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Pseudo-spectral versus Intelligent Optimal Control: Exo-Atmospheric Guidance Accuracy

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ABSTRACT

Exo-atmospheric guidance is a nonlinear finite-horizon optimal control problem with constraints on endpoint and control signals. The nonlinear optimization is iterative and takes a substantial amount of time to converge. Pseudospectral (PS) methods, due to coarse discretization, are favored to be a good choice for reducing the computation load. In this respect, the Legendre-Gauss (LG) and Legendre- Gauss-Lobbato (LGL) techniques are tried. Moreover, the genetic algorithm (GA) is also utilized for reference. In this case, the results show that LG PS indices are better than the GA both in terms of accuracy and convergence time. Nevertheless, while LGL PS performs faster than GA but actual (not approximate) system simulation is not accurate and further modifications are required for successful operation.

Keywords: Optimal control, Intelligent optimization, Collocation methods, Legendre Pseudo-spectral, Guidance

1. INTRODUCTION

A global payload delivery flight consists of endo, exo, re-entry, and terminal phases where each requires appropriate methods of guidance. In the exo-atmospheric part, the vehicle advances in the vacuum where it is controlled by divert-attitude control rather than by aerodynamic control surfaces. With no path constraints, the guidance objective is concluding the endpoint conditions [1] which is a finite horizon optimal control problem. In the direct formula, the original optimality equations are used while in indirect routine the reformulated equations by employing the Lagrange multiplier and the Hamiltonian function are employed. In any case, the problem is discretized and it is solved by iterative nonlinear programming algorithms.

The global optimization methods find the solution by intelligently searching the problem space and are used to solve real-life complex problems arising from different fields such as economics, engineering, politics, management, and engineering. Hyper-Spherical Search method has been used in [2] to tune PSS of power systems. In [3], the genetic algorithm has been practiced for optimal tuning of guidance law. Intelligent tinning of a robust μ-PID controller has been applied in [4].

There are various classical optimization methods, which seek the optimal solution by tracing the derivative of the cost function. In all, the discretization step size is a problem where fine segmentation slows the convergence rate and coarse ones damage the correctness. Types of coarse splitting such as Gauss, Gauss–Lobatto, and Gauss-Radau are suggested in the so-called Pseudo-spectral methods that to some extent prevent perfection loss. Among them is the Gauss–Lobatto that gives the highest accuracy for polynomial integrands and produce significantly smaller mesh sizes. In [5], an optimal control problem is solved by discretizing the equations at a series of Legendre-Gauss-Lobatto points, then the trajectory states are approximated by using local Hermite interpolating polynomials. A robust pseudospectral method is presented in [6] for milling operations considering uncertainties in both modal parameters and cutting coefficients. Multi-objective gearshift optimization with Legendre pseudo-spectral method for seamless two-speed transmission has been discussed in [7]. Apart from the global allocation methods, there are local ones that divide the integral range into multiple segments and for each segment, Gauss points are assigned [8].

Several vacuum guidance approaches have been published. The earliest one is the Iterative Guidance Mode which was designed for the Saturn class vehicles [9]. An application of the pseudo-spectral model predictive method for exo-atmospheric guidance has been discussed in [10]. Comparison of the Legendre–Gauss pseudospectral and Hermite–Legendre–Gauss–Lobatto methods for low-thrust spacecraft trajectory optimization has been addressed in [11]. Time-energy optimal guidance strategy for realistic interceptor using the pseudospectral method is the subject of study in [12]. The optimal midcourse trajectory planning approach that considers the capture region of the terminal guidance and is based on the Gauss pseudospectral method has been detailed in [13].

In this paper, the exo-atmospheric guidance using an approximate solution by LG and LGL Pseudo-spectral methods are presented and the results are compared by the performance of the intelligent GA (Genetic algorithm). The study indicates that the LG PS outcome is promising but while the LGL PS method behaves faster than GA, its accuracy is not acceptable.

The paper presentation is as follows. In Section 2, the vehicle flight dynamics and optimal control paradigms are briefly elaborated. The pseudo-spectral methods are discussed in short terms in Section 3. The intelligent control and numerical simulation results are detailed in Section 4 and lastly, the conclusion is presented in Section 5.

