the combined genetic algorithm and convex programming techniques. Moreover, some numerical and practical examples are given to illustrate the effectiveness of the obtained theoretical results.

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Switched systems; time-varying delay; multiple discontinuous Lyapunov function (MDLF) sliding mode control; mode dependent average dwell time (MDADT)

1. Introduction
In recent years, the switched systems, as a significant class of hybrid systems have been widely studied, and many impressive results have been reported (Briat, 2017; Wang et al., 2019a; Wu et al., 2017). Many real-world systems can be modelled as switched systems, multi-agent systems (Dong et al., 2017), control in networks (Su et al., 2018), chemical process systems (Niu et al., 2015), flight control systems (Lian et al., 2016), and communication systems (Su et al., 2017). For the switched systems, because of the complicated behaviour caused by the interaction between the continuous dynamics and discrete switching, the problem of stability is more problematic to study. Mainly in the continuous-time stability, analysis tools in the Lyapunov sense include common the Lyapunov function (CLF) approach and multiple Lyapunov function (MLF) approach. In practice, it is impossible to always ensure the stability of a switched system under arbitrary switching signals, i.e. not all subsystems can share a CLF. Consequently, it is reasonable to consider stability with constrained switching signals. MLF approaches not only have flexibility but also can reduce the conservativeness to analyse switched systems using constrained switching (Liberzon & Morse, 1999). Recently, (Zhao et al., 2017) developed multiple discontinuous Lyapunov function (MDLF) approach for stability analysing with a designed switching strategy where fast switching and slow switching are respectively applied to unstable and stable subsystems, and each Lyapunov function is piecewise continuous throughout the dwell time for an activated system mode. The tighter bounds on the average dwell time or mode-dependent average dwell time can be obtained by using the proposed MDLF approach.

Switching signals as an essential factor in switched systems, determine the dynamic behaviour of a switched system in most cases. Researches in the field of stability analysis with constrained switching have been mostly motivated by state-dependent switching, time-dependent switching, and a combination of them. However, the time-dependent switching strategy has been recognised to be more flexible and efficient than state-dependent switching in stability analysis (Zhao et al., 2017). Usual time-dependent switching signals consist of dwell time (DT) switching signal,
average dwell time (ADT) switching signal and mode-dependent average dwell time (MDADT) switching signal, which MDADT is a special class of ADT switching signal, and the only difference is that a separate ADT is considered for each subsystem. In (Xiang & Xiao, 2014), using DT, a sufficient condition ensuring the asymptotic stability of switched systems with all subsystems unstable, has been proposed. Stability analysis for switched positive linear systems with average dwell time switching is revisited and discussed in both continuous-time and discrete-time contexts (Yin et al., 2017b). Some extended results on DT, and ADT approach applied to switched systems can be referred to (Briat, 2017; Etienne et al., 2019; Jin et al., 2018; Yuan & Wu, 2015). In practice, the MDADT approach is more applicable than DT, and ADT approaches, as the switching rule allows each subsystem to specify its individual ADT (Zhao et al., 2012). Some results on MDADT approach applied to switched systems are presented in (Fei et al., 2017; Wang et al., 2019a; Zhai et al., 2016; Zhao et al., 2017).

On the other hand, a time-delay widely exists in many control systems, such as aircraft, chemical or process control systems, and communication networks, either in the state, the control input, or the measurements (Yin et al., 2017b). Hence, the study of time-delay systems has been of great interest in many branches of science and engineering applications. For instance, stability analysis of time-varying delay systems, using free-matrix-based integral inequality is provided in (Zeng et al., 2015) and a novel Lyapunov functional using matrix-refined-function for the stability of time-varying delay systems is provided in (Lee & Park, 2017). In the stability analysis of time-varying delay systems, Lyapunov–Krasovskii functional (LKF) is commonly used. LKF is often sufficient but not necessary, which brings conservativeness inevitably. In general, the conservativeness is caused by two factors: the selection of appropriate LKF and the estimation of \( \int_a^b x^T(s)Yx(s)ds \). In creating LKF, we should select more vectors that can reflect the state variables of the system, such as adding some integral terms or quadratic integral terms to LKF or dividing the delay into several parts by time delay decomposition, which can generate more state vectors of the system. Many studies have been done on time-varying delay systems to achieve relaxed stability conditions (Lee et al., 2017; Lee et al., 2018; Park et al., 2015; Wang et al., 2019b). Recently, (Lee et al., 2018) have developed new methods to obtain less conservative delay-dependent stability conditions. The main idea of these methods is to use an equivalent form of the Bessel–Legendre inequality to find a precise bound of the quadratic integral function in the LKF. Switched systems with time delay have a strong engineering background in power systems and multi-rate control systems (Mahmoud, 2010). Therefore, it is exciting and challenging to investigate the stability problem of switched time-delay systems. From the studies conducted, we can mention (Zamani et al., 2013; Zamani et al., 2014; Zamani et al., 2015; Zamani & Shafiee, 2014; Zamani & Shafiee, 2015) in singular systems, which is provided the stability analysis of a various class of singular switched delayed systems. Moreover, in (Ma et al., 2016; Zong et al., 2015), the problem of finite-time \( H_{\infty} \) control is addressed for a class of switched nonlinear systems with time delay.

Since the 1950s, sliding mode control has proven to be an effective robust control scheme for nonlinear systems and imperfectly modelled systems. In the past decade, sliding mode control has been successfully applied to a variety of practical engineering systems such as robot manipulators, aircraft, spacecraft, underwater vehicles, electrical motors, power systems, and automotive engines (Azar & Zhu, 2015; Utkin et al., 2017). The main idea of sliding mode control is to utilise a discontinuous control to force the system state trajectories to some predefined sliding surfaces on which the system has desired properties such as stability, disturbance rejection capability, and tracking ability (Shtessel et al., 2014). Many significant results have been reported for this kind of control scheme. To express a few, in particular, sliding mode control has been investigated for uncertain systems (Ginoya et al., 2014), stochastic systems (Wu et al., 2017), Markovian jump systems (Li et al., 2016) and singular systems (Wang et al., 2018). There are some papers which have presented sliding mode control for the continuous-time switched system (Wu et al., 2011; Wu & Lam, 2008; Yin et al., 2017a), and discrete-time switched system (Su et al., 2017; Su et al., 2018), but switching signal with MDADT property, time-varying delay, and consisting unstable subsystems have not been fully investigated.

In this paper, we are concerned with the problem of stability analysis and sliding mode control of a continuous-time switched system, including stable and unstable subsystems and time-varying delay in
the states. By developing a multiple discontinuous Lyapunov function approach and using the mode-dependent average dwell time (MDADT) switching, a set of sufficient conditions is first proposed to guarantee the exponential stability of the unforced system. Here, to obtain less conservative conditions, we introduce relaxed stability conditions by using an equivalent form of the Bessel–Legendre inequality. A set derived, meanwhile, these obtained conditions are not expressed in terms of LMIs; therefore we use the combined genetic algorithm and convex programming techniques, to cast them into minimisation problems subject to LMI constraints (Lam & Seneviratne, 2007), which can be solved numerically with no trouble. Then, the reaching motion is synthesised such that the trajectories of the resulting closed-loop system can be driven onto prescribed sliding surfaces and sustained there for all subsequent time. As a final point, a numerical example is given to show the effectiveness of the obtained results.

The rest of this paper is organised as follows. Some necessary concepts of switched delay systems are reviewed in Section 2. In Section 3, exponential stability conditions for autonomous switched delay systems with MDADT switching signals are first established. These conditions are relaxed by using the equivalent form of the Bessel–Legendre inequality. Our main results of the sliding mode control problem present in Section 4. A Numerical example is given in Section 5, and the paper is concluded in Section 6.

Notations: The notations used all over the paper are as follows. The superscript “T” stands for matrix transposition; \( \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \) stand for n-dimensional Euclidean space and the space of \( m \times n \) matrices with real entries; the notation \( P > 0 \) refers to a real symmetric and positive definite; \( I \) and \( 0 \) denote the identity and zero matrices, respectively; \( \text{diag}(\cdot) \) stands for a block-diagonal matrix; \( \min(P) \) refers to the minimum eigenvalue of symmetric matrix \( P \). A function \( \alpha : [0, \infty) \to [0, \infty) \) is of class \( \mathcal{K} \) if it is continuous, strictly increasing, and \( \alpha(0) = 0 \). Class \( \mathcal{K}_\infty \) denotes the subset of \( \mathcal{K} \) composed of all those boundless functions. In symmetric block matrices or lengthy matrices, we use a character (*) to characterise a term that is induced by symmetry. Matrices, if their dimensions are not clearly presented, are assumed compatible for algebraic operations. \( || \cdot || \) denotes the Euclidean norm.

2. System description and preliminaries

Consider a class of switched time-varying delay systems as following equation:

\[
\dot{x}(t) = A(\beta)x(t) + A_d(\beta)x(t - d(t)) + B(u(t, \beta)) + F(\beta)f(t)
\]

\[
x(t) = \phi(t), \ t \in [-d, 0],
\]

where \( x(t) \in \mathbb{R}^n, u(t, \beta) \in \mathbb{R}^m, \) and \( f(t) \in \mathbb{R}^p \) denote the state, the control input, and the nonlinearity, respectively. \( \{(A(\beta), A_d(\beta), F(\beta)) : \beta \in \mathcal{S}\} \) is a family of matrices parameterised by an index set \( \mathcal{S} = \{1, 2, \ldots, m\} \), in which \( m > 1 \) represents the number of sub-systems. \( B \) is the input matrix and assumed to be of full column rank. In here, \( \mathcal{S} = \mathcal{G} \cup \mathcal{U} \) where \( \mathcal{G} \) and \( \mathcal{U} \) signify the sets composed of stable subsystems and unstable subsystems, respectively and \( \beta(t) : [0, \infty) \to \mathcal{S} \) represents a switching signal, which is a piecewise constant function of time \( t \) that might depend on \( t \) or \( x(t) \) or both. From now on, at a given time \( t \), the value of \( \beta(t) \), denoted by \( \beta \) for simplicity. In addition, \( \phi(t) \) is an initial vector-valued continuous function on \([-d, 0] \) for a known constant \( d > 0 \), \( d(t) \) denotes the time-varying delay satisfying either \( (A1) \) or \( (A2) \) as stated below:

\[(A1)0 \leq d(t) \leq d, \ \dot{d}(t) \leq \sigma,
\]

\[(A2)0 \leq d(t) \leq d, \ \sigma_1 \leq \dot{d}(t) \leq \sigma_2 \leq 1.
\]

For the nonlinearity \( f(t) \), we suppose that

\[||F(\beta)f(t) \leq \delta(\beta)|| \forall \beta \in \mathcal{S},
\]

where \( \delta(\beta) > 0 \) are scalars. In accordance with the switching signal \( \beta \), we have the switching sequence \( \{(\beta_0, t_0), (\beta_1, t_1), (\beta_2, t_2), \ldots, (\beta_k, t_k), \ldots, |\beta_k \in \mathcal{S}, k = 0, 1, 2, \ldots\} \) with \( t_0 = 0 \), means that the \( \beta_k \)-th sub-system is activated when \( t \in [t_k, t_{k+1}) \). In (1), when \( B = 0 \) the following free system is given:

\[
\dot{x}(t) = A(\beta)x(t) + A_d(\beta)x(t - d(t))
\]

\[
x(t) = \phi(t), \ t \in [-d, 0].
\]

It should be emphasised from the theory of delay differential equations (Hale et al., 1993) that the existence of the solutions of a non-switched linear delay system is guaranteed by a continuous and piecewise differentiable initial condition. Consider the following non-switched delay system:

\[
\dot{x}(t) = Ax(t) + A_dx(t - d)
\]
\[ x(\theta) = \phi(\theta), \theta \in [-d, 0]. \]

when \( \phi(0) \) is continuous then there exists a unique solution \( x(\phi) \) defined on \([-d, \infty)\) that coincides with \( \phi \) on \([-d, 0]\). By the Lagrange's formula, this solution is given by

\[
x(t) = \exp^{At}x(0) + \int_0^t \exp^{A(t-\theta)A_d}x(\theta - d)d\theta \\
= \exp^{At}x(0) + \int_{-d}^{t-d} \exp^{A(t-\theta-d)A_d}x(\theta)d\theta.
\]

This is carried over to linear switched-delay systems since the state does not experience any jump at the switching instants, based on (Mahmoud, 2010). Now assume \( t \in [t_k, t_{k+1}) \), for a switched time-varying delay systems, we have

\[
\exp^{-A(\beta(t))t}x(t) = \exp^{-A(\beta(t))t}A(\beta(t))x(t) \\
+ \exp^{-A(\beta(t))t}A_d(\beta(t))x(t - d(t)),
\]

which yields

\[
\exp^{-A(\beta(t))t}x(t) - \exp^{-A(\beta(t))t}A(\beta(t))x(t) \\
= \exp^{-A(\beta(t))t}A_d(\beta(t))x(t - d(t))
\]

Therefore,

\[
\frac{d}{dt}(\exp^{-A(\beta(t))t}x(t))) \\
= \exp^{-A(\beta(t))t}A_d(\beta(t))x(t - d(t))
\]

Consider that \( A(\beta(t)) \) is a constant matrix for \( t \in [t_k, t_{k+1}) \). By integrating in \([t_k, t]\), we get

\[
x(t) = \exp^{-A(\beta(t))t}x(t_k) \\
+ \int_{t_k}^{t} \exp^{A(\beta(t))(t-\theta)}A_d(\beta(t))x(\theta - d(\theta))d\theta,
\]

where

\[
x(t_k) = \exp^{-A(\beta(t_{k-1}))t_k}x(t_{k-1}) \\
+ \int_{t_k}^{t_k} \exp^{A(\beta(t_{k-1}))\theta-\theta)}A_d(\beta(t_{k-1})) \\
x(\theta - d(\theta))d\theta,
\]

and yields

\[
x(t) = \exp^{-A(\beta(t))t}\exp^{-A(\beta(t_{k-1}))t_k}x(t_{k-1}) \\
+ \exp^{-A(\beta(t))t} \int_{t_k}^{t} \exp^{A(\beta(t_{k-1}))\theta-\theta)}A_d(\beta(t_{k-1}))(\theta - d(\theta))d\theta \\
+ \int_{t_k}^{t} \exp^{A(\beta(t_{k-1}))\theta-\theta)}A_d(\beta(t_k))(\theta - d(\theta))d\theta,
\]

where

\[
x(t_{k-1}) = \exp^{-A(\beta(t_{k-2}))t_{k-1}}x(t_{k-2}) \\
+ \int_{t_{k-2}}^{t_{k-1}} \exp^{A(\beta(t_{k-2}))\theta-\theta)}A_d(\beta(t_{k-2})) \\
x(\theta - d(\theta))d\theta,
\]

and again yields,

\[
x(t) = \exp^{-A(\beta(t))t}\exp^{-A(\beta(t_{k-1}))t_k}x(t_{k-1}) \\
+ \exp^{-A(\beta(t))t} \int_{t_{k-1}}^{t} \exp^{A(\beta(t_{k-1}))\theta-\theta)}A_d(\beta(t_{k-1}))(\theta - d(\theta))d\theta \\
+ \int_{t_{k-1}}^{t} \exp^{A(\beta(t_{k-1}))\theta-\theta)}A_d(\beta(t_k))(\theta - d(\theta))d\theta.
\]

By a recursive procedure, and since the state does not experience any jump at the switching instants and \( d(t) \) is a continuous positive bounded function, this solution is given by the following.

\[
x(t) = \prod_{i=1}^{k} \exp^{-A(\beta(t_{i}))t}x(0) + \exp^{-A(\beta(t_{k}))t} \\
\times \sum_{i=1}^{k} \left( \prod_{j=i}^{k-1} \exp^{-A(\beta(t_{j}))t_{j+1}} \int_{t_{j-1}}^{t_{j}} \exp^{A(\beta(t_{j-1}))\theta-\theta)}A_d(\beta(t_{j-1}))(\theta - d(\theta))d\theta \right) \\
+ \int_{t_{k}}^{t} \exp^{A(\beta(t_{k}))\theta-\theta)}A_d(\beta(t_k))(\theta - d(\theta))d\theta.
\]

Now, considering above systems, the following definitions and lemmas are made through the paper.
Definition 1: (Liberzon & Morse, 1999) Switched system (3) is said to be exponentially stable under the switching signal $\beta(t)$ if there exist constants $\varepsilon > 0$ and $k > 0$ such that, for any compatible initial condition $\phi(t)$ the solution $x(t)$ to the switched system (3) satisfies

$$||x(t)|| \leq ke^{-\varepsilon(t-t_0)}||x(t_0)|| \quad \forall t \geq t_0.$$  

Definition 2: (Zhao et al., 2012): For a switching signal $\beta(t)$ and any time interval $[t_1, t_2]$, $N_{\beta i}(t_1, t_2)$ represents the number of times that the $i$th subsystem is activated, and $T_i(t_1, t_2)$ represents the sum of the running time of the $i$th subsystem, $i \in S$. We say that $\beta(t)$ has a mode-dependent average dwell (MDADT) time $\tau_{ai}$ if there exist constants $N_0$ and $\tau_{ai}$ such that:

$$N_{\beta i}(t_1, t_2) \leq N_0 + \frac{T_i(t_1, t_2)}{\tau_{ai}}, \forall t_2 \geq t_1 \geq 0.$$ 

Definition 3: (Zhao et al., 2017): For a switching signal $\beta(t)$ and any time interval $[t_1, t_2]$, $N_{\beta i}(t_1, t_2)$ represents the number of times that the $i$th subsystem is activated, and $T_i(t_1, t_2)$ represents the sum of the running time of the $i$th subsystem, $i \in S$. If there exist constants $N_0$ and $\tau_{ai}$ such that:

$$N_{\beta i}(t_1, t_2) \geq N_0 + \frac{T_i(t_1, t_2)}{\tau_{ai}}, \forall t_2 \geq t_1 \geq 0,$$

then we say that the constant $\tau_{ai}$ is the MDADT of the fast switching signal $\beta(t)$.

