Journal of Industrial and Systems Engineering
Vol. 13, No. 3, pp. 193-215
Summer (July) 2021

On the effect of measurement errors and auto-correlation on the performance of Hotelling's T^2 control chart

Hosseinali Beydaghi[1], Amirhossein Amiri[1*], Zahra Jalilibal[1], Reza Kamranrad[2]

[1]*Department of Industrial Engineering, Shahed University, Tehran, Iran*
[2]*Department of Industrial Engineering, Semnan University, Semnan, Iran*

hosseinalibeydaghi7395@gmail.com, amiri@shahed.ac.ir, zjalili222@alumni.ut.ac.ir, rezakamranrad@yahoo.com

Abstract

Independency of observations is one of the fundamental assumptions in control charts. However, in some processes this assumption is violated and data are auto-correlated. Also, it is assumed that the measurement errors are absent in measurement system while, this assumption is usually violated. The existence of the auto-correlation and measurement errors causes the poor performance of the control charts. In other words, the average run length in the case of out-of-control (OC) situations increases in the presence of auto-correlation and measurement errors. In this paper, the effect of auto-correlation and measurement errors on the performance of Hotelling's T^2 control charts in Phase II in multivariate normal processes is investigated in terms of average run length (ARL) criterion. The first order auto-regressive model as auto-correlation structure between observations within each sample is discussed in this paper. To decrease the effect of auto-correlation and measurement errors on the performance of the Hotelling's T^2 control chart, jump strategy and multiple measurements methods are applied, respectively. The effect of auto-correlation and measurement errors, individually and simultaneously, as well as the performance of the suggested methods to address these effects is appraised through simulation studies and a numerical example. The effect of number of measurements and jumps on the ARL values of the proposed control chart is also evaluated. Results show the acceptable performance of the multiple measurements and jumps methods in diminishing the effect of measurement errors and auto-correlation, respectively. At last, a real case is presented to show the application of the proposed scheme.

Keywords: Average run length, jump strategy, measurement errors, multiple measurements, multivariate control chart, the first order auto-regressive model

1-Introduction

In recent years, with the advancement of technology and data collection methods, the time between successive sampling of products has been decreased and as a result the independency of observations is violated and become auto-correlated. On the other hand, in some processes, the measurement system is not accurate enough and causes measurement errors in the collected observations. The presence of auto-correlation and measurement errors greatly affects the performance of control charts and increases the false alarm rate of control charts, which leads to misleading results.

---

*Corresponding author
ISSN: 1735-8272,

For example, Lina and Woodall (2001) demonstrated that when the observed values have the measurement errors, the control limits of the $\bar{X}$ control chart are affected and the power of the control chart decreases. Maravelakis et al. (2004) also checked the performance of the exponentially weighted moving average (EWMA) control chart in the existence of measurement errors and demonstrated that the observed values in the presence of the measurement errors affect the control limits. They introduced multiple measurements method to neutralize the negative effect of the measurement errors. Abbasi (2010) also checked the effect of the two-component measurement errors on the performance of the EWMA control chart and demonstrated that applying multiple measurements decreases the effect of measurement errors. Maravelakis (2012), checked the effect of measurement errors on the performance of Cumulative Sum control chart. Noorossana and Zerehsaz (2015) investigated the effect of the measurement errors on the performance of the EWMA-3 control charts for monitoring the profiles and demonstrated that the measurement errors affect the IC and OC ARL of the control chart in Phase II. Lina et al. (2001) examined the effect of measurement errors on the performance of multivariate control charts and demonstrated that the measurement errors affect the performance of the chi-squared control chart. Ding and Zeng (2015) also investigated the effect of measurement errors in multi-stage processes. Regarding the fact that the quality characteristics were adjusted in different stages of production with the regression model, they demonstrated that the measurement errors influence the estimated parameters of the regression model.