2. EXO-ATMOSPHERIC OPTIMAL CONTROL

2-1. Vehicle dynamics

The entire flight path has been shown in Fig. 1. In the midcourse the atmospheric effects are low and flight in the vacuum is conducted with no affects from aerodynamic forces as it has been expressed by the following equations [10],

$$
\begin{aligned}
&\dot{r} = v\sin\theta \\
&\dot{v} = \frac{T\cos\alpha\cos\sigma}{m} - g\frac{r_e^2}{r^2}\sin\theta & \dot{\phi} &= \frac{v\cos\theta\sin\psi}{r} \\
&\dot{\theta} = \frac{T\sin\alpha}{mv} + \left(\frac{v}{r} - g\frac{r_e^2}{r^2 v}\right)\cos\theta & \dot{\lambda} &= \frac{v\cos\theta\cos\psi}{r\cos\phi} \\
&\dot{\psi} = \frac{T\cos\alpha\sin\sigma}{mV\cos\theta} - \frac{v\cos\theta\sin\psi\tan\phi}{r} & \dot{m} &= -k
\end{aligned}
\tag{1}
$$

where r is the radial distance from the center of the earth to the vehicle, v is the velocity, θ is the flight path angle, ψ is the heading angle, φ is the latitude, λ is the longitude, m is the mass of the vehicle, σ is the bank angle α is the pitch angle of the jets command.

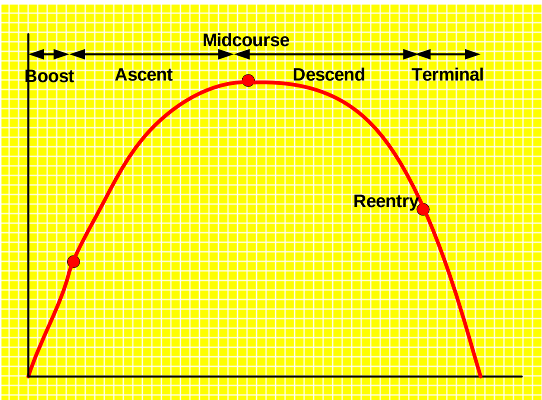

Fig. 1. Flight trajectory

Considering the longitudinal motion (1) of the vehicle on a freeway while assuming no interaction with other vehicles, the state equation of the vehicle at instant *t* can be represented by

$$\dot{x}(t) = f(x,u) \tag{2}$$

Because of the absence of atmosphere, the flight has no path constraints, but only limited by the physical constraint of control and endpoint conditions as follows:

$$
\begin{aligned}
&r_f - r_i \approx 0, V_f - V_i \approx 0, \theta_f - \theta_i \approx 0 \\
&\psi_f - \psi_i \approx 0, \phi_f - \phi_i \approx 0, \lambda_f - \lambda_i \approx 0
\end{aligned}
\tag{3}
$$

2-2. **Optimal control**

The finite horizon optimal control of the dynamic system (2) means minimizing the cost function,

$$\min_u J = h\left(x_f, t_f\right) + \int_{t_0}^{t_f} g\left(x,t,u\right) dt$$
$$g(.) = x(t)^T Qx(t) + u(t)^T Ru(t) \quad (4)$$

subject to the system constraints such as,

$$\dot{x} = f\left(x,u,t\right) \quad (5)$$
$$x_{t_f} = x_f,\ x_{t_0} = x_0$$
$$|u| < \pm 90 \quad (6)$$

There are various ways to numerically solve an optimal control problem. In the direct method which is conceptually simpler, the original optimization formulation (4) is manipulated. Various discretization styles exist, most of them stemming from the vast field of numerical solution of differential equations; especially collocation methods have been popular, due to the high order of numerical accuracy they possess.

In indirect methods, the problem is reformulated using the λ Lagrange multiplier and H Hamiltonian function as given below,

$$\min_u J = h\left(x_f, t_f\right) + \int_{t_0}^{t_f} [g\left(.\right) + \lambda'(f\left(x,u,t\right) - \dot{x})]dt$$
$$H(.) = g(.) + \lambda' f(.)$$

where the solution should satisfy the following Euler-Lagrange equations,

$$f\left(x,u,t\right) - \dot{x} = 0, \quad \frac{\partial H}{\partial x} - \dot{\lambda} = 0, \quad \frac{\partial H}{\partial u} = 0$$

Compared to the calculus of variations based indirect methods, direct methods do not require the derivation of the 1rst order necessary conditions and are less sensitive to the initial guess [11].