For stability analysis, for any time interval $[t_k, t_{k+1}]$ between two consecutive switching instances, the interval is divided into $G_{\beta(t_k)}$ divisions, and the length of each section is denoted by $H_{\beta(t_k)}^i, \forall i \in \{1, 2, \ldots, G_{\beta(t_k)}\}$ as stated in (Zhao et al., 2017). For this purpose, define the $f_{\beta i}^t(\beta(t)) = \sum_{j=1}^{G_{\beta(t_k)}} H_{\beta(t_k)}^i$, where $f_{\beta i}^t(\beta(t_k)) = 0, \forall i \in \{0, 1, 2, \ldots, G_{\beta(t_k)}\}$, and denote $L_{\beta i}(t_k) = [t_k + f_{\beta i}^t(\beta(t_k)), t_k + f_{\beta i}^t(\beta(t_k)) + 1], \forall i \in \{0, 1, 2, \ldots, G_{\beta(t_k)} - 1\}$. Then, the time interval $[t_k, t_{k+1})$ can be described as $[t_k, t_{k+1}) = \bigcup_{i \in \{0, 1, 2, \ldots, G_{\beta(t_k)} - 1\}} L_{\beta i}(t_k)$.

Lemma 1: (Zhao et al., 2017): consider the following switched system,

$$\dot{x}(t) = f_{\beta(t)}(x(t)). \quad (4)$$

If for $\beta \in \mathcal{G}$, there exist scalars $\alpha_\beta > 0, 0 < \eta_\beta \leq 1, \mu_\beta > 1$, satisfying $(\eta_\beta)^{G_{\beta}-1}\mu_\beta > 1$ and also for $\beta \in \mathcal{U}$, there exist scalars $\alpha_\beta < 0, 0 < \eta_\beta < 1, \mu_\beta < 1$. If there exist a set of continuously differentiable non-negative functions $V_{\beta i}(x(t)), \beta \in \mathcal{S}, i \in \mathcal{R}_{\beta(t_k)}$, and two class $\mathcal{K}_\infty$ functions $\kappa_1$ and $\kappa_2$, such that for all $\beta \in \mathcal{S}$,

$$\kappa_1(x(t)) \leq V_{\beta i}(x(t)) \leq \kappa_2(x(t)),$$ 

and for all $\beta \in \mathcal{U}$,

$$\dot{V}_{\beta i}(x(t)) + \alpha_\beta V_{\beta i}(x(t)) \leq 0,$$ 

where $V_{\beta i}(x(t)) = x(t)^T P_i x(t), i \in \mathcal{R}_{\beta(t_k)}$, $P_i$ and $F_i$ are the Lyapunov matrices and $\alpha_\beta$ is the Lyapunov exponent of $\beta$. Then, the fast subsystem is exponentially stable under $\beta(t)$ if there exist constants $\kappa_1$ and $\kappa_2$, such that

$$V_{\beta i}(x(t)) \leq \alpha_\beta V_{\beta i}(x(t)) \leq \kappa_2(x(t)),$$  

then switched system (4) is exponentially stable for any MDADT switching signals satisfying

$$\left\{ \begin{array}{l}
\tau^a_\beta \geq \frac{\ln \mu_\beta + (G_{\beta} - 1) \ln \eta_\beta}{\alpha_\beta}, \beta \in \mathcal{G}, \\
\tau^a_\beta \leq \frac{\ln \mu_\beta + (G_{\beta} - 1) \ln \eta_\beta}{\alpha_\beta}, \beta \in \mathcal{U}.
\end{array} \right. \quad (5)$$

According to Lemma 1, the switching low depends on the switching frequency to ensure system stability. Switching frequency in stable subsystems should not be less than a certain value (this value is denoted by $\tau^{\text{as}}_\beta$ for $\beta \in \mathcal{G}$). For unstable subsystems, the switching frequency must be greater than a certain value (this value is denoted by $\tau^{\text{as}}_\beta$ for $\beta \in \mathcal{U}$). These two values are achieved with the help of slow switching and fast switching, which are expressed in Definition 2 and Definition 3, respectively. It should be noted that in this method, it is sufficient to know the stability and instability of the active subsystem, which in itself is an advantage.

It is worth mentioning, since $(\eta_\beta)^{G_{\beta}-1}\mu_\beta > 1$ for $\beta \in \mathcal{G}$, then $\ln (\eta_\beta)^{G_{\beta}-1}\mu_\beta > 0$. So, we have $\ln \mu_\beta + (G_{\beta} - 1) \ln \eta_\beta > 0$. On the other hand, for $\beta \in \mathcal{G}$, $\alpha_\beta > 0$. So, $\tau^{\text{as}}_\beta > 0$ for $\beta \in \mathcal{G}$.

For $\beta \in \mathcal{U}$, we have $0 < \eta_\beta \leq 1, 0 < \mu_\beta < 1$. So, $\ln \mu_\beta < 0$ and $\ln \mu_\beta + (G_{\beta} - 1) \ln \eta_\beta < 0$. Hence, we have $\ln \mu_\beta + (G_{\beta} - 1) \ln \eta_\beta < 0$. On the other hand, for $\beta \in \mathcal{U}, \alpha_\beta < 0$. So finally, $\tau^{\text{as}}_\beta > 0$ for $\beta \in \mathcal{U}$.
Lemma 2: (Wu et al., 2018): For a given matrix $R = R^T > 0$, there exists a matrix $X$ such that the inequality

$$-\int_a^b \dot{X}^T(s) R \dot{X}(s) ds \leq \xi_N(t)^T \mathcal{Z}(X) \xi_N(t),$$

that is known as affine Bessel–Legendre inequality, holds for all continuously differentiable function $X(0)$ in $[a, b] \rightarrow \mathbb{R}^n$ and all integer $N \in \mathbb{N}$, where

$$\xi_N(t) = \begin{cases} 
\left[ \begin{array}{c}
X^T(b) \\
X^T(a) \\
\ldots \\
\frac{1}{b-a} \Omega_{N-1}^T
\end{array} \right] , & N = 0 \\
\left[ \begin{array}{c}
X^T(b) \\
X^T(a) \\
\ldots \\
\frac{1}{b-a} \Omega_{N}^T
\end{array} \right] , & N > 0
\end{cases}$$

$$\Omega_k = \int_a^b L_k(s) R x(s) ds,$$

$$L_k(u) = (-1)^K \sum_{l=0}^{K} \left[ (-1)^l \begin{pmatrix} k \\ l \end{pmatrix} \left( \frac{u-a}{b-a} \right)^l \right]$$

$$\mathcal{Z}(X) = X H_N + H_N^T X^T - (b-a)x \dot{R} X^T,$$

$$\dot{R} = \text{diag} \left( R^{-1}, \frac{1}{3} R^{-1}, \ldots, \frac{1}{2N+1} R^{-1} \right),$$

$$H_N = \begin{bmatrix} \Gamma_{N}^T(0) & \Gamma_{N}^T(1) & \ldots & \Gamma_{N}^T(N) \end{bmatrix},$$

$$\Gamma_{N}(K) = \begin{cases} 
[I - I], & N = 0 \\
[I - (-1)^{K+1} I] & \gamma_{NK}^0 I \\
\ldots & \gamma_{NK}^{N-1} I
\end{cases},$$

$$\gamma_{NK}^i = \begin{cases} 
-(2i+1) (1 - (-1)^{K+i}), & i \leq K \\
0, & i \geq K + 1
\end{cases}.$$

In the continuation, to demonstrate the stability of these particular systems, motivated by the method presented in (Zhao et al., 2017) an approach is given to address many of the problems and challenges associated with delayed switched systems. Then, we decided to relax these stability conditions of the delayed switching systems, using Bessel–Legendre inequality in constructing LKF. In the second part, a sliding mode controller is designed for these specific systems.

3. Stability analysis

Now, we establish exponential stability conditions for the switched system (3) by using the mode-dependent average dwell time approach and the multiple discontinuous Lyapunov function technique. The main results are given as the following theorems.

Theorem 1: Consider the switched system (3). Suppose (A1) holds and for given scalars $\alpha_\beta > 0$, $0 < \eta_\beta \leq 1$, $\mu_\beta > 1$, $\beta \in \mathcal{G}$ satisfying $(\eta_\beta)\gamma_{\beta}^{-1} \mu_\beta > 1$, and $\alpha_\beta < 0$, $0 < \eta_\beta \leq 1$, $0 < \mu_\beta < 1$, $\beta \in \mathcal{U}$ if there exist matrices $P^i_\beta > 0$, $Q^i_\beta > 0$, $R^i_\beta > 0$ and $X^i_\beta, \gamma^i_\beta, \beta \in \mathcal{S}$, $i \in \mathcal{R}_{\beta}(I_2)$, such that, $\forall i \in \mathcal{R}_{\beta}(I_2)$,

$$\begin{align*}
&\begin{cases}
\dot{P}^i_\beta \leq \eta_\beta P_{I-1}^i_\beta, & \beta \in \mathcal{S}, i \neq 0 \\
\dot{Q}^i_\beta \leq \eta_\beta Q_{I-1}^i_\beta, & \beta \in \mathcal{S}, i \neq 0 \\
\dot{R}^i_\beta \leq \eta_\beta R_{I-1}^i_\beta, & \beta \in \mathcal{S}, i \neq 0 \\
\end{cases} \quad (6.a) \\
&\begin{cases}
\dot{P}^\beta_\gamma \leq \mu_\beta P_{\gamma}^{-1}_\beta, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
\dot{Q}^\beta_\gamma \leq \mu_\beta Q_{\gamma}^{-1}_\beta, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
\dot{R}^\beta_\gamma \leq \mu_\beta R_{\gamma}^{-1}_\beta, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
\end{cases} \quad (6.b)
\end{align*}$$

and $\forall \beta \in \mathcal{S}$ and $\forall i \in \mathcal{R}_{\beta}(I_2)$,

$$\begin{pmatrix}
\dot{\pi}^1_1(\beta) & \dot{\pi}^1_{12}(\beta) & dA^T(\beta) R_\beta^i & dX_\beta^i \\
\ast & \dot{\pi}^1_{22}(\beta) & dA_d^i(\beta) R_\beta^i & d\gamma_\beta^i \\
\ast & \ast & -dR_\beta^i & 0 \\
\ast & \ast & \ast & -d\alpha_\beta d R_\beta^i
\end{pmatrix} < 0$$

where

$$\begin{align*}
\dot{\pi}^1_1(\beta) &= \left( P^i_\beta A(\beta) + X^i_\beta \right)^T + Q^i_\beta + \alpha_\beta I^p_\beta \\
\dot{\pi}^1_{12}(\beta) &= P^i_\beta A_d(\beta) + \gamma^i_\beta - X^i_\beta \\
\dot{\pi}^1_{22}(\beta) &= - (1 - \sigma) \dot{\gamma}_\beta^i - \gamma^i_\beta - \gamma^i_{\beta} T
\end{align*}$$

then the switched system (3) is exponentially stable for any MDADT switching signals satisfying (5).

Proof: Consider the multiple discontinuous Lyapunov functional candidate for the switched system (3) with the following form:

$$V^i_\beta(x(t)) = V^i_{\beta 1}(x(t)) + V^i_{\beta 2}(x(t)) + V^i_{\beta 3}(x(t)),$$
We know from the Newton–Leibniz formula that
\[ x(t) - x(t - d(t)) = \int_{t-d(t)}^{t} \dot{x}(s) ds. \]
Then, for any matrices with appropriate dimension
\[ Z^j(\beta) = [X^1_i T Y^i_{\beta} T]^T, \]
we have
\[ 2e^{-\alpha_d t} \phi^T(t) Z^j(\beta) \]
\[ \left[ x(t) - x(t - d(t)) - \int_{t-d(t)}^{t} \dot{x}(s) ds \right] = 0, \quad (12) \]
where \( \phi(t) = [x^T(t) x^T(t - d(t))]^T. \) Regarding
(9)–(11) and (12), we have
\[ \dot{V}_\beta^i(x(t)) + \alpha_\beta V_\beta^i(x(t)) \]
\[ \leq \phi^T(t) [\Pi^i_\beta + de^{-\alpha_d t} Z^j(\beta) R^{-1}_\beta^{-1} Z^T(\beta)] \phi(t) \]
\[ - \int_{t-d(t)}^{t} e^{-\alpha_d s} [Z^T(\beta) \varphi(t) + R^i_\beta \dot{x}(s)]^T \]
\[ \times R^{-1}_\beta [Z^T(\beta) \varphi(t) + R^i_\beta \dot{x}(s)] ds, \quad (13) \]
where
\[ \Pi^i_\beta = \left[ \pi^i_{11}(\beta) \pi^i_{12}(\beta) \right], \]
with
\[ \pi^i_{11}(\beta) = (P^i_\beta A(\beta) + e^{-\alpha_d t} X^j_\beta) + (P^i_\beta A(\beta) \]
\[ + e^{-\alpha_d t} X^j_\beta)^T + Q^i_\beta + \alpha_\beta P^i_\beta \]
\[ + dA^T(\beta) R^{-1}_\beta A(\beta), \]
\[ \pi^i_{12}(\beta) = P^i_\beta A_d(\beta) + e^{-\alpha_d t} Y^i_\beta - e^{-\alpha_d t} X^i_\beta \]
\[ + dA^T(\beta) R^{-1}_\beta A_d(\beta) \]
\[ \pi^i_{22}(\beta) = -e^{-\alpha_d t} Y^i_\beta - e^{-\alpha_d t} Y^i_\beta^T - (1 - \sigma)e^{-\alpha_d t} Q^i_\beta \]
\[ + dA^T(\beta) R^{-1}_\beta A_d(\beta). \]
Note that
\[ \int_{t-d(t)}^{t} e^{-\alpha_d s} [Z^T(\beta) \varphi(t) + R^i_\beta \dot{x}(s)] c^T R^{-1}_\beta \]
\[ [Z^T(\beta) \varphi(t) + R^i_\beta \dot{x}(s)] ds \leq 0. \quad (14) \]
Carrying out a congruence transformation to (7) by diag(1, 1, 1, \( e^{\alpha_d t} 1 \)) and considering \( X^j_\beta = e^{-\alpha_d t} X^j_\beta, \)
\( Y^i_\beta = e^{-\alpha_d t} Y^i_\beta, \) by Schur complement, (7) implies
\[ \Pi^i_\beta + de^{-\alpha_d t} Z(\beta) R^{-1}_\beta Z^T(\beta) \leq 0. \quad (15) \]
Consequently, it follows from (13), (14), and (15) that
\[ \dot{V}^i_\beta(x(t)) + \alpha_\beta V^i_\beta(x(t)) \leq 0, \quad \forall \beta \in \mathcal{S}, i \in \mathcal{R}(\beta), \quad (16) \]
According to (6.a) and (8), \( \forall \beta \in \mathcal{S}, i \neq 0 \) we have
\[ \dot{V}^{i_1}_\beta(x(t + j^i_\beta)) - \eta_\beta V^{i_1-1}_\beta(x(t + j^i_\beta)) \]
\[ \leq x^T(t) [P^i_\beta - \eta_\beta P^i_\beta^{-1}] x(t) \leq 0, \]
\[ \dot{V}^{i_2}_\beta(x(t + j^i_\beta)) - \eta_\beta V^{i_2-1}_\beta(x(t + j^i_\beta)) \]
\[ \leq \int_{t-d(t)}^{t} e^{\alpha_d t} x^T(s) [Q^i_\beta - \eta_\beta Q^i_\beta^{-1}] x(s) ds \leq 0, \]
According to (6.b) and (8), \( \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{S} \), \( \beta \neq \gamma \) we have