Maragah and Woodal (1992) studied effect of auto-correlated data on the performance of the $\bar{X}$ control chart and showed that auto-correlation results in increasing the rate of false alarm. Franco et al. (Franco et al., 2014)used the jump strategy to ameliorate the performance of the $\bar{X}$ control chart and demonstrated that using the jump strategy can improve the performance of the $\bar{X}$ control chart in the presence of auto-correlation. Goswami and Dutta (2014) applied EWMA control chart for monitoring the auto-correlated data processes. Leoni et al. (2015) investigated the jump strategy to address the effect of auto-correlation on the performance of T^2 control chart. Kalgondaand and Kulkarni (2004) monitored the multivariate auto-correlated processes. Soleimani and Noorossana (2014) studied the multivariate simple linear profiles monitoring when auto-correlation exists between profiles. Also, Soleimani et al. (2009) and Soleimani et al. (2014), Khedmati and Niaki(2016), Kazemzadeh et al. (2010)studied simple linear and polynomial profiles monitoring in the existence of auto-correlation as well.

Costa and Castagliola (2011) checked the effect of auto-correlation and measurement errors on the performance of the $\bar{X}$ control chart and demonstrated that the performance of $\bar{X}$ control chart is influenced by the auto-correlation structure as well as measurement errors. Xiaohong and Zhaojun (2009) checked the effect of auto-correlation and measurement errors on the performance of the CUSUM control chart. Sabahno et al. (2019a) appraised the performance of variable sampling intervals Hotelling's T^2 control chart in the presence of measurement errors. Sabahno et al. (2019b) studied the effect of measurement errors on the performance of variable parameters (VP) $\bar{X}$ control chart. Sabahno et al. (2018) investigated the effect of measurement errors on the performance of variable sample size Hotelling's T^2 control chart. Amiri et al. (2016) appraised the effect of measurement errors on the performance of EWMA control chart for simultaneous monitoring of mean and variability of multivariate normal processes. Maleki et al. (2017) provided a review paper in the field of measurement errors in statistical process monitoring (SPM). Ghasghaei et al. (2016) investigated the effect of measurement errors on the performance of control chart for simultaneous monitoring of mean and variability of the process by using ranked set sampling (RSS).

Shishebori and Hamedani (2010) evaluated the effect of gauge measurement capability and dependency measure of process variables on the MCp. Chattinnawat and Belin (2017) assessed the performance of the Hotelling's T^2 control chart for monitoring individual observations in Phase II under multivariate normal inspection errors. Rakhmawati et al. (2020) compared the performance of the generalized confidence interval (GCI) and the modified sampling distribution (MSD) methods for assessing process capability in the presence of gauge measurement.

Most of the papers, which discussed the effect of measurement errors on the performance of different control charts, assumed that the observations within each sample are independent. However, when the time between successive sampling collapse, the observations are auto-correlated. As mentioned before, the auto-correlation and measurement errors lead to negative performance of control charts. We discuss a multivariate normal process and investigate the effect of the auto-correlation and measurement errors on the in-control(IC) and OC ARL performance of chi-square control charts, in this paper. In other words, the effect of measurement errors on chi-square chart performance for monitoring auto-correlated multivariate normal process, is investigated.

In particular, the first-order auto-regressive (AR(1)) time series is applied for modeling the auto-correlation structure, in this paper. Also, to decrease the effect of measurement errors, multiple measurements method has been used. Moreover, the jump strategy is applied to address the auto-correlation effect. The structure of the paper is as follows: In section 2, the effect of measurement errors on the performance of the chi-square chart is appraised. In section 3, the auto-correlation effect on the performance of the chi-square chart is evaluated under the AR(1) model. Also, in this section, the jump strategy is presented to neutralize the negative effect of auto-correlation structure. In Section 4, the performance of the chi-square chart under multiple measurements and jump strategy is examined. In Section 5, simulation studies are executed to appraise the performance of chi-square control chart in the existence of auto-correlation and measurement errors individually and simultaneously. In section 6, a real case study is presented. Conclusion and a suggestion for future research are given in the final section.

2-The effect of auto-correlation on the performance of chi-square control chart

Time series is a sequence of observations, usually arranged in terms of time, especially at equal intervals, but sorting may be due to other dimensions such as distance. The inherent nature of time series is the auto-correlation of its observations. Therefore, the order of observation is important. In this section, we suppose that the measurement procedure is precise, and there are no measurement errors at each sampling. The replication is equal to one. The time series, which is investigated in this section, is a static time series model, which means that the distribution and corresponding parameters are constant over time.