3. PSEDOSPECTRL METHODS

Pseudo-spectral optimization methods solve dynamic optimization problems by discretizing the state-space, creating a discretized version of a continuous problem. The resulting discretized optimization problems are solved by standard codes of nonlinear optimization. These methods use orthogonal collocation points to achieve accurate quadrature approximations.

f(t) function approximation:

Nonlinear functions can be approximated by the summation of weighted basis polynomials as given below,

$$f(t) \approx \sum_{i=1}^{N} a_k \phi_k(t)$$

where ϕ_k is the global basis functions and α_k is the weights. For periodic signals, it is the Fourier series with a global trigonometric basis which can model the function perfectly. For nonperiodic signals, a polynomial basis has been found appropriate and so the method is called pseudo-spectral.

Similar to other nonlinear functions approximations, the objective is to determine the coefficients α_k so the residual becomes zero at the set of collocation points.

$$R(t_i) = 0, \quad i = 1, \ldots, N$$

The basis function can be the Lagrange polynomials,

$$f(t) \approx \sum_{i=1}^{N} a_k L_k(t), \qquad L_i(t) = \prod_{k=1, k\neq i}^{N} \frac{t - t_k}{t_i - t_k}, \qquad L_i(t) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

The Lagrange polynomials are advantageous because their coefficients are equal to the value of the approximating polynomial at the collocation points,

$$a_k = f(t_k)$$

Integral approximation and collocation points

Quadrature method is a common approach for a definite integral approximation as expressed below,

$$\int_{-1}^{1} f(t)dt = \int_{-1}^{1} \sum_{i=1}^{N} L_i(t) f(t_i) dt \approx \sum_{i=1}^{N} \omega_i f(t_i), \quad \omega_i = \int_{-1}^{1} L_i(t) dt$$

where ω_i's are the quadrature weights and t_i's are the quadrature points or nodes. For reducing the mash size while preserving the computation accuracy, Gauss, Gauss-Lobbot, and Gaus-Radu quadratures have been suggested. An example of a 7 point Gauss and Gauss-Lobbto quadratures and their relevant weights have been shown in Fig. 2.

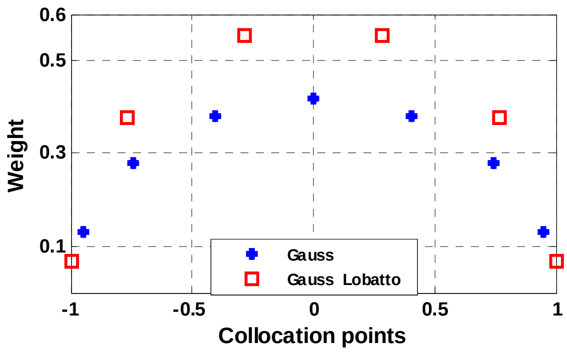

Fig. 2. The Gauss and Gauss-Lobbato points and weights

The N collocation points exist in the domain [-1, 1], therefore, for a real problem the points are mapped onto a real-time domain $[t_0, t_f]$ using the following transformation,

$$t = \frac{(t_f - t_0)\tau + (t_f + t_0)}{2}$$

$\dot{x}$ Derivative approximation

The derivative of the state, $\dot{x}$ is approximated as the exact derivative of the interpolating Lagrange polynomials. So it is computed as below,

$$\frac{dx}{dt}(t_k) \approx \sum_{i=1}^{N} x(t_i) D_{ki}, \qquad k = 1, \dots, N$$

$$D_{ki} = \frac{dL_i(t)}{dt}(t_k)$$

where D_{ki} is the derivative of the Lagrange interpolating polynomials.

In the following, two of the established methods are reviewed.

3-1. Differential Legendre Gauss (DLG) PS control

In this approach, the Gauss quadrature method is employed and the collocation points are determined as the zeros of the Nth degree Legendre polynomial P_N where the weights are computed using,

$$\omega_i = \frac{2}{1 - t_i^2} \left[\dot{P}_N(t_i) \right]^2$$

In this respect, the discretized form of (4) is expressed by [14],

$$J = h(x_f, t_f) + \frac{(t_f - t_0)}{2} \sum_{k=1}^{N} \omega_k \cdot g(x_k, u_K, \tau_k)$$

where the integral t_0 to t_f range is transformed to -1 to 1 band and the overall path is partitioned into N segments. The initial and final point constraints (6) and the final value computation are simply defined by,

$$\phi(x_0,t_0,x_f,t_f)=0, \qquad x_f = x_0 + \frac{(t_f - t_0)}{2}\sum_{k=1}^{N}\omega_k \cdot f(x_k,u_k,\tau_k)$$