\[
V_{\beta 1}^0(x(t)) - \mu_\beta V_{\beta 1}^{G_{\beta} - 1}(x(t)) = x^T(t)[p_0^\beta - \mu_\beta P_{\beta}^{G_{\beta} - 1}]x(t) \leq 0,
\]

\[
V_{\beta 2}^0(x(t)) - \mu_\beta V_{\beta 2}^{G_{\beta} - 1}(x(t)) = \int_{t-d(t)}^{t} e^{\theta (s-t)}[Q_0^\beta - \mu_\beta Q_{\beta}^{G_{\beta} - 1}]x(s) ds \leq 0,
\]

\[
V_{\beta 3}^0(x(t)) - \mu_\beta V_{\beta 3}^{G_{\beta} - 1}(x(t)) = \int_{t-d(t)}^{t} e^{\theta (s-t)}x^T(s)[R_0^\beta - \mu_\beta R_{\beta}^{G_{\beta} - 1}]x(s) ds \leq 0,
\]

which implies

\[
V_{\gamma}^0(x(t)) - \mu_\gamma V_{\gamma}^{G_{\gamma} - 1}(x(t)) \leq 0,
\]

\( \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \). (17)

According to (6.c) and (8), \( \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \) we have

\[
V_{\gamma 1}^0(x(t)) - \mu_\gamma V_{\gamma 1}^{G_{\gamma} - 1}(x(t)) = x^T(t)[P_0^\gamma - \mu_\gamma P_{\gamma}^{G_{\gamma} - 1}]x(t) \leq 0,
\]

\[
V_{\gamma 2}^0(x(t)) - \mu_\gamma V_{\gamma 2}^{G_{\gamma} - 1}(x(t)) = \int_{t-d(t)}^{t} e^{\theta (s-t)}[Q_0^\gamma - \mu_\gamma Q_{\gamma}^{G_{\gamma} - 1}]x(s) ds \leq 0,
\]

\[
V_{\gamma 3}^0(x(t)) - \mu_\gamma V_{\gamma 3}^{G_{\gamma} - 1}(x(t)) = \int_{t-d(t)}^{t} e^{\theta (s-t)}x^T(s)[R_0^\gamma - \mu_\gamma R_{\gamma}^{G_{\gamma} - 1}]x(s) ds \leq 0,
\]

which implies

\[
V_{\gamma}^0(x(t)) - \mu_\gamma V_{\gamma}^{G_{\gamma} - 1}(x(t)) \leq 0,
\]

\( \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \). (19)

According to Theorem 1 in (Zhao et al., 2017), and considering (16)–(19), if \( \tau_\beta^a \) satisfy (5), on time interval \([0, T]\), we have

\[
V_{\beta(T^-)}(T^-) \leq \exp \left\{ \sum_{\beta \in \mathcal{G}} (N_{0\beta} \ln \mu_\beta(\eta_\beta)^{\beta_{\beta} - 1}) \right\}
\]

\[
+ \sum_{\beta \in \mathcal{U}} (N_{0\beta} \ln \mu_\beta(\eta_\beta)^{\beta_{\beta} - 1})
\]

\[
\times e^{\max_{\beta \in \mathcal{U}} \left( \frac{(\ln \mu_\beta(\eta_\beta)^{\beta_{\beta} - 1})}{\tau^\beta_{\beta}} - \alpha_{\beta} \right) T_{\beta(T,0)}}
\]

\[
\times (\eta_{\beta(0)(\eta_\beta)^{\beta_{\beta} - 1} - 1} V_{\beta(0)}(\alpha(0))
\]

Which \( N_{0\beta} \) is a constant number. So, one can conclude, \( V_{\beta(T^-)}(x(T^-)) \) converges to zero as \( T \to \infty \) for any MDADT switching signal satisfying (5). Finally, by Definition 1 and the first equation of Lemma 1, we can get that switched system (3) is exponentially
stable, and the exponential decay rate is equal to
\[
\max_{\beta \in S}\left\{ \frac{\left((\ln \mu_\beta(\eta_\beta)G_{\beta}^{-1})/\tau_\beta^a\right) - \alpha_\beta}{\beta} \right\}.
\]

As mentioned before, the basic idea of MLF is that multiple Lyapunov functions, which correspond to each subsystem or certain region in the state space, are pieced together to produce a non-traditional Lyapunov function whose overall energy decreases to zero along the system state trajectories.

In equation (6-a), from the energy point of view shows the energy changes at the moments that are broken for each sub-system of the Lyapunov function. Note that these are not the switching time of the subsystems but are the switching time of the Lyapunov function in each subsystem. This relationship shows the decreasing trend of energy in these moments. The equation (6-b) indicates that Lyapunov’s energy is limited in switching moments when switching occurs from a stable subsystem to other subsystems. The equation (6-c) indicates the energy depletion of Lyapunov function at the switching instants when a switching to an unstable subsystem is occurred, as well as the relation (7) indicates the depletion or increasing energy of each subsystem. Equation (7) also states that if the subsystems are stable, their changes are reduced to zero under an exponential function, and if the subsystems are unstable, they tend to be infinite under one cover.

**Remark 1:** In Theorem 1, subsystems can be stable and unstable. For stable subsystems, \(\alpha_\beta\) may be positive or negative. However, for unstable subsystems \(\alpha_\beta\) must be negatively obtained. Note that for the stability of the switched system, at least an \(\alpha_\beta\) has to be positive. This means, if there are a number of subsystems, then at least one of them must be stable. It is also worth noting that two unstable subsystems cannot be switched subsequently.

**Remark 2:** In the case of switched system without delay, the Theorem 1 of this paper can be reduced to Theorem 2 in (Zhao et al., 2017). By considering \(d(t) = 0\), the Lyapunov function of the new problem will be as follows

\[
V_{\beta i}(x(t)) = V_{\beta 1}^i(x(t)) + V_{\beta 2}^i(x(t)) + V_{\beta 3}^i(x(t)),
\]

\[
\forall t \in L_{\beta(t)}^i, i \in R_{\beta(t)}, \beta \in S,
\]

where

\[
V_{\beta 1}^i(x(t)) = x^T(t)P_{\beta i}^i x(t), \quad V_{\beta 2}^i(x(t)) = 0, \quad V_{\beta 3}^i(x(t)) = 0.
\]

It is obvious when \(V_{\beta 2}^i(x(t)) = 0, V_{\beta 3}^i(x(t)) = 0\), results in \(Q_{\beta}^i = 0, R_{\beta}^i = 0\). Then stability criteria will be as follows that is similar to Theorem 2 in (Zhao et al., 2017).

**Corollary 1:** For given scalars \(\alpha_\beta > 0, 0 < \eta_\beta \leq 1, \mu_\beta > 1, \beta \in G\) satisfying \((\eta_\beta G_{\beta}^{-1} - \mu_\beta > 1, \alpha_\beta < 0, 0 < \eta_\beta \leq 1, 0 < \mu_\beta < 1, \beta \in U\) if there exist matrices \(P_{\beta}^i > 0\) and \(X_{\beta}^i, Y_{\beta}^i, \beta \in S, i \in R_{\beta(t)}\), such that, \(\forall i \in R_{\beta(t)}\)

\[
\begin{align*}
A^T(\beta) P_{\beta}^i + P_{\beta}^i A(\beta) &\leq -\alpha_\beta P_{\beta}^i, \quad \beta \in S, \\
P_{\beta}^i &\leq \eta_\beta P_{\beta}^{i-1}, \quad \beta \in S, \quad i \neq 0, \\
\mu_\beta P_{\gamma}^i &\leq \mu_{\beta} G_{\beta}^{-1}, \quad (\beta, \gamma) \in G \times S, \beta \neq \gamma, \\
P_{\gamma}^0 &\leq \mu_{\gamma} P_{\beta}^{i-1}, \quad (\beta, \gamma) \in G \times U,
\end{align*}
\]

then the switched system (3) with \(A_d(\beta) = 0\) is exponentially stable for any MDADT switching signals satisfying (5).

**Remark 3:** In the study of switched time-delay systems, a lot of interesting work has been done in the field of stability and stabilisation. In (Chiou et al., 2011), the stability of switched time-delay systems, under a state-driven switching law is explored. The stability problem for a class of switched positive time-delay systems is studied in (Zhao et al., 2013). The fault-tolerant control problem of a class of uncertain switched linear time-delay systems is studied in (Jin et al., 2018). It is worthwhile to point out that, in the above-mentioned references, the case of constant delay is only considered, and the studied systems do not contain time-varying delay while the time-varying delay is an important issue in delayed systems. In the case of time-varying delay, the stability with ADT switching, for a class of uncertain switched systems with nonlinear perturbations, is studied in (Zhang & Yu, 2019). In (Zhao et al., 2013), for a class of switched positive time-delay systems, the stability problem is studied. The fault-tolerant control problem of a class of uncertain switched linear time-delay systems is studied in (Jin et al., 2018) with ADT switching. Note that in the above references, the considered systems assume that all the subsystems are stable,
which greatly limits the scope of switching systems. For the switched time-delay systems with unstable subsystems, preliminary research has been done, see (Li et al., 2018; Zamani et al., 2014). As stated previously, the MDADT switching approach is more applicable. In (Hou & Zong, 2018), the stability problem for switched time-delay systems with MDADT switching is explored. In this reference, although unstable subsystems are also considered, there is still conservativeness in the determination of dwell time and delay bands.

In our work, the switched system with time-varying delays, consist of stable and unstable subsystems, is considered. The switching signal follows the MDADT strategy. And above all, inspired by the proposed method of MDLF in (Zhao et al., 2017), and using a form of Bessel–Legendre inequality in constructing LKF, to reduce conservativeness, in the determination of dwell time and delay bands respectively, we carry out this study. Compared with the relevant literature, our main advantage and contributions lie in three aspects as follows. (1) The stability issue of switched systems with time-varying delay is studied by employing the MDLF approach and using affine Bessel–Legendre inequality in constructing LKF, which is the first attempt in this area; (2) New sufficient criteria are derived, ensuring the stability of the switched time-delay system. This offers a tighter dwell time-bound with less conservativeness. (3) Mostly in literature, subsystems are considered only stable, which is a restriction. However, in this paper, subsystems can be both stable and unstable.

By using affine Bessel–Legendre inequality that introduced in Lemma 2, we obtain a relaxed and less conservative stability criterion of the switched time-varying delay systems as follows.

**Theorem 2:** Consider the switched system (3). Suppose (A2) holds and for given scalars $\alpha_\beta > 0$, $0 < \eta_\beta \leq 1$, $\mu_\beta > 1$, $\beta \in \mathcal{G}$ satisfying $(\eta_\beta)G_{\beta}^{-1} \mu_\beta > 1$, and $\alpha_\beta < 0$, $0 < \eta_\beta \leq 1$, $0 < \mu_\beta < 1$, $\beta \in \mathcal{U}$, if there exist matrices $P^i_\beta > 0$, $Q^i_\beta > 0$, $S^i_\beta > 0$, $R^i_\beta > 0$, and $X^i_\beta$, $X^i_{\beta_0}$, $\beta \in \mathcal{S}$, $i \in \mathcal{R}_{\beta_0(t_k)}$, such that $\forall i \in \mathcal{R}_{\beta_0(t_k)}$

\[
\begin{align*}
    P^0_\beta &\leq \mu_\beta P^0_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
    Q^0_\beta &\leq \mu_\beta Q^0_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
    S^0_\beta &\leq \mu_\beta S^0_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
    R^0_\beta &\leq \mu_\beta R^0_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
\end{align*}
\]

(20.a)

\[
\begin{align*}
    P^\beta &\leq \mu_\beta P^\beta_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \\
    Q^\beta &\leq \mu_\beta Q^\beta_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \\
    S^\beta &\leq \mu_\beta S^\beta_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \\
    R^\beta &\leq \mu_\beta R^\beta_{\beta_0}, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \\
\end{align*}
\]

(20.b)

\[
\begin{align*}
    \phi^\beta(d(t), \bar{d}(t)) &= e^{\alpha_\beta d} \left[ (G_0^T(d(t)))P^\beta_\beta G_1(d(t))) \\
    &+ (G_0^T(d(t)))P^\beta_\beta G_1(d(t))) \right] \\
    &+ e^{\alpha_\beta d} e^T_1 (Q^\beta_\beta + S^\beta_\beta) e_1 - \bar{d}(t) e^T_2 Q^\beta_\beta e_2 \\
    &- e^T_3 S^\beta_\beta e_3 + e^{\alpha_\beta d} d_0^T(\beta) R^\beta_\beta e_0(\beta), \\
\end{align*}
\]

(22)

and for $\alpha_\beta < 0$,

\[
\begin{align*}
    \phi^\beta(d(t), \bar{d}(t)) &= e^{\alpha_\beta d} \left[ (G_0^T(d(t)))P^\beta_\beta G_1(d(t))) \\
    &+ (G_0^T(d(t)))P^\beta_\beta G_1(d(t))) \right] \\
    &+ e^{\alpha_\beta d} e^T_1 (Q^\beta_\beta + S^\beta_\beta) e_1 - \bar{d}(t) e^T_2 Q^\beta_\beta e_2 \\
    &- e^T_3 S^\beta_\beta e_3 + e^{\alpha_\beta d} d_0^T(\beta) R^\beta_\beta e_0(\beta), \\
\end{align*}
\]

(23)

and for $\alpha_\beta < 0$,
in which \(e_l(l = 1, \ldots, 5) \in \mathbb{R}^{n \times 5n}\) are elementary matrices, (for example \(e_3 = (0_{n \times 2n}, I_n, 0_{n \times 2n})\), \(e_0(\beta) = A(\beta)e_1 + A_2(\beta)e_2\). Then, the switched system (3) is exponentially stable for any MDADT switching signals satisfying (5).

**Proof:** Consider the multiple discontinuous Lyapunov functional candidate for switched system (3) as follows:

\[
V^i_{\beta}(x(t)) = V^i_{\beta_1}(x(t)) + V^i_{\beta_2}(x(t)) + V^i_{\beta_3}(x(t)) + V^i_{\beta_4}(x(t)), \quad \forall \, t \in L^i_{\beta}(i), \, i \in \mathcal{R}_{\beta}(t), \beta \in \mathcal{S} \tag{24}
\]

\[
V^i_{\beta_1}(x(t)) = \zeta^T(t)P^i_{\beta}\zeta(t),
\]

\[
V^i_{\beta_2}(x(t)) = \int_{t-d(t)}^t e^{\alpha_\beta(s-t)}x^T(s)Q^i_{\beta}s(s)ds,
\]

\[
V^i_{\beta_3}(x(t)) = \int_{t-d}^t e^{\alpha_\beta(s-t)}x^T(s)S^i_{\beta}s(s)ds,
\]

\[
V^i_{\beta_4}(x(t)) = \int_{t-d}^0 \int_{t-theta}^t e^{\alpha_\beta(s-t)}x^T(s)R^i_{\beta}s(s)ds \, dt,
\]

where

\[
\zeta(t) = \begin{pmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-d) \end{pmatrix}^T.
\]

Let functional \(W^i_{\beta}(x(t))\) be as follows

\[
W^i_{\beta}(x(t)) = \dot{V}^i_{\beta}(x(t)) + \alpha_\beta V^i_{\beta}(x(t)). \tag{25}
\]

Differentiating (24) with respect to the time variable \(t\) and considering (25), if \(\alpha_\beta > 0\) we have

\[
W^i_{\beta}(x(t)) \leq 2\dot{\zeta}^T(t)P^i_{\beta}\zeta(t) + \alpha_\beta \zeta^T(t)P^i_{\beta}\zeta(t) + x^T(t)Q^i_{\beta}s(t) - e^{-\alpha_\beta d} \dot{d}(t) x^T(t-d(t))Q^i_{\beta}s(t-d(t)) + x^T(t)S^i_{\beta}s(t) - e^{-\alpha_\beta d} \dot{d}(t) x^T(t-d)S^i_{\beta}s(t-d) + dx^T(t)R^i_{\beta}s(t) - \int_{t-d}^t e^{-\alpha_\beta d} \dot{x}^T(s)R^i_{\beta}s(s)ds
\]

therefore

\[
W^i_{\beta}(x(t)) \leq \xi^T(t)\phi^i_{\beta}(d(t), \dot{d}(t))\xi(t) - \int_{t-d}^t e^{-\alpha_\beta d} \dot{x}^T(s)R^i_{\beta}s(s)ds, \tag{26}
\]

where \(\phi^i_{\beta}(d(t), \dot{d}(t))\) is given in (22), and

\[
\xi(t) = \begin{pmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-d) \end{pmatrix}.
\]

Dividing the interval \([t-d, t]\) of the integral term in (26) into two intervals \([t-d, t-d(t)]\) and \([t-d(t), t]\), and applying Lemma 2 to each of them when \(N = 1\) incomes

\[
-\int_{t-d}^t e^{-\alpha_\beta d} \dot{x}^T(s)R^i_{\beta}s(s)ds \leq -\xi^T(t)\psi^i_{\beta}(t, d(t))\xi(t), \tag{28}
\]

where

\[
\psi^i_{\beta}(t, d(t), d(t)) = \pi^T_1 (\lambda^i_{1\beta}M + M^T \lambda^i_{1\beta}^T) - d(t) \lambda^i_{1\beta} R^i_N^{-1}(\beta) \lambda^i_{1\beta}^T \pi^T_1 + \pi^T_2 (\lambda^i_{2\beta}M + M^T \lambda^i_{2\beta}^T) - (d(t) \lambda^i_{2\beta} R^i_N^{-1}(\beta) \lambda^i_{2\beta}^T) \pi^T_2. \tag{29}
\]

Incorporating (26)–(28) gives \(W^i_{\beta}(x(t)) \leq \xi^T(t)\phi^i_{\beta}(d(t), d(t))\psi^i_{\beta}(t, d(t))\xi(t)\). As regard to condition that \(\phi^i_{\beta}(d(t), \dot{d}(t)) = e^{-\alpha_\beta d} \psi^i_{\beta}(t, d(t))\leq 0\), is affine with respect to \(d(t)\) and \(d(t)\), it can be satisfied for all \((d(t), \dot{d}(t)) \in [0, d] \times [\sigma_1, \sigma_2]\). If it is satisfied at the vertices of the interval \([0, d] \times [\sigma_1, \sigma_2]\), then with the help of Schur complement, the conditions can be derived as the form of LMIs in (21).

Now, if \(\alpha_\beta < 0\), by differentiating (24) with respect to the time variable \(t\) and considering (25), we have

\[
W^i_{\beta}(x(t)) \leq 2\dot{\zeta}^T(t)P^i_{\beta}\zeta(t) + \alpha_\beta \zeta^T(t)P^i_{\beta}\zeta(t) + x^T(t)Q^i_{\beta}s(t) - \dot{d}(t) x^T(t-d(t))Q^i_{\beta}s(t-d(t)) + x^T(t)S^i_{\beta}s(t) - \dot{d}(t) x^T(t-d)S^i_{\beta}s(t-d) + dx^T(t)R^i_{\beta}s(t) + \int_{t-d}^t e^{-\alpha_\beta d} \dot{x}^T(s)R^i_{\beta}s(s)ds
\]

and

\[
W^i_{\beta}(x(t)) \leq 2\dot{\zeta}^T(t)P^i_{\beta}\zeta(t) + \alpha_\beta \zeta^T(t)P^i_{\beta}\zeta(t) + x^T(t)Q^i_{\beta}s(t) - \dot{d}(t) x^T(t-d(t))Q^i_{\beta}s(t-d(t)) + x^T(t)S^i_{\beta}s(t) - \dot{d}(t) x^T(t-d)S^i_{\beta}s(t-d) + dx^T(t)R^i_{\beta}s(t) + \int_{t-d}^t e^{-\alpha_\beta d} \dot{x}^T(s)R^i_{\beta}s(s)ds
\]
\[ -e^{-\alpha \beta} x^T (t - d) S_{\beta} \dot{x}(t - d) + d \dot{x}^T (t) R_{\beta} \dot{x}(t) - \int_{t - d}^{t} \dot{x}^T (s) R_{\beta} \dot{x}(s) ds, \]

therefore

\[ W_{\beta}^i(x(t)) \leq \xi^T (t) \phi_{\beta} (d(t), \dot{d}(t)) \xi(t) - \int_{t - d}^{t} \dot{x}^T (s) R_{\beta} \dot{x}(s) ds, \] (30)

where \( \phi_{\beta} (d(t), \dot{d}(t)) \) and \( \xi(t) \) are given in (2) and (27), respectively. Dividing the interval \([t - d, t]\) of the integral term in (30) into two intervals \([t - d, t - d(t)]\) and \([t - d(t), t]\), and applying Lemma 2 to each of them when \( N = 1 \) incomes

\[ - \int_{t - d}^{t} \dot{x}^T (s) R_{\beta} \dot{x}(s) ds \leq -\xi^T (t) \{ \psi_{\beta}^i (t, d(t)) \} \xi(t), \] (31)

where \( \psi_{\beta}^i (t, d(t)) \) is given in (29). Considering (30) and (31) we have

\[ W_{\beta}^i(x(t)) \leq \xi^T (t) \{ \phi_{\beta}^i (d(t), \dot{d}(t)) - \psi_{\beta}^i (t, d(t)) \} \xi(t). \] (32)

Similar to the previous one, with the help of Schur complement, the conditions \( \phi_{\beta}^i (d(t), \dot{d}(t)) - \psi_{\beta}^i (t, d(t)) \leq 0 \) can be derived as the form of LMI in (21).

Consequently, we obtain

\[ V_{\beta}^i(x(t)) + \alpha_{\beta} V_{\beta}^i(x(t)) < 0, \quad \forall \beta \in \mathcal{S}, i \in R_{\beta}(t), \] (33)

According to (20.a) and (24), \( \forall \beta \in \mathcal{S}, i \neq 0 \) we have

\[ V_{\beta 1}^i(x(t_k + j_{\beta}^i)) - \eta_{\beta} V_{\beta 1}^{i-1}(x(t_k + j_{\beta}^i)) \leq \xi^T (t) [P_{\beta}^0 - \eta_{\beta} P_{\beta}^{i-1}] \xi(t) \leq 0, \]

\[ V_{\beta 2}^i(x(t_k + j_{\beta}^i)) - \eta_{\beta} V_{\beta 2}^{i-1}(x(t_k + j_{\beta}^i)) \leq \int_{t - d(t)}^{t} e^{\alpha_{\beta}(s-t)} x^T(s) [Q_{\beta}^0 - \eta_{\beta} Q_{\beta}^{i-1}] x(s) ds \leq 0, \]

\[ V_{\beta 3}^i(x(t_k + j_{\beta}^i)) - \eta_{\beta} V_{\beta 3}^{i-1}(x(t_k + j_{\beta}^i)) \leq \int_{t - d(t)}^{t} e^{\alpha_{\beta}(s-t)} x^T(s) [S_{\beta}^0 - \eta_{\beta} S_{\beta}^{i-1}] x(s) ds \leq 0, \]

\[ V_{\beta 4}^i(x(t_k + j_{\beta}^i)) - \eta_{\beta} V_{\beta 4}^{i-1}(x(t_k + j_{\beta}^i)) \leq \int_{t - d(t)}^{t} e^{\alpha_{\beta}(s-t)} x^T(s) [R_{\beta}^0 - \eta_{\beta} R_{\beta}^{i-1}] x(s) ds \leq 0, \]

so

\[ [V_{\beta 1}^i(x(t_k + j_{\beta}^i)) + V_{\beta 2}^i(x(t_k + j_{\beta}^i)) + V_{\beta 3}^i(x(t_k + j_{\beta}^i)) + V_{\beta 4}^i(x(t_k + j_{\beta}^i)) \]

\[ + V_{\beta 1}^{i-1}(x(t_k + j_{\beta}^i)) - \eta_{\beta} [V_{\beta 1}^{i-1}(x(t_k + j_{\beta}^i)) + V_{\beta 2}^{i-1}(x(t_k + j_{\beta}^i)) + V_{\beta 3}^{i-1}(x(t_k + j_{\beta}^i)) + V_{\beta 4}^{i-1}(x(t_k + j_{\beta}^i)) \]

\[ + V_{\beta 1}^{i-1}(x(t_k + j_{\beta}^i))] \leq 0, \]

which results in

\[ V_{\beta}^i(x(t_k + j_{\beta}^i)) - \eta_{\beta} V_{\beta}^{i-1}(x(t_k + j_{\beta}^i)) \leq 0, \quad \forall \beta \in \mathcal{S}, i \neq 0. \] (34)

According to (20.b) and (24), \( \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \) we have

\[ V_{\beta 1}^0(x(t)) - \mu_{\beta} V_{\gamma 1}^{G_{\gamma}^{-1}}(x(t)) \]

\[ = \xi^T(t) [P_{\gamma}^0 - \mu_{\beta} P_{\gamma}^{G_{\gamma}^{-1}}] \xi(t) \leq 0, \]

\[ V_{\beta 2}^0(x(t)) - \mu_{\beta} V_{\gamma 2}^{G_{\gamma}^{-1}}(x(t)) \]

\[ = \int_{t - d(t)}^{t} e^{\alpha_{\beta}(s-t)} x^T(s) [Q_{\gamma}^0 - \mu_{\beta} Q_{\gamma}^{G_{\gamma}^{-1}}] x(s) ds \leq 0, \]

\[ V_{\beta 3}^0(x(t)) - \mu_{\beta} V_{\gamma 3}^{G_{\gamma}^{-1}}(x(t)) \]

\[ = \int_{t - d(t)}^{t} e^{\alpha_{\beta}(s-t)} x^T(s) [S_{\gamma}^0 - \mu_{\beta} S_{\gamma}^{G_{\gamma}^{-1}}] x(s) ds \leq 0, \]

\[ V_{\beta 4}^0(x(t)) - \mu_{\beta} V_{\gamma 4}^{G_{\gamma}^{-1}}(x(t)) \]

\[ = \int_{t - d(t)}^{t} e^{\alpha_{\beta}(s-t)} x^T(s) [R_{\gamma}^0 - \mu_{\beta} R_{\gamma}^{G_{\gamma}^{-1}}] x(s) ds \leq 0, \]

therefore

\[ [V_{\beta 1}^0(x(t)) + V_{\beta 2}^0(x(t)) + V_{\beta 3}^0(x(t)) + V_{\beta 4}^0(x(t))] \]

\[ - \mu_{\beta} [V_{\gamma 1}^{G_{\gamma}^{-1}}(x(t)) + V_{\gamma 2}^{G_{\gamma}^{-1}}(x(t)) + V_{\gamma 3}^{G_{\gamma}^{-1}}(x(t))] \]

\[ + V_{\gamma 4}^{G_{\gamma}^{-1}}(x(t))] \leq 0, \]

which implies

\[ V_{\beta}^0(x(t)) - \mu_{\beta} V_{\gamma}^{G_{\gamma}^{-1}}(x(t)) \leq 0, \quad \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \] (35)
According to (20.c) and (24), $\forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{U}$ we have

$$V^0_{\gamma 1}(x(t)) - \mu_{\gamma} V^0_{\beta 1}(x(t)) = \xi^T(t) [P^0_{\gamma} - \mu_{\gamma} P^G_{\beta}^{-1}] \xi(t) \leq 0,$$

$$V^0_{\gamma 2}(x(t)) - \mu_{\gamma} V^0_{\beta 2}(x(t)) = \int_0^t e^{\mu_{\gamma} (s-t)} x^T(s) [Q^0_{\gamma} - \mu_{\gamma} Q^G_{\beta}^{-1}] x(s) ds \leq 0,$$

$$V^0_{\gamma 3}(x(t)) - \mu_{\gamma} V^0_{\beta 3}(x(t)) = \int_0^t e^{\mu_{\gamma} (s-t)} x^T(s) [S^0_{\gamma} - \mu_{\gamma} S^G_{\beta}^{-1}] x(s) ds \leq 0,$$

$$V^0_{\gamma 4}(x(t)) - \mu_{\gamma} V^0_{\beta 4}(x(t)) = \int_{-d}^0 \int_0^t e^{\mu_{\gamma} (s-t)} x^T(s) [R^0_{\gamma} - \mu_{\gamma} R^G_{\beta}^{-1}] x(s) ds \, d\theta \leq 0,$$

Therefore,

$$[V^0_{\gamma 1}(x(t)) + V^0_{\gamma 2}(x(t)) + V^0_{\gamma 3}(x(t)) + V^0_{\gamma 4}(x(t))] - \mu_{\gamma} [V^0_{\beta 1}(x(t)) + V^0_{\beta 2}(x(t)) + V^0_{\beta 3}(x(t)) + V^0_{\beta 4}(x(t))]$$

$$+ V^G_{\beta 1}(x(t)) \leq 0,$$

which yields

$$V^0_{\gamma}(x(t)) - \mu_{\gamma} V^G_{\beta}(x(t)) \leq 0, \forall (\beta, \gamma) \in \mathcal{G} \times \mathcal{U}. \quad (36)$$

According to Theorem 1 in (Zhao et al., 2017), and considering (33)–(36), if $\tau^a_i$ satisfy (5), on time interval $[0; T]$, we have

$$V_{\beta(T^-)}(T^-) \leq \exp \left\{ \sum_{\beta \in \mathcal{G}} (N_0 \beta \ln \mu_{\beta} (\gamma_{\beta}) G^G_{\beta}^{-1}) \right\}$$

$$+ \sum_{\beta \in \mathcal{U}} (N_0 \beta \ln \mu_{\beta} (\gamma_{\beta}) G^G_{\beta}^{-1})$$

$$\max_{\beta \in \mathcal{S}} \left\{ \left( \frac{n_{\beta}(\eta_{\beta}) G^G_{\beta}^{-1} - \alpha_{\beta}}{\gamma_{\beta}} \right) T_{\beta}(T, 0) \right\} \times \exp \left\{ \sum_{\beta \in \mathcal{G}} (N_0 \beta \ln \mu_{\beta} (\gamma_{\beta}) G^G_{\beta}^{-1}) \right\} \times (\eta_{\beta(0)} G^G_{\beta(0)} - 1) V^0_{\beta(0)}(x(0))$$

which, $N_0 \beta$ is a constant number. So, one can conclude, $V_{\beta(T^-)}(x(T))$ converges to zero as $T \to \infty$ for any MDADT switching signal satisfying (5). Finally, by Definition 1 and the first equation of Lemma 1, we can get that switched system (3) is exponentially stable, and the exponential decay rate is equal to

$$\max \left\{ (\ln \mu_{\beta} (\gamma_{\beta}) G^G_{\beta}^{-1}) / T_{\beta}(T, 0) - \alpha_{\beta} \right\}. \quad \Box$$

Both Theorems 1 and 2 address the issue of stability. But the method of constructing LKF, which is a well-known and important issue in delayed systems, the calculations, and the relations obtained, are completely different, so none can be considered a subset of another.