Discretization of ẋ is conducted using the following equation,

$$\frac{2}{(t_f - t_0)}\bar{D}_i \cdot x_0 + \frac{2}{(t_f - t_0)}\sum_{k=1}^{N} D_{ik} \cdot x_k = f(x_k,u_k,t_k), \qquad i = 1,\cdots,N$$

The first term is to consider the initial point which is not in the set of Gauss points.

3-2. **Legendre-Gauss-Lobatto (LGL) PS optimal control**

This method uses the Gauss-Lobbato quadrature method. The collocation points are the zeros of the derivative of the Legendre polynomial of degree N – 1 plus the two endpoints, -1, 1 which are calculated as below [14],

$$\omega_i = \frac{2}{N(N-1)P_{N-1}^2(t_i)}, \quad t_i \neq \pm 1,$$
$$\omega_i = \frac{2}{N(N-1)}, \qquad t_i = \pm 1$$

Then the integral cost function (5) is approximated by,

$$J = \Phi(X_N,t_f) + \frac{(t_f - t_0)}{2}\sum_{k=1}^{N} g(x_k,u_K,\tau_k)\cdot \omega_k$$

The derivative matrix, *D,* composes of the derivative of the Lagrange polynomials at the GL points. Therefore, the resulting algebraic constraints are

$$\frac{2}{(t_f - t_0)}\sum_{i=1}^{N} D_{ki} \cdot X_i = f(x_k,u_k,\tau_k), \quad k = 1,\cdots,N$$

The boundary constraints (6) are enforced using the boundary points of the approximating polynomial for the state, x_1, and x_N,

$$\phi(x_1,t_0,x_N,t_f) = 0$$

4. Simulations

The vehicle dynamics are simulated in MATLAB based on the model parameters borrowed from [10] that have been depicted in Table I.

TABLE I. The vehicle and flight parameters

	Variables	Initial condition	Terminal condition
r	Altitude	118 km+Re	145 km+Re
v	Velocity	5400 m/sec	6700 m/sec
θ	Flight path angle	10°	4°
ψ	Azimuth angle	-0.03°	-0.1°
φ	Latitude	2°	2°
λ	Longitude	0.0004°	2°
m	Vehicle mass	17000 kg	10000 kg
Re		6378km	
T		890kN	
ṁ		-200kg/s	

The vehicle is expected to reach the final point at a fixed time with a minimum final error. Considering ṁ, the flight time is t_f=35s. The flight cost function is defined by,

$$J = \frac{1}{N} * \left(x(t_f) - x_f\right) Q \left(x(t_f) - x_f\right)$$
$$Q = diag([10^{-4};10^{-3};10^{2};10^{3};10^{5};10;0])$$

The algorithms are weighted up based on J and the convergence time. To have a comparison index, fist a GA algorithm is run and the results are used for the evaluation of the pseudospectral algorithms

4-1. **Test 1:GA-PSO**

Among the metaheuristic algorithms, the Genetic algorithm (GA) is inspired by the biological evolution process. The Genetic algorithm is an intelligent search algorithm that has been developed to imitate the mechanics of natural selection and natural genetics through the following steps [15, 16]:

Generate a random initial population
While stop conditions met
    Compute the fitness function, f_i
    Normalize f_i and rank them
    Select parents
    Do crossover
    Apply mutation
    Form a new population including Best solutions
end

For applying GA to the exo-atmospheric guidance, the integral interval is partitioned into 6 equally spaced segments formed by 7 nodes. Matlab ga is used for executing the search for the optimal solution. After several runs, the average elapsed time of 21s is obtained with an average of J=2.27 for minim normalized endpoint error. The system states and the controls have been shown in Fig 3 where the initial and the endpoints are marked distinctively. The simulation is exact, not approximate as ode45 does not rely on the PS nodes, rather it uses whatever step sizes are required for accurate computation.