**Remark 4:** In Lemma 2, we have additional decision variables $X$ in $Z(X) = X_N + H^T_N X_T - (b - a) X \dot{R} X^T$ where

$$H_N = \left[ \Gamma^T_N(0) \quad \Gamma^T_N(1) \quad \ldots \quad \Gamma^T_N(N) \right]^T$$

$$\Gamma_N(K) = \begin{cases} [I - 1], & N = 0 \\ [I (-1)^K I + \gamma_{N0}^0 I], & N > 0 \end{cases}$$

if $N = 0$ then $X$ is matrix with size $n \times 2n$ and $N_v = 2n^2$ decision variable will be added. If $N = 1$ then $N_v = 6n^2$ decision variable and if $N = 2$ then $N_v = 12n^2$ decisions variable will be added, respectively. In the general case, $(N + 1)(N + 2)n^2$ decision variable will be added in which $n$ is the number of states. Consider that $X$ is not a symmetric matrix, generally. From optimisation theory, we know that more decision variables yield a larger solution area and give less conservative optimisation conditions. Now consider Theorem 2 where $N = 1$. In this theorem $P_{\beta}^0 > 0$, $Q_{\beta}^0 > 0$, $S_{\beta}^0 > 0$, $R_{\beta}^0 > 0$ have to be obtained which includes $((5n + 1)(5n + 2)/2)$, $((4n + 1)(4n + 2)/2)$, $((3n + 1)(3n + 2)/2)$, $((n + 1)(n + 2)/2)$ decision variable, respectively for a fix $i$, $\beta$ and also $X^i_{\beta}$ and $\gamma^i_{\beta}$ include $3n \times 2n$ decision variables. By considering $i$, $\beta$, we have $m \times i \times (4n^2 + 7n)$ where $i \in \mathcal{R}_{\beta(t_k)} = \{0, 1, 2, \ldots, G_{\beta(t_k)} - 1\}$. By adding $\mu_{\beta}$ and $\eta_{\beta}$, we have $m \times i \times (4n^2 + 7n) + 2m$ decision variables in which $i \in \mathcal{R}_{\beta(t_k)} = \{0, 1, 2, \ldots, G_{\beta(t_k)} - 1\}$. Also, using the proposed multiple discontinuous Lyapunov function technique makes more decision variables depending on the number of each interval division $G_{\beta(t_k)}$ where $i \in \mathcal{R}_{\beta(t_k)} = \{0, 1, 2, \ldots, G_{\beta(t_k)} - 1\}$. 


Remark 5: It is worth mentioning that in our switching mechanisms in Theorems 1 and 2, using the MDADT method, we design fast switching and slow switching for unstable and stable subsystems, respectively. It not only gives lower bounds that stable subsystems should dwell on but also provides the upper bounds that the MDADT of unstable modes cannot exceed. Such a switching strategy enables us to easily balance the dwell time between unstable subsystems and stable subsystems in a mode-dependent manner. This switching strategy cannot be applied to some other time-dependent switching signals like DT, ADT, due to the fact that they are set in a mode-dependent manner. On the other hand, it must be said that the MDLF in mentioned Theorems, is only a piecewise continuous function during the dwell time on each mode, unlike some common MLFs for switched systems, which requires that the Lyapunov function for each mode is continuously differentiable during the running time. Based on such a Lyapunov functional, tighter bounds on dwell time can be obtained, which undoubtedly enhances the application flexibility in practice. In MDLF if we choose $G_\beta = 1$, which means that there is only one continuous Lyapunov function between two consecutive switching, then the MDLF directly reduces MLF. Thus, MLF is a special case of MDLF. One of the challenging issues of the MDLF is that all subsystems cannot be unstable without considering the case that when switched systems composed of stable and unstable subsystems, which may bring some limitations in actual operations.

Remark 6: In Theorems 1 and 2, computable sufficient conditions formulated in the form of LMIs. When solving these LMIs, the parameters $\alpha_\beta$, $G_\beta$, $\eta_\beta$, and $\mu_\beta$ should be given in advance. To search tighter bounds on MDADT, the following procedures are introduced in (Zhao et al., 2017). To choose the corresponding parameters for the LMIs: First, noticing that $G_\beta$ do not affect the feasibility of the LMIs, so if the computer computation ability allows, $G_\beta$ can be chosen big enough, but otherwise, $G_\beta$ can be selected bigger for those modes needing tighter bounds on dwell time and smaller for the other modes to prevent the complexity of computation. Second, the convergence rate (or divergence rate) $\alpha_\beta$, can be estimated based on the eigenvalues of each subsystem. Third, setting the step lengths and the initial values of $\eta_\beta$ and $\mu_\beta$, we can apply a two-layer loop programme under the condition $(\eta_\beta)^{G_\beta^{-1}}\mu_\beta > 1$ to solve the LMI conditions. Then, tighter bounds on MDADT can be identified among the obtained feasible solutions. It should be pointed out that from the energy view, the parameter $\eta_\beta$ also influence the decay rate $\alpha_\beta$ of the MDLF. So that smaller $\eta_\beta$ can cause the infeasibility problem of the conditions. Hence, $G_\beta$ and $\eta_\beta$ should be carefully designed depending on the practical situations. However, it is worth noting that obtaining minimal MDADT is still an unsolved problem.

4. Stabilisation analysis

In this section, based on the mentioned approach in the previous section, a sliding mode controller is designed to guarantee the exponential stability of the switched time varying delay system.

In order to obtain a regular form of systems (1), we can define a nonsingular matrix $T$ such that $TB = \begin{pmatrix} 0_{(n-m)\times m} & B_1 \end{pmatrix}^T$ where $B_1 \in \mathbb{R}^{m \times m}$ is nonsingular. For convenience, let us choose $T = \begin{pmatrix} U_1 & U_2 \end{pmatrix}^T$ where $U_1 \in \mathbb{R}^{n \times (n-m)}$ and $U_2 \in \mathbb{R}^{n \times m}$ are two sub-blocks of a unitary matrix resulting from taking the singular value decomposition of $B$, that is $B = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} 0_{(n-m)\times m} & \Gamma \end{pmatrix} V^T$ in which $\Gamma \in \mathbb{R}^{m \times m}$ is a diagonal positive-definite matrix and $V \in \mathbb{R}^{m \times m}$ is a unitary matrix. Then by the state transformation $z(t) = Tx(t)$, system (1) becomes

$$\dot{z}(t) = TA(\beta)T^{-1}z(t) + TA_d(\beta)T^{-1}z(t-d(t)) + TB(u(t, \beta) + F(\beta)f(t)).$$  \hspace{1cm} (37)

Now, let $z(t) = \begin{pmatrix} z_1^T(t) & z_2^T(t) \end{pmatrix}^T$ with $z_1 \in \mathbb{R}^{n-m}$ and $z_2 \in \mathbb{R}^m$ and

$$TA(\beta)T^{-1} = \begin{pmatrix} A_{11}(\beta) & A_{12}(\beta) \\ A_{21}(\beta) & A_{22}(\beta) \end{pmatrix},$$

$$TA(\beta)^{-1}T^{-1} = \begin{pmatrix} A_{11}(\beta) & A_{12}(\beta) \\ A_{21}(\beta) & A_{22}(\beta) \end{pmatrix},$$

$$TA_d(\beta)T^{-1} = \begin{pmatrix} A_{d11}(\beta) & A_{d12}(\beta) \\ A_{d21}(\beta) & A_{d22}(\beta) \end{pmatrix}.$$

Then system (37) turns into the following regular form:

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} A_{11}(\beta) & A_{12}(\beta) \\ A_{21}(\beta) & A_{22}(\beta) \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$$
observability, detectability, etc. Since the delay system introduction of appropriate concepts of stabilizability, large classes of linear systems for which these evolve in time are particularly appealing due to their ability to deal with time-varying systems, showing a robust behaviour (Camacho et al., 2007).

Consequently, specific models, analyses, and controllers must be designed to account for the infinite-dimensional nature of such systems. Even for linear models, the design of controllers is not obvious, mainly because applying the existing necessary and sufficient stability conditions is very tricky (Perruquet & Barbot, 2002). Although the sliding mode control has been extended to infinite-dimensional systems, the combination of delay phenomenon in switched systems makes the situation much more complex, and the concrete results are scarce.

In the following theorem, we analyse the stability of the sliding mode dynamics in (40) based on the result obtained in Theorem 1.

**Theorem 3:** Suppose (A1) holds and for given scalars \( \alpha > 0 \), \( 0 < \eta \leq 1 \), \( \mu > 1 \), \( \beta \in \mathcal{G} \) satisfying \((\eta)^{\beta-1} \mu > 1\), and \( \alpha < 0 \), \( 0 < \eta \leq 1 \), \( 0 < \mu < 1 \), \( \beta \in \mathcal{U} \) if there exist matrices \( P_\beta > 0 \), \( Q_\beta^i > 0 \), \( R_\beta^i > 0 \), \( S_\beta^i > 0 \), \( \gamma_\beta^i > 0 \), \( \gamma_\beta^i \), \( \lambda_\beta \), \( \eta_\beta \), \( \gamma_\beta \), \( \beta \in \mathcal{S} \), \( i \in \mathcal{R}_\beta(t_k) \), such that, \( \forall i \in \mathcal{R}_\beta(t_k) \)

\[
\begin{bmatrix}
-R_\beta^i & P_\beta \\
\ast & -J_\beta^i
\end{bmatrix} < 0, \beta \in \mathcal{S}
\]

\[
P_\beta P_\beta = I, \quad R_\beta^i S_\beta^i = I, \quad J_\beta^i S_\beta^i = I, \beta \in \mathcal{S}
\]

\[
\begin{align}
Q_\beta^i & \leq \eta_\beta Q_\beta^i, & \beta \in \mathcal{S}, i \neq 0 \\
S_\beta^i & \leq \eta_\beta S_\beta^i, & \beta \in \mathcal{S}, i \neq 0 \\
R_\beta^i & \leq \eta_\beta R_\beta^i, & \beta \in \mathcal{S}, i \neq 0
\end{align}
\]

\[
\begin{align}
P_\gamma & \leq \mu \gamma P_\beta, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
P_\mu & \leq \mu \gamma P_\beta, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
Q_\gamma^i & \leq \mu \gamma Q_\gamma^i, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
S_\gamma^i & \leq \mu \gamma S_\gamma^i, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
R_\gamma^i & \leq \mu \gamma R_\gamma^i, & (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma
\end{align}
\]
and also $\forall \beta \in \mathcal{S}$ and $\forall i \in \mathcal{R}_{\beta(t_k)}$

$$
\begin{pmatrix}
\hat{x}_{11}^i(\beta) & \hat{x}_{12}^i(\beta) & d(A_{11}(\beta)P_\beta^i - A_{12}(\beta)K_\beta^i + \chi_\beta^i) & dX_\beta^i \\
* & \hat{x}_{22}^i(\beta) & d(A_{d11}(\beta)P_\beta^i - A_{d12}(\beta)K_\beta^i) & dY_\beta^i \\
* & * & -dR^i_\beta & 0 \\
* & * & * & -de^{\alpha_d d}z_\beta^i
\end{pmatrix}
< 0
$$

(44)

where

$$
\hat{x}_{11}^i(\beta) = (A_{11}(\beta)P_\beta^i - A_{12}(\beta)K_\beta^i + \chi_\beta^i) + (A_{11}(\beta)P_\beta - d(A_{12}(\beta)K_\beta^i + \chi_\beta^i)^T + Q_\beta^i + \alpha_d P_\beta
$$

$$
\hat{x}_{12}^i(\beta) = (A_{d11}(\beta)P_\beta^i - A_{d12}(\beta)K_\beta^i) + Y_\beta^T - \chi_\beta^i,
$$

$$
\hat{x}_{22}^i(\beta) = -(1 - \sigma)e^{-\beta_d d}Q_\beta^i - Y_\beta^i - Y_\beta^T,
$$

then the sliding mode dynamics in (40) is exponentially stable for any MDADT switching signals satisfying (5). Furthermore, if the conditions above are feasible, the matrix $C(\beta)$ in (39) is given by $C(\beta) = K_\beta P_\beta^{-1}$ that is the sliding surfaces can be designed as

$$
S_\beta(t) = K_\beta P_\beta^{-1} z_1(t) + z_2(t)
$$

$$
= K_\beta P_\beta z_1(t) + z_2(t) = 0 \quad (45)
$$

Proof: According to Theorem 1, we know that if there exist matrices $P_\beta > 0, Q_\beta^i > 0, \forall \beta \in \mathcal{S}$ and $\forall i \in \mathcal{R}_{\beta(t_k)}$, such that $\gamma, \chi_\beta^i$ and $R_\beta^i > 0$

$$
\begin{cases}
Q_\beta^i \leq \eta_\beta Q_\beta^{i-1}, \quad \beta \in \mathcal{S}, i \neq 0 \\
R_\beta^i \leq \eta_\beta R_\beta^{i-1}, \quad \beta \in \mathcal{S}, i \neq 0
\end{cases}
\quad (46.a)
$$

$$
\begin{cases}
P_\beta \leq \mu_\beta P_\gamma, \quad (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
Q_\beta^0 \leq \mu_\beta Q_\gamma^{G_\gamma-1}, \quad (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma \\
R_\beta^0 \leq \mu_\gamma R_\gamma^{G_\gamma-1}, \quad (\beta, \gamma) \in \mathcal{G} \times \mathcal{S}, \beta \neq \gamma
\end{cases}
\quad (46.b)
$$

$$
\begin{cases}
P_\gamma \leq \mu_\gamma P_\beta, \quad (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \\
Q_\beta^0 \leq \mu_\beta Q_\beta^{G_\beta-1}, \quad (\beta, \gamma) \in \mathcal{G} \times \mathcal{U} \\
R_\beta^0 \leq \mu_\gamma R_\beta^{G_\beta-1}, \quad (\beta, \gamma) \in \mathcal{G} \times \mathcal{U}
\end{cases}
\quad (46.c)
$$

Performing a congruence transformation to (47) with $diag(P_\beta, P_\beta^i, R_\beta^i, P_\beta)$, we have

$$
\begin{pmatrix}
\hat{x}_{11}^i(\beta) & \hat{x}_{12}^i(\beta) & d(A_{11}(\beta)P_\beta^i - A_{12}(\beta)K_\beta^i) & dX_\beta^i \\
* & \hat{x}_{22}^i(\beta) & d(A_{d11}(\beta)P_\beta^i - A_{d12}(\beta)K_\beta^i) & dY_\beta^i \\
* & * & -dR^i_\beta & 0 \\
* & * & * & -de^{\alpha_d d}z_\beta^i
\end{pmatrix}
< 0
$$

(47)

where

$$
\hat{x}_{11}^i(\beta) = (P_\beta(A_{11}(\beta) - A_{12}(\beta)C(\beta)) + \chi_\beta^i) + (P_\beta(A_{11}(\beta) - A_{12}(\beta)C(\beta)) + \chi_\beta^i)^T + Q_\beta^i + \alpha_d P_\beta
$$

$$
\hat{x}_{12}^i(\beta) = P_\beta(A_{d11}(\beta) - A_{d12}(\beta)C(\beta)) + Y_\beta^T - \chi_\beta^i,
$$

then the sliding mode dynamics in (40) is exponentially stable for any MDADT switching signals satisfying (5). Now, we define the following matrices:

$$
\mathcal{P}_\beta \triangleq P_\beta^{-1}, \quad \mathcal{Q}_\beta^i \triangleq R_\beta^{i-1}, \quad \mathcal{Q}_\beta \triangleq P_\beta Q_\beta P_\beta
$$

$$
\chi_\beta^i \triangleq \mathcal{P}_\beta^i \chi_\beta^i \mathcal{P}_\beta, \quad Y_\beta^i \triangleq \mathcal{P}_\beta^i Y_\beta^i \mathcal{P}_\beta, \quad K_\beta \triangleq C(\beta)\mathcal{P}_\beta
$$

$$
\forall \beta \in \mathcal{S}, i \in \mathcal{R}_{\beta(t_k)} \quad (48)
$$

\[\square\]
∀ β ∈ S and i ∈ ℜβ(tk)
\[ P_β^j Y_β^{j-1} P_β \geq \exists_β^j. \]  
(50)

By noting (48), and letting \( J_β^j = \exists_β^{j-1} \), and using Schur complement, condition (50) is converted to
\[ \begin{pmatrix} -Y_β^{j-1} & P_β^{-1} \\ * & -\exists_β^{j-1} \end{pmatrix} \leq 0, \]
(51)

which implies (41). Besides, considering (46.a)–(46.c) and noting (48), we have (43.a)–(43.c). This completes the proof. □

Now, it is time to analyse the stability of the sliding mode dynamics in (40) based on the result obtained in Theorem 2, which stated below.

**Theorem 4:** Suppose (A2) holds and for given scalars \( α_β > 0, 0 < η_β \leq 1, μ_β > 1, β \in \mathcal{G} \) satisfying \( (η_β)^{G_β-1} μ_β > 1 \), and \( α_β < 0, 0 < η_β \leq 1, 0 < μ_β < 1, β \in \mathcal{U} \), if there exist matrices \( P_β > 0, Q_β^i > 0, S_β^i > 0, R_β^i > 0 \) and \( 0 < \beta \leq 0, I_{β}^i, i \in \mathcal{S} \), \( i \in \mathcal{R}_β(tk) \), such that \( \forall i \in \mathcal{R}_β(tk) \)

\[
\begin{align*}
Q_β^i & \leq η_β Q_β^{i-1}, \quad β ∈ \mathcal{S}, i \neq 0 \\
S_β^i & \leq η_β S_β^{i-1}, \quad β ∈ \mathcal{S}, i \neq 0 \\
P_β^i & \leq η_β P_β^{i-1}, \quad β ∈ \mathcal{S}, i \neq 0 \\
0^i & \leq μ_β P_γ, \quad (β, γ) ∈ \mathcal{G} × \mathcal{S}, β ≠ γ \\
Q_β^0 & \leq μ_β Q_γ^{G_γ-1}, \quad (β, γ) ∈ \mathcal{G} × \mathcal{S}, β ≠ γ \\
S_β^0 & \leq μ_β S_γ^{G_γ-1}, \quad (β, γ) ∈ \mathcal{G} × \mathcal{S}, β ≠ γ \\
P_β^0 & \leq μ_β P_γ^{G_γ-1}, \quad (β, γ) ∈ \mathcal{G} × \mathcal{S}, β ≠ γ \\
\end{align*}
\]
(52.a)

\[
\begin{align*}
P_γ & \leq μ_γ P_β, \quad (β, γ) ∈ \mathcal{G} × \mathcal{U} \\
Q_γ & \leq μ_γ Q_β^{G_β-1}, \quad (β, γ) ∈ \mathcal{G} × \mathcal{U} \\
S_γ & \leq μ_γ S_β^{G_β-1}, \quad (β, γ) ∈ \mathcal{G} × \mathcal{U} \\
P_γ & \leq μ_γ P_β^{G_β-1}, \quad (β, γ) ∈ \mathcal{G} × \mathcal{U} \\
\end{align*}
\]
(52.b)

and also \( ∀ β ∈ S \) and \( ∀ i ∈ \mathcal{R}_β(tk) \) and \( j, k = 1, 2 \)

\[
\begin{pmatrix}
φ_β^i(d(t), σ_k − π_1^T (\alpha_1^i M + M^T \alpha_2^i) \sigma_1) \\
−π_2^T (\alpha_1^i M + M^T \alpha_2^i) σ_2 \\
−dR_N(β)
\end{pmatrix} < 0,
\]
(53)

where \( π_1, π_2, R_N(ν), G_0(\dot{d}(t)), G_1(d(t)) \), and \( M \) are previously defined in Theorem 2 and for \( α_β > 0 \)

\[ φ_β^i(d(t), \dot{d}(t)) = e^{α_β d} [(G_0^T (d(t)) P_β G_1(d(t))) \\
+ (G_0^T (d(t)) P_β G_1(d(t)))^T ] + e^{α_β d} e_T (Q_β^i + S_β^i) e_1 − \tilde{d}(t) e_T^2 Q_β^i e_2 \\
− e_T^2 S_β^i e_3 + e^{α_β d} de_T (β) R_β^i e_0(β),
\]

and for \( α_β < 0 \)

\[ φ_β^i(d(t), \dot{d}(t)) = (G_0^T (d(t)) P_β G_1(d(t))) \\
+ (G_0^T (d(t)) P_β G_1(d(t)))^T \\
+ e_T (Q_β^i + S_β^i) e_1 − \tilde{d}(t) e_T^2 Q_β^i e_2 \\
− e^{−α_β d} e_T S_β^i e_3 + de_T (β) R_β^i e_0(β),
\]

\[ e_l(l = 1, \ldots, 5) ∈ \mathbb{R}^{n × 5n} \] are elementary matrices, and

\[ e_0(β) = (A_{11}(β) − A_{12}(β) C(β)) e_1 \\
+ (A_{d11}(β) − A_{d12}(β) C(β)) e_2, \]
(54)

then the sliding mode dynamics in (40) is exponentially stable for any MDADT switching signals satisfying (5).

**Proof:** According to Theorem 2, we know that if there exist matrices \( P_β > 0, Q_β^i > 0, S_β^i > 0, R_β^i > 0 \) and \( 0 < \beta \leq 0 \), \( 0 < η_β \leq 1, μ_β > 1 \), \( β ∈ \mathcal{S}, i \in \mathcal{R}_β(tk) \), such that \( \forall i ∈ \mathcal{R}_β(tk) \) satisfying (52.a)–(52.c) and (53), then the sliding mode dynamics in (40) is exponentially stable for any MDADT switching signals satisfying (5). This completes the proof. □

**Remark 8:** It should be noted that since the parameter \( C(β) \) in sliding mode dynamics, is designed for each subsystem, and in accordance with \( C(β) = K_β P_β^{-1} \), \( C(β) \) is dependent on the \( P_β \) and consequently is dependent on the \( P_β \), hence \( V_β^i(x(t)) = x^T(t) P_β^i x(t) \) is continues for each subsystems.

**Remark 9:** In general, the designed sliding mode controller stabilises all subsystems, and in the next step, the entire system. However, in some systems, the controller may not be able to provide the desired stability result alone. In this case, fast switching and slow switching are implemented, and by limiting the dwell time interval of unstable subsystems and increasing the dwell time interval of stable subsystems respectively, they provide more efficient results of stability.
**Remark 10:** Note that the obtained conditions in Theorem 3 and Theorem 4 are not all in LMI form due to (42) and (53), respectively that cannot be solved directly using LMI procedures. However, for variables that cause the inequalities to be converted to BMI, values can be considered with respect to the existing conditions as the default, and by trial and error, the inequalities are solved by the usual method of LMI. But, there is a need for algorithms that can directly solve such BMI problems. One way to solve the BMI optimisation problem is to use the genetic algorithm (GA). Since the BMI optimisation problem in the class of NP is hard, which means it is difficult to solve in case the number of variables is large, the efficient BMI algorithm solving is obtained using a combination with other methods (Kawanishi & Ikuyama, 2005).

Alternating minimisation (AM) is another popular solution method and has been widely used because of its simplicity and effectiveness. For the AM methods, decision variables are divided into two groups. By fixing one group of variables, the other group of variables forms an LMI problem, which is convex and can be solved efficiently. Decision variables in separate groups are then determined alternately during the solving process of LMI (Chiu, 2016). with the result obtained in (Lam & Seneviratne, 2007), we can use a method that combines GA algorithms and AM techniques in the form of convex programming (for example, LMI solvers) to solve BMI-based stability analysis, which will be further developed.

By combining GA and LMI algorithm, the procedure of finding the solution is shown in Figure 1. Considering the Figure 1, the procedure on solving the solution is summarised as the following algorithm.

---

**Algorithm I (GA-based BMI Solver)**

**Step 1.** Initialise an arbitrary value for $P_s = [C(\beta)]$.

**Step 2.** Use GA to generate the potential solution of $P_s = [C(\beta)]$ which will be kept constant and fed to an LMI solver in the subsequent stage.

**Step 3.** The LMI solver will solve the solution $P_m = [P_\beta, Q_\beta, S_\beta, R_\beta, \lambda_1, \lambda_2]$ in (52)–(54), to the LMI problem based on the fixed value of $P_s$ generated by the GA in Step 2. The LMI problem is generally denoted by $L(P_m, P_s) + zI > 0$, where $P_m$ denotes the solution of the LMI problem, and $z$ is a scalar.

**Step 4.** If there exist a negative $z$ such that $L(P_m, P_s) + zI > 0$, it implies that both $P_m$ and $P_s$ satisfy the stability conditions. The problem-solving process stops. In the combined GA and convex programming process, $z$ is taken as a fitness function to indicate the degree of satisfaction of both $P_m$ and $P_s$ to the inequality problem. When $z$ is negative, the small value of $z$ gives a better solution of $P_m$ and $P_s$. Consequently, the solution solving problem can be expressed as a minimisation problem (minimise the value of $z$). A stop criterion will be set to stop the problem-solving process, e.g. a predefined number of iterations is reached.

**Step 5.** If the stop criterion is not met, return to Step 2.

---

**Figure 1.** Procedure of the combined GA and convex programming technique.
In Algorithm I, one can use condition of Theorem 3 instead of Theorem 4 and extract a similar GA-based BMI Algorithm to extract the variables mentioned in Theorem 3.

Now, we want to synthesise a sliding mode controller to drive the system trajectories onto the predefined sliding surfaces $S_\beta(t) = 0$ in (45) which give the following result.

**Theorem 5:** Suppose that (41)–(44) have solutions $P_\beta > 0$, $P_\beta > 0$, $Q_\beta > 0$, $R_\beta > 0$, $M_\beta > 0$, $J_\beta > 0$, $\gamma_i > 0$, $\lambda_\beta$, $\gamma_\beta$, $\lambda_\beta$ and the linear sliding surfaces is given by (45). Then the trajectories of the closed loop system (38) can be driven onto the sliding surfaces $S_\beta(t) = 0$ in finite time with the control $u(t, \beta)$ as follows:

$$u(t, \beta) = -B_1^{-1}[K_\beta P_\beta^{-1}[A_{11}(\beta)z_1(t) + A_{12}(\beta)z_2(t) + A_{11}(\beta)z_1(t - d(t))] + A_{12}(\beta)z_2(t - d(t))] + A_{21}(\beta)z_1(t - d(t))] + A_{22}(\beta)z_2(t) + A_{21}(\beta)z_1(t - d(t))] + A_{22}(\beta)z_2(t - d(t)) - (\rho(\beta) + \delta(\beta))\text{sign}(B_1^TS_\beta(t)), \quad (55)$$

where $\rho(\beta) > 0$, $\beta \in S$, are constants.

**Proof:** We prove that the control law (55) cannot only drive the system trajectories onto the linear sliding surfaces but also keep it there for all subsequent time. Consider the switching function $S_\beta(t) = K_\beta P_\beta^{-1}z_1(t) + z_2(t)$ and choose the Lyapunov functional $W_\beta(t) = (1/2)S_\beta^T(t)S_\beta(t)$. Thus, along with the solution to the system in (42) for a fixed $\beta$ we have

$$\dot{W}_\beta(t) = S_\beta^T(t)\dot{S}_\beta(t) = S_\beta^T(t)(K_\beta P_\beta^{-1}z_1(t) + z_2(t)) = S_\beta^T(t)(K_\beta P_\beta^{-1}[A_{11}(\beta)z_1(t) + A_{12}(\beta)z_2(t) + A_{11}(\beta)z_1(t - d(t))] + A_{12}(\beta)z_2(t - d(t))] + A_{21}(\beta)z_1(t) + A_{22}(\beta)z_2(t) + A_{21}(\beta)z_1(t - d(t))] + A_{22}(\beta)z_2(t - d(t)) + B_1u(t, \beta) + B_1F(\beta)f(t)).$$

Substituting the control law (55) into the above equation and noting that $||S_\beta^T(t)B_1|| < ||S_\beta^T(t)B_1||$, we have

$$W_\beta(t) = S_\beta^T(t)[-B_1(\rho(\beta) + \delta(\beta))\text{sign}(B_1^TS_\beta(t)) + B_1F(\beta)f(t)] = -S_\beta^T(t)B_1(\rho(\beta) + \delta(\beta))\text{sign}(B_1^TS_\beta(t)) - F(\beta)f(t)] \leq -\rho(\beta)\text{sign}(S_\beta^T(t)B_1) + \delta(\beta)S_\beta^T(t)B_1 \leq -\rho(\beta)S_\beta^T(t)B_1$$

$$\leq -\rho(\beta)\sqrt{\lambda_{\text{min}}(B_1B_1^T)}S_\beta^T(t) \leq -\rho(\beta)\sqrt{\lambda_{\text{min}}(B_1B_1^T)(S_\beta^T(t)S_\beta(t))^{1/2}} \leq -\sqrt{2\lambda_{\text{min}}(B_1B_1^T)\min_{\beta \in S}(\rho(\beta))W_\beta(t)^{1/2}} \leq 0,$$

where $\rho \equiv \sqrt{2\lambda_{\text{min}}(B_1B_1^T)\min_{\beta \in S}(\rho(\beta)) > 0}$. As in the proof of Theorem 1, for an arbitrary piecewise constant switching signal $\beta$, and for any $t > 0$, we let $0 = t_0 < t_1 < t_2 < \ldots < t_k < \ldots$, $k = 0, 1, 2, \ldots$, to denote the switching points of $\beta$ over the interval $(0, t)$. The $\beta_k$ subsystem is activated when $t \in [t_k, t_{k+1})$. Integrating $\dot{W}_\beta(t) \leq -\rho W_\beta(t)^{1/2}$ from $t_k$ to $t$, for $t \in L_{\beta(t_k)}^{G_{\beta(t_k)}^{-1}}$, we have

$$W_\beta(t)^{1/2} - W_\beta(t_k + J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1/2} \leq -1/2(\rho(t + J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1}) \leq -1/2(\rho(t + J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1} - t_k - J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1/2}$$

$$\vdots$$

$$W_\beta(t_k + J_{\beta(t_k)}^{G_{\beta(t_k)}})^{1/2} - W_\beta(t_k + J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1/2} \leq -1/2(\rho(t_k + J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1}) \leq -1/2(\rho(t_k + J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1} - t_k - J_{\beta(t_k)}^{G_{\beta(t_k)}})^{-1/2}$$

Summing the terms on both sides of (56) gives

$$W_\beta(t)^{1/2} - W_\beta(t_k)^{1/2} \leq -1/2\rho(t - t_k). \quad (57)$$

Continue the process above, in $L_{\beta(t_k-1)}^{G_{\beta(t_k-1)}^{-1}}$, we have

$$W_\beta(t_k)^{1/2} - W_\beta(t_k-1)^{1/2} \leq -1/2\rho(t_k - t_k-1) \frac{G_{\beta(t_k-1)}^{G_{\beta(t_k-1)}^{-1}}}{(56)}$$

$$W_\beta(t_k-1)^{1/2} - W_\beta(t_k-1-1)^{1/2} \leq -1/2\rho(t_k - t_k-1 - 1) \frac{G_{\beta(t_k-1-1)}^{G_{\beta(t_k-1)}^{-1}}}{(56)}$$

$$\vdots$$

$$W_\beta(t_0)^{1/2} - W_\beta(0)^{1/2} \leq -1/2\rho(t_0 - 0) \frac{G_{\beta(0)}^{G_{\beta(0)}^{-1}}}{(56)}$$