4-2. **Test 2: LG PS**

In this test, the DG PS method is used with N=7 equivalent to 8 segments. The elapsed time of the algorithm execution is 11.5s. The optimization result for the approximate system is J =7e-8 which is perfect for the approximate system and objective function as is shown in Fig. 4. However, when the simulation of the exact system using the obtained control signal is performed, J rises to 3.7 which is failrly acceptable with respect to the GA result which is as low as 2.27. The exact system responses have been shown in Fig. 4 with dashed lines. Meanwhile, using the linear interpolated input rises the cost function to the unappropriate J=54.

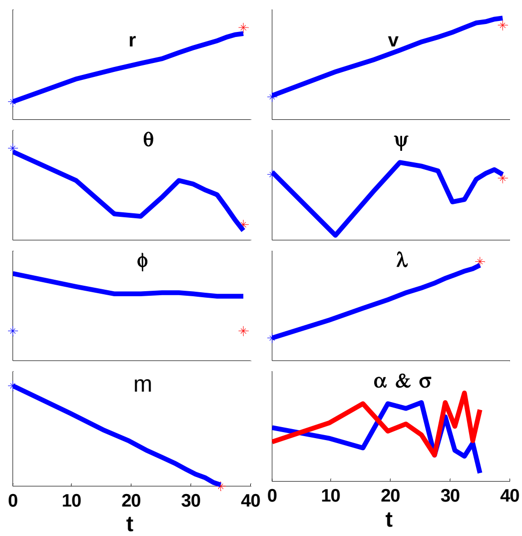

Fig. 3. Genetic algorithm result for optimal guidance

4-3. **Tets 3:LGL PS**

In another test, the Legendre Gauss Lobboto (LGL) PS optimal control is tried. The number of nodes is 7 which makes 6 segments. It took 15s for the algorithm to converge to the minimum of J= 3.33 which is pretty good. The result has been illustrated in Fig. 5. However, when an exact system simulation is conducted, utilizing the obtained control signal, the endpoint normalized error of J= 37 is obtained which is higher than the one obtained by GA.

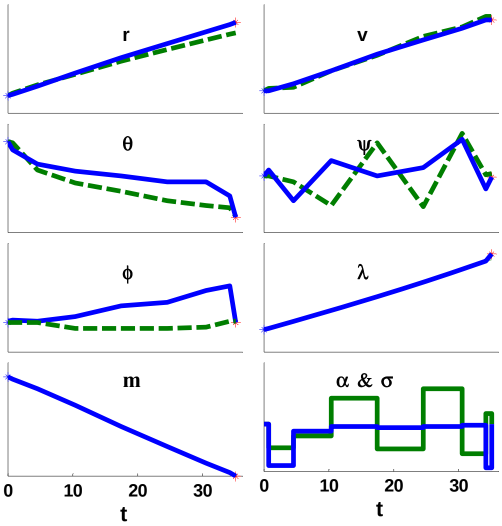

Fig. 4. DG PS Algorithm results: approximate (solid) and exact (dashed) system responses.

By increasing the number of nodes to 40, as is expected better results are obtained at the cost of higher convergence time which mounts to 52.71s. In this test, J=6.53 is obtained from the simulation of the approximate system, and J = 9.5175 for its real counterpart.

CONCLUSION

In this paper, the pseudospectral methods of optimal control are used for exo-atmospheric guidance. The algorithm accuracy and convergence time are compared with the intelligent GA algorithm. The LG PS is %50 faster than GA and both almost meet the endpoint similarly. On the other hand, the LGLPS overtakes the GA algorithm in converges rate but fails in providing accurate results.

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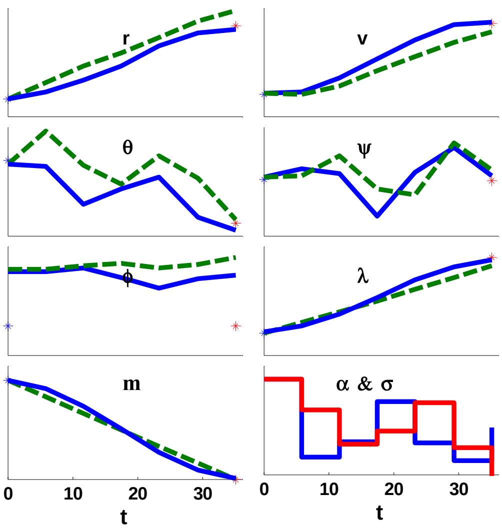

Fig. 5. LGL PS guidance a) approximated and b) exact system responses

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