Therefore, we have

$$W_\beta(t)^{1/2} - W_\beta(0)^{1/2} \leq -1/2\rho(t - 0) \frac{G_{\beta(0)}^{G_{\beta(0)}^{-1}}}{(56)}$$

which implies that

$$W_\beta(t) \leq W_\beta(0)\text{sign}(B_1^TS_\beta(t)) + \delta(\beta)S_\beta^T(t)B_1 \leq -\rho(\beta)\text{sign}(S_\beta^T(t)B_1) + \delta(\beta)S_\beta^T(t)B_1 \leq -\rho(\beta)S_\beta^T(t)B_1 \leq -\rho(\beta)\sqrt{\lambda_{\text{min}}(B_1B_1^T)}S_\beta^T(t) \leq -\rho(\beta)\sqrt{\lambda_{\text{min}}(B_1B_1^T)(S_\beta^T(t)S_\beta(t))^{1/2}} \leq -\sqrt{2\lambda_{\text{min}}(B_1B_1^T)\min_{\beta \in S}(\rho(\beta))W_\beta(t)^{1/2}} \leq 0,$$
\[ W_\beta(t_k-1 + j_\beta(t_{k-1})^{-1})^{1/2} - W_\beta(t_k-1 + j_\beta(t_{k-1})^{-1})^{1/2} \leq -1/2 \rho(j_\beta(t_{k-1}) - j_\beta(t_{k-1})). \]

Summing the terms on both sides of (58) gives
\[ W_\beta(t_{k-1})^{1/2} - W_\beta(t_{k-1})^{1/2} \leq -1/2 \rho(t_k - t_{k-1}). \]  

Considering (59) for \( k = 0, 1, 2, \ldots \), gives
\[ W_\beta(t_0)^{1/2} - W_\beta(t_0)^{1/2} \leq -1/2 \rho(t_1 - 0). \]  

Summing the terms on both sides of (57) and (60) gives
\[ W_\beta(t)^{1/2} - W_\beta(0)^{1/2} \leq -1/2 \rho t. \]  

It can be seen from (61), there exists a time \( t^* \leq 2W_\beta(0)^{1/2}/\rho \) such that \( W_\beta(t) = 0 \), and consequently \( S_\beta(t) = 0 \) for \( t \geq t^* \) which means the system trajectories can reach onto the predefined sliding surfaces in a finite time. Since the reaching condition \( S_\beta^T(t)\dot{S}_\beta(t) < 0 \) holds, the system trajectories will be kept within the sliding surfaces for all subsequent time. This completes the proof. \( \square \)

Synthesising a sliding mode controller to drive the system trajectories onto the predefined sliding surfaces \( S_\beta(t) = 0 \) in (45) with consideration Theorem 4, gives the following result.

**Theorem 6:** Suppose that (52)–(54) have solutions \( P_\beta > 0, Q_\beta > 0, S_\beta > 0, R_\beta^i > 0, \beta_{1i}^2, \beta_{2i}^2, \) and the linear sliding surfaces is given by (45). Then the trajectories of the closed loop system (38) can be driven onto the sliding surfaces \( S_\beta(t) = 0 \) in finite time with the control \( u(t, \beta) \) as follows:

\[
\begin{align*}
    u(t, \beta) &= -B_1^{-1}(K_\beta P_\beta^{-1}[A_{11}(\beta)z_1(t) + A_{12}(\beta)z_2(t) + A_{d11}(\beta)z_1(t) - d(t))] \\
    &+ A_{d12}(\beta)z_2(t - d(t)) + A_{d21}(\beta)z_1(t - d(t)] + A_{d22}(\beta)z_2(t - d(t) \right)
\end{align*}
\]

where \( \rho(\beta) > 0, \beta \in \mathcal{S} \) are constants and \( d \) is an known upper bound for delay. Now, consider the switching function \( S_\beta(t) = K_\beta P_\beta^{-1}z_1(t) + z_2(t) \) and assume \( \sup_{s \in [-d, 0]}z_1(t + s) \leq r_1(t) - r_1 \) and \( \sup_{s \in [-d, 0]}z_2(t + s) \leq r_2(t) - r_2 \), where an adaptive law is used to estimate \( r_1 \) and \( r_2 \). Here, choose the Lyapunov functional \( W_\beta(t) = (1/2)S_\beta^T(t)S_\beta(t) + (1/2)(r_1(t) - r_1)^2 + (1/2)(r_2(t) - r_2)^2 \). Thus, along with the solution to the system in (42) for a fixed \( \beta \) we have

\[
\begin{align*}
    \dot{W}_\beta(t) &= S_\beta^T(t)\dot{S}_\beta(t) + r_1(t)(r_1(t) - r_1) \\
    &= -\rho(\beta)(\beta)\text{sign}(B_1^T S_\beta(t)),
\end{align*}
\]

**Proof:** By replacing \( C(\beta) \) instead of \( K_\beta P_\beta^{-1} \) in Theorem 5, the proof of this theorem is similar to the proof of Theorem 5. Therefore, the proof is omitted, here. \( \square \)

Notice that, in Theorem 5 and Theorem 6, each subsystem has different parameters and constants (e.g. \( A_{nn}(\beta), \rho(\beta), \delta(\beta) \)) in the control input structure. On the other hand, in this work, the controllers are considered synchronous, and for each subsystem, a separate controller is designed. Therefore, the control input is related to \( \beta \), so it is written as \( u(t, \beta) \).

It is worth noting, that the sliding mode controller in (55) and (62) is more applicable only when the time-varying delay \( d(t) \) is explicitly known a priori, since there exist \( z_1(t - d(t)) \) and \( z_2(t - d(t)) \) in (55) and (62). However, in some practical situations, the information for delay \( d(t) \) is unavailable or difficult to measure. One way to overcome this is to use an adaptive sliding mode controller for the system (38), which is to use a controller that just involves the upper bound of the delay. In this case, there is no need to know the law of variation, which is convenient for robustness issues. To do this, consider the following controller instead of (55):

\[
\begin{align*}
    u(t, \beta) &= -B_1^{-1}(K_\beta P_\beta^{-1}[A_{11}(\beta)z_1(t) + A_{12}(\beta)z_2(t) \\
    &+ A_{d11}(\beta)z_1(t) - d(t))] \\
    &+ A_{d12}(\beta)z_2(t - d(t)) + A_{d21}(\beta)z_1(t - d(t)] \\
    &+ A_{d22}(\beta)z_2(t - d(t) \right)
\end{align*}
\]

where \( \rho(\beta) > 0, \beta \in \mathcal{S} \) are constants.
\[+ \dot{r}_2(t)(r_2(t) - r_2)\]
\[= S_\beta^T(t)(K_\beta P_\beta^{-1}z_1(t) + \dot{z}_2(t))\]
\[= S_\beta^T(t)\{K_\beta P_\beta^{-1}[A_{11}(\beta)z_1(t) + A_{12}(\beta)z_2(t) + A_{d11}(\beta)z_1(t - d(t)) + A_{d12}(\beta)z_2(t - d(t))] + A_{d21}(\beta)z_1(t - d(t)) + A_{d22}(\beta)z_2(t - d(t))\} + B_1u(t, \beta) + B_1F(\beta)f(t)\]
\[+ \dot{r}_1(t)(r_1(t) - r_1) + \dot{r}_2(t)(r_2(t) - r_2)\]
\[= -S_\beta^T(t)B_1\{(\rho(\beta) + \delta(\beta))\text{sign}(B_1^TS_\beta(t)) - m_1(\beta)(K_\beta P_\beta^{-1}) - m_2(\beta) - F(\beta)f(t)\}
\[+ \dot{r}_1(t)(r_1(t) - r_1) + \dot{r}_2(t)(r_2(t) - r_2),\]

where \(m_1 = (((|A_{d11}(\beta)||\sup_{s\in[-d_0]}|z_1(t + s)||) /B_1) + (((|A_{d12}(\beta)||\sup_{s\in[-d_0]}|z_2(t + s)||)) /B_1))\) and \(m_2 = (((|A_{d21}(\beta)||\sup_{s\in[-d_0]}|z_1(t + s)||)) /B_1) + (((|A_{d22}(\beta)||\sup_{s\in[-d_0]}|z_2(t + s)||)) /B_1).\)

We know that \(m_2(\beta) \geq 0\), but negativity and positiveness of \(m_1(\beta)\) depends on \(K_\beta P_\beta^{-1}\). We have:

\[\dot{W}_\beta(t) \leq -S_\beta^T(t)B_1\{(\rho(\beta) + \delta(\beta))\text{sign}(B_1^TS_\beta(t)) - \left(\frac{A_{d11}(\beta)(r_1(t) - r_1)}{B_1} + \frac{A_{d12}(\beta)(r_2(t) - r_2)}{B_1}\right)(K_\beta P_\beta^{-1})\}
\[+ \dot{r}_1(t)(r_1(t) - r_1) + \dot{r}_2(t)(r_2(t) - r_2)\]
\[= -S_\beta^T(t)B_1\{(\rho(\beta) + \delta(\beta))\text{sign}(B_1^TS_\beta(t)) - (\frac{A_{d11}(\beta)(r_1(t) - r_1)}{B_1} + \frac{A_{d12}(\beta)(r_2(t) - r_2)}{B_1})(K_\beta P_\beta^{-1})\}
\[+ \dot{r}_1(t)(r_1(t) - r_1) + \dot{r}_2(t)(r_2(t) - r_2)\]
\[= -S_\beta^T(t)B_1\{(\rho(\beta) + \delta(\beta))\text{sign}(B_1^TS_\beta(t)) - (\frac{A_{d11}(\beta)(r_1(t) - r_1)}{B_1} + \frac{A_{d12}(\beta)(r_2(t) - r_2)}{B_1})(K_\beta P_\beta^{-1})\}
\[+ \dot{r}_1(t)(r_1(t) - r_1) + \dot{r}_2(t)(r_2(t) - r_2)\]

With the adaptive law \(\dot{r}_1(t) = -S_\beta^T(t)A_{d11}(\beta)(K_\beta P_\beta^{-1})\) and \(\dot{r}_2(t) = -S_\beta^T(t)A_{d12}(\beta)(K_\beta P_\beta^{-1})\), we have:

\[\dot{W}_\beta(t) \leq -(\rho(\beta) + \delta(\beta))|S_\beta^T(t)B_1| + \delta(\beta)|S_\beta^T(t)B_1|\]
\[\leq -\rho(\beta)|S_\beta^T(t)B_1|\]
\[\leq -\rho(\beta)\sqrt{\lambda_{\min}(B_1B_1^T)}|S_\beta^T(t)|\]
\[= -\rho(\beta)\sqrt{\lambda_{\min}(B_1B_1^T)}(S_\beta^T(t)S_\beta(t))^{1/2}\]
\[\leq -\sqrt{2\lambda_{\min}(B_1B_1^T)}\min_{\forall \beta \in S}(\rho(\beta))W_\beta(t)^{1/2}\]
\[\Delta = -\rho W_\beta(t)^{1/2} < 0,\]

where \(\rho \triangleq \sqrt{2\lambda_{\min}(B_1B_1^T)}\min_{\forall \beta \in S}(\rho(\beta)) > 0\). As in the proof of Theorem 5, for an arbitrary piecewise constant switching signal \(\beta\), and for any \(t > 0\), it can be seen there exists a time \(t^* \leq 2W_\beta(0)^{1/2}/\rho\) such that \(W_\beta(t) = 0\), and consequently \(S_\beta(t) = 0\) for \(t \geq t^*\). This completes the proof. \(\square\)
5. Numerical example

In this section, we present some examples illustrating how our Theorems can be applied to a time-varying delay switched system. In the first part, where Example 1 and Example 2 are presented, the stability analysis and comparison of our results with existing work are considered, and in the second part, where Example 3 and Example 4 are presented, stabilisation is provided using slide mode control. Note that the practical application of the presented results is given in Example 4.

5.1. Stability

Example 1: Consider the switched delay system (3) as in (Wu & Lam, 2008), with two subsystems as follows

Subsystem 1 : \( A(1) = \begin{pmatrix} -0.4 & 0.2 \\ 0.2 & -0.3 \end{pmatrix}, \)

\( A_d(1) = \begin{pmatrix} -0.2 & 0 \\ 0.1 & -0.4 \end{pmatrix} \)

and \( d = 1.2, \sigma = 0.3 \). We compare our results with the one in (Wu & Lam, 2008). By using Theorem 1, and choosing \( \alpha_1 = \alpha_2 = 0.5, G_1 = G_2 = 3 \), and considering Table 1, the LMIs conditions (6)–(7) will be feasible, in both cases 1 and 2. The comparison results are listed in Table 2.

It can be seen that the obtained results in this paper under both cases, provide better outcomes in determining the tighter band on dwell time, to ensure the stability of the system. We can obtain the state response, related to case 1 as shown in Figure 2 with initial state condition \( x(\theta) = (10 \ -5)^T, \theta \in [-1.2, 0] \), and choosing \( \tau^a_1 \geq \tau^a_{\beta_1} = 0.29, \tau^a_2 \leq \tau^a_{\beta_2} = 0.14 \). We can see that the switched system is stable under designed MDADT switching signal. According to \( \max_{\beta \in S} \{ ((\ln \mu_\beta (\eta_\beta))^{G_\beta - 1})/\tau^a_\beta - \alpha_\beta \} \), the exponential decay rate is \( \max \{-0.0106 \ -0.0816\} \), so it is equal to \(-0.0816\).

Example 2: Consider the switched delay system (3) as in (Zhao et al., 2013), with two subsystems as follows

Subsystem 2 : \( A(2) = \begin{pmatrix} -0.2 & 0.3 \\ 0.2 & -0.7 \end{pmatrix}, \)

\( A_d(2) = \begin{pmatrix} -0.3 & 0.1 \\ 0 & -0.2 \end{pmatrix} \)

and \( d = 1.2, \sigma = 0.3 \). We compare our results with the one in (Wu & Lam, 2008). By using Theorem 1, and choosing \( \alpha_1 = \alpha_2 = 0.5, G_1 = G_2 = 3 \), and considering Table 1, the LMIs conditions (6)–(7) will be feasible, in both cases 1 and 2. The comparison results are listed in Table 2.

![Figure 2. Simulation result for Example 1. States of the closed-loop system with designed MDADT switching signal.](image-url)
Table 3. Design parameters.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Case 1, $d = 2$</th>
<th>Case 2, $d = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Zhao et al., 2013)</td>
<td>$\mu = 2$</td>
<td>$\mu = 2$</td>
</tr>
<tr>
<td>$\alpha = 0.11$</td>
<td>$\alpha = 0.05$</td>
<td></td>
</tr>
<tr>
<td>Theorem 1</td>
<td>$\mu_1 = \mu_2 = 1.4$</td>
<td>$\mu_1 = \mu_2 = 1.6$</td>
</tr>
<tr>
<td>$\eta_1 = 0.98, \eta_2 = 0.89$</td>
<td>$\eta_1 = 0.98, \eta_2 = 0.92$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_1 = \alpha_2 = 0.1$</td>
<td>$\alpha_1 = \alpha_2 = 0.1$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. Comparative results of $\tau^*_{\beta}$.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Zhao et al., 2013)</td>
<td>$\tau^*_{1\beta} = 6.3$</td>
<td>$\tau^*_{1\beta} = 13.86$</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>$\tau^<em>_{2\beta} = 2.96, \tau^</em>_{2\beta} = 1.03$</td>
<td>$\tau^<em>_{2\beta} = 4.29, \tau^</em>_{2\beta} = 3.03$</td>
</tr>
</tbody>
</table>

$$A_d(1) = \begin{pmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{pmatrix}$$

Subsystem 2: $A(2) = \begin{pmatrix} -2 & 3 \\ 1 & -4.2 \end{pmatrix}$,

$$A_d(2) = \begin{pmatrix} 0.1 & 0.8 \\ 0.3 & 0.1 \end{pmatrix}$$

We compare our results with the one in (Zhao et al., 2013). By using Theorem 1, and choosing $G_1 = G_2 = 3$, and considering Table 3, which represents the design parameters, the LMIs conditions (6)–(7) will be feasible, in both cases 1 and 2. The comparison results are listed in Table 4.

It is clear that the obtained results in this paper are less conservative in determining the tighter dwell time band than those in (Zhao et al., 2013). We can obtain the state response, related to case 1 as shown in Figure 3 with initial state condition $x(\theta) = (5 \ 8)^T$, $\theta \in [-2, 0]$, and choosing $\tau^*_1 \geq \tau^*_1 = 3$, $\tau^*_2 \leq \tau^*_2 = 1.1$. We can see that the switched system is stable under designed MDADT switching signal. According to $\max_{\beta \in S} \left\{ (\ln \mu_\beta (\eta_\beta)/\tau^*_{\beta}) - \alpha_\beta \right\}$, the exponential decay rate is $\max \{ -0.0013, -0.006 \}$, so it is equal to $-0.006$.

5.2. Stabilisation

Example 3: Consider the switched delay systems (1) with three modes and parameters as follows

Subsystem 1 (Stable):

$$A_d(1) = \begin{pmatrix} 0 & -0.04 & 0.06 \\ -0.05 & 0 & 0.03 \\ -0.1 & 0.3 & 0.5 \end{pmatrix}, F(1) = 1.6,$$

$$A_d(2) = \begin{pmatrix} -1.5 & 0.1 & 0 \\ -0.2 & -2 & 0 \\ -0.5 & 1 & 1 \end{pmatrix}, F(2) = 2,$$

Subsystem 3 (Stable):

$$(5 \ 8)^T, \theta \in [-2, 0], \text{ and choosing } \tau^*_1 \geq \tau^*_1 = 3, \tau^*_2 \leq \tau^*_2 = 1.1. \text{ We can see that the switched system is stable under designed MDADT switching signal. According to } \max_{\beta \in S} \left\{ (\ln \mu_\beta (\eta_\beta)/\tau^*_{\beta}) - \alpha_\beta \right\}, \text{ the exponential decay rate is } \max \{ -0.0013, -0.006 \}, \text{ so it is equal to } -0.006.$$
\[
A(3) = \begin{pmatrix}
-2 & 0.1 & 0 \\
-0.2 & -0.8 & 0 \\
0.08 & 0.04 & -1.3
\end{pmatrix},
\]
\[
A_d(3) = \begin{pmatrix}
-0.6 & 0.3 & 0.9 \\
0.5 & 0 & 0.03 \\
-0.1 & 0.4 & 0.6
\end{pmatrix},
\]
and \( f(t) = 0.5e^{-t}\sin(t) \), \( B = (0 \ 0 \ 2)^T \), \( d = 0.11, \sigma = 0.01 \). Giving initial state condition \( x(\theta) = (-2 \ 2 \ 4)^T, \theta \in [-2, 0] \), to the switched system with \( u(t) = 0 \), and under the random switching signal, the switched system will be unstable as shown in Figure 4.

Therefore, the purpose is to design a sliding mode controller \( u(t) \) such that the closed-loop system is stable, under designed MDADT switching. By choosing \( \alpha_1 = 0.9, \alpha_2 = -3, \alpha_3 = 0.8, G_1 = G_2 = G_3 = 3, \mu_1 = \mu_3 = 6, \mu_2 = 0.71, \) and \( \eta_1 = \eta_2 = \eta_3 = 0.7 \) Conditions (45)–(50) will be feasible, which gives MDADT switching signals as \( \tau^*_1 = 1.98, \tau^*_2 = 0.351, \tau^*_3 = 1.348 \) and control parameter as
\[
\mathcal{P}_1 = \begin{pmatrix}
32.427 & -0.037 \\
-0.037 & 32.561
\end{pmatrix},
\]
\[
\mathcal{P}_2 = \begin{pmatrix}
12.813 & -0.656 \\
-0.566 & 13.624
\end{pmatrix},
\]
\[
\mathcal{P}_3 = \begin{pmatrix}
31.8 & -0.021 \\
-0.021 & 32.98
\end{pmatrix},
\]
\[
\mathcal{K}_1 = (0.107 \ 0.098), \quad \mathcal{K}_2 = (-1.400 \ -0.865), \quad \mathcal{K}_3 = (-1.725 \ 1.1911)
\]

According to (45), we have
\[
S_1(t) = \mathcal{K}_1 \mathcal{P}_1^{-1} z_1(t) + z_2(t)
\]
\[
= (0.0033 \ 0.0030 \ 1) x(t)
\]
\[
S_2(t) = \mathcal{K}_2 \mathcal{P}_2^{-1} z_1(t) + z_2(t)
\]
\[
= (-0.1128 \ -0.069 \ 1) x(t)
\]
\[
S_3(t) = \mathcal{K}_3 \mathcal{P}_3^{-1} z_1(t) + z_2(t)
\]
\[
= (-0.0543 \ -0.036 \ 1) x(t)
\]

The rest of the task is to design a sliding mode controller such that the system trajectories can be driven onto the predefined sliding surfaces (64)–(66) and stays there for all subsequent times. When delay \( d(t) \) is explicitly given as \( d(t) = 0.1 + 0.01\sin(t) \), the sliding mode controller in (55) can be calculated as
\[
u(t, 1) = -\frac{1}{2} \begin{pmatrix}
0.046 & 0.006 & -1 \\
-0.1 & 0.299 & 0.5
\end{pmatrix} x(t)
\]
\[
+ \begin{pmatrix}
\delta(1) + 0.8 \sin(S_\beta(t)) \\
\delta(2) + 1 \sin(S_\beta(t))
\end{pmatrix}
\]
\[
u(t, 2) = -\frac{1}{2} \begin{pmatrix}
-0.317 & 1.127 & 1 \\
-0.444 & 1.031 & 0.912
\end{pmatrix} x(t)
\]
\[
+ \begin{pmatrix}
\delta(1) + 0.8 \sin(S_\beta(t)) \\
\delta(2) + 1 \sin(S_\beta(t))
\end{pmatrix}
\]
\[
u(t, 3) = -\frac{1}{2} \begin{pmatrix}
0.181 & 0.006 & -1.3
\end{pmatrix} x(t)
\]

Figure 4. Simulation result for Example 3. States of the open-loop system with random switching signal.
\[ \begin{pmatrix} 0.131 & 0.024 & -1.348 \end{pmatrix} x(t - d(t)) \]
\[ - (\delta(3) + 0.9) \text{sign}(S_{\beta}(t)) \]
\[ \quad \text{(69)} \]

Set \( \delta(1) = \delta(2) = \delta(3) = 1 \). To avoid the control signals from chattering, we replace \( \text{sign}(S_{\beta}(t)) \) with \( (S_{\beta}(t))/(0.02 + |S_{\beta}(t)|) \). Figure 5 shows the state responses of the closed-loop switched system with (67)–(69) and \( \tau_{1}^{a} \geq \tau_{1}^{a*} = 1.225 \), \( \tau_{2}^{a} \leq \tau_{2}^{a*} = 0.345 \), \( \tau_{3}^{a} \geq \tau_{3}^{a*} = 1.4 \). According to \( \max_{\beta \in S} \left\{ \left( \ln \mu_{\beta} (\eta_{\beta})^{G_{\beta}^{-1}} / \tau_{\beta}^{a} \right) - \alpha_{\beta} \right\} \), the exponential decay rate is \( \max\{-0.02, -0.06, -0.03\} \), so it is equal to \(-0.06\). The switched control input and the sliding functions are given in Figures 6 and 7, respectively.

**Example 4:** A model of combustion in rocket motor chambers (Zheng & Frank, 2002), is considered here for stabilisation with sliding mode control, to illustrate the effectiveness of the obtained results in practical applications. This model represents a liquid monopropellant rocket motor with a pressure feeding system. Under the assumption of nonsteady flow and lumped lag factor, an appropriate linearised model can be in

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**Figure 5.** Simulation result for Example 3. States of the closed-loop system with designed MDADT switching signal.

**Figure 6.** Simulation result for Example 3. Switched control input.
the form of switched delay systems (1) with the following coefficients (Mahmoud, 2010):

\[
A(\beta) = \begin{pmatrix}
\rho \beta - 1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1/\eta J \\
-\frac{p}{2J(1-\xi)} & 0 & -\frac{1}{J(1-\xi)} & \frac{1}{J(1-\xi)} \\
0 & \frac{1}{E_e} & -\frac{1}{E_e} & 0
\end{pmatrix},
\]

\[
A_d(\beta) = \begin{pmatrix}
-\rho \beta & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
B = \begin{pmatrix}
0 \\
-\frac{1}{\eta J} \\
0 \\
0
\end{pmatrix},
\]

and \( f(t) = 0.2e^{-t}\sin(t) \), where \( \beta \) represents the mode of operation, and Tables 5 and 6 show the data and Parameters of Example 4.

Giving initial state condition \( x(\theta) \) = (1 1 1 1)\( T \), \( \theta \in [-1,0] \), and \( d = 1 \), \( \sigma = 0.5 \), to the switched system with \( u(t) = 0 \), and under the random switching signal, the switched system will be unstable as shown in Figure 8.

Therefore, the purpose is to design a sliding mode controller \( u(t) \) such that the closed-loop system is stable, under designed MDADT switching. By choosing \( \alpha_1 = \alpha_2 = \alpha_3 = 0.01 \), \( \mu_1 = \mu_2 = \mu_3 = 1.3 \), and \( \eta_1 = \eta_2 = \eta_3 = 0.88 \). Conditions (45)–(50) will be feasible, which gives MDADT switching signals as \( \tau_1^{a*} = \tau_2^{a*} = \tau_3^{a*} = 0.67 \) and control parameter as

\[
\mathcal{P}_1 = \begin{pmatrix}
0.045 & 0.001 & 0.006 \\
0.006 & -0.006 & 0.027 \\
0.060 & -0.001 & 0.009 \\
0.009 & -0.014 & 0.033
\end{pmatrix},
\]

\[
\mathcal{P}_2 = \begin{pmatrix}
0.046 & 0.001 & 0.006 \\
0.006 & -0.006 & 0.027 \\
0.001 & 0.051 & -0.006 \\
0.009 & -0.014 & 0.033
\end{pmatrix},
\]

\[
\mathcal{P}_3 = \begin{pmatrix}
-6.252 & 2.832 & 1.414 \\
-6.023 & 2.616 & 1.224 \\
-6.415 & 3.012 & 1.632
\end{pmatrix},
\]

\[
K_1 = (\begin{pmatrix}
-6.252 & 2.832 & 1.414
\end{pmatrix}),
\]

\[
K_2 = (\begin{pmatrix}
-6.023 & 2.616 & 1.224
\end{pmatrix}),
\]

\[
K_3 = (\begin{pmatrix}
-6.415 & 3.012 & 1.632
\end{pmatrix}).
\]

Table 6. Parameters of illustrative example 4.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>Pressure exponent of the combustion process</td>
<td>Stated in Table 5</td>
</tr>
<tr>
<td>( E_e )</td>
<td>Line elasticity parameter</td>
<td>2</td>
</tr>
<tr>
<td>( \xi )</td>
<td>Fractional length for pressure supply</td>
<td>0.1</td>
</tr>
<tr>
<td>( J )</td>
<td>Line inertia</td>
<td>2</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Ratio of steady-state pressure and steady-state injector pressure drop</td>
<td>2</td>
</tr>
</tbody>
</table>
According to (45), we have

\[ S_1(t) = K_1 P_1^{-1} z_1(t) + z_2(t) = \begin{pmatrix} -0.2667 & 0.1271 & -0.0171 & 1 \end{pmatrix} x(t) \] (70)

\[ S_2(t) = K_2 P_2^{-1} z_1(t) + z_2(t) = \begin{pmatrix} -0.3486 & 0.2278 & -0.0511 & 1 \end{pmatrix} x(t) \] (71)

\[ S_3(t) = K_3 P_3^{-1} z_1(t) + z_2(t) = \begin{pmatrix} -0.2795 & 0.1363 & -0.0147 & 1 \end{pmatrix} x(t) \] (72)

The rest of the task is to design a sliding mode controller such that the system trajectories can be driven onto the predefined sliding surfaces (70)–(72) and stays there for all subsequent times. When delay \(d(t)\) is explicitly given as \(d(t) = 0.5 + 0.5 \sin(t)\), the sliding mode controller in (55) can be calculated as

\[ u(t, 1) = \frac{-1}{5} \{(-0.5423 & 0.6413 & -0.1271 
-0.6827) x(t) + (-0.2534 & 0 & 0 
-0.2667) x(t - d(t)) - (\delta(1) + 1.5) \text{sign}(S_\beta(t)) \} \] (73)

\[ u(t, 2) = \frac{-1}{5} \{(-0.573 & 0.8112 & 0.2278 & 0.7834 
-0.1896 & 0.5556 & 0 & -0.9042) x(t) + (-0.2342 & 0.5556 & 0 & -0.8351) x(t - d(t)) - (\delta(3) + 1.3) \text{sign}(S_\beta(t)) \} \] (74)

\[ u(t, 3) = \frac{-1}{5} \{(-0.5975 & 0.6293 & 0.1363 & -0.6919 
-0.2534 & 0.5556 & 0 & -0.8351) x(t) + (-0.2342 & 0.5556 & 0 & -0.8351) x(t - d(t)) - (\delta(3) + 1.3) \text{sign}(S_\beta(t)) \} \] (75)

Set \(\delta(1) = \delta(2) = \delta(3) = 1\). To avoid the control signals from chattering, we replace \(\text{sign}(S_\beta(t))\) with \(((S_\beta(t))/(0.02 + \text{abs}(S_\beta(t))))\). Figure 9 shows the state responses and phase trajectories of the closed-loop switched system with (73)–(75) and \(\tau_{a1} \geq \tau_{a*1} = 0.8, \tau_{a2} \geq \tau_{a*2} = 0.8, \tau_{a3} \geq \tau_{a*3} = 0.8\). The switched control input and the sliding functions are given in Figure 10, respectively.

**Remark 11:** The practical applications of the introduced system, namely the delayed switching system, have been studied in the literature using simulation and numerical analysis. Among them, we can mention (Mahmoud, 2010), in which a model of combustion in rocket motor chambers is considered for feedback stabilisation. This model is in the form of the class of linear switched time-delay systems. The mechanical rotational cutting process, considering the failure in its actuators during system operation, can be modelled as a class of linear switched time-delay systems. The stability of these systems has been investigated in (Sun et al., 2006), using the state feedback control and ADT approach. The stabilisation of networked systems with time-varying delays in the form of the class of linear switched time-delay systems, is studied.
in (Ma & Zhao, 2015). Recently, in (Kani et al., 2019), the stability analysis of a model of high-speed super cavitation vehicles in the form of switched time-delay systems with time-delayed state-dependent switching conditions is proposed.

6. Conclusion
In this paper, stability analysis and sliding mode controller were investigated for a class of switched time-varying delay systems. The stability issue of switched systems with time-varying delay was studied by employing the multiple discontinuous Lyapunov function approach, and using the mode-dependent average dwell time (MDADT) switching regime, sufficient conditions have been proposed to guarantee the exponential stability of the unforced system with existence unstable subsystems, which offers a tighter dwell time-bound with less conservativeness. These conditions were relaxed by using an equivalent form of the Bessel–Legendre inequality in constructing Lyapunov–Krasovskii functional, which was the first attempt in this area. Existence sufficient conditions of a reduced-order sliding mode dynamics were derived, and by using the combined genetic algorithm and convex programming techniques, the desired sliding surfaces conditions became into the form of LMIs. To illustrate the effectiveness of the obtained theoretic
results, and to clarify the differences with other works, the proposed method was compared with the existing methods. The comparison showed that a tighter bound of dwell time, can be achieved with the proposed method, and showed that with the increase of the delay limit, the stability of the system with the proposed method is still maintained, which indicates less conservativeness in results. Finally, a numerical example showing the practical application of the proposed results was given.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

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References