Often with a finite number of observations, a parametric pattern of finite order is made to express a time series process. In this section, the AR(1) model is discussed. Suppose that in time i (i=1,2,…), the successive observations in k^{th} quality characteristic $\left\{x_{i,1,k}, x_{i,2,k}, ..., x_{i,n,k}\right\}$ are modeled by the AR(1) model.

$$x_{i,j,k} - \mu_{0,k} = \phi(x_{i,(j-1),k} - \mu_{0,k}) + \varepsilon_{i,j,k}, i = 1,2,3,..., j = 1,2,...,n, k = 1,2,..., p. \tag{1}$$

$x_{i,j,k}$ is the actual value of the j^{th} observation in k^{th} quality characteristic at time i . $\mu_{0,k}$ is the mean of the k^{th} quality characteristic, ε_{ijk} is the error term of j^{th} (j=1,2,…,n) observation in k^{th} quality characteristic at time i ,which follows an independent normal distribution with mean zero and variance σ_k^2 , and ϕ_k is the auto-correlation coefficient of the AR(1) model in k^{th} quality characteristic. Note that samples at different times are independent. The variance of x_{ijk} , $\text{var}(x_{ijk})$, is computed as equation (2): (Alwan and Radson, 1992)

$$\text{var}(x_{ijk}) = \frac{\sigma_k^2}{1-\phi_k^2}. \tag{2}$$

Also, the standard deviation of mean sample is equal to:

$$\sigma\left(\bar{x}_{ik}\right) = \frac{\sigma_k}{\sqrt{n}C_{2k}}, k = 1,2,..., p, \tag{3}$$

where $\bar{x}_{ik}$ is the mean sample and it is computed as $\bar{x}_{ik} = \left(x_{i,1,k} + ... + x_{i,n,k}\right)/n$,

and C_{2k} related to k^{th} quality characteristic is equal to equation (4):

$$C_{2\text{k}} = \sqrt{\frac{n}{n + 2\left(\Phi_k^{n+1} - n\Phi_k^2 + (n-1)\Phi_k \big/ (\Phi_k - 1)^2\right)}}. \tag{4}$$

The proposed statistic for monitoring the mean process vector is as follows:

$$T_i^2 = \left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)^t \boldsymbol{\Sigma}_{\overline{\mathbf{x}}}^{-1} \left(\overline{\mathbf{x}}_i - \boldsymbol{\mu}_0\right), \tag{5}$$

where $\overline{\mathbf{x}}_i = (\overline{\text{x}}_{i1}, \overline{\text{x}}_{i2}, ..., \overline{\text{x}}_{ip})^T$, $\mu_0 = (\mu_1, \mu_2, ..., \mu_p)$.

The variance- covariance matrix of $\overline{\mathbf{x}}_i$ vector is as below:

$$\boldsymbol{\Sigma}_{\overline{\mathbf{x}}} = \begin{pmatrix} \sigma_{\overline{\text{x}}_1}^2 & \sigma_{\overline{\text{x}}_1\overline{\text{x}}_2} & \cdots & \sigma_{\overline{\text{x}}_1\overline{\text{x}}_p} \\ \sigma_{\overline{\text{x}}_2\overline{\text{x}}_1} & \sigma_{\overline{\text{x}}_2}^2 & \cdots & \sigma_{\overline{\text{x}}_2\overline{\text{x}}_p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\overline{\text{x}}_p\overline{\text{x}}_1} & \sigma_{\overline{\text{x}}_p\overline{\text{x}}_2} & \cdots & \sigma_{\overline{\text{x}}_p}^2 \end{pmatrix}. \tag{6}$$

Note that in this paper, it is assumed that ϕ_k (k=1,2,…,p) values are equal to ϕ .

Values of $\sigma_{\overline{x}_1\overline{x}_2}, ..., \sigma_{\overline{x}_p\overline{x}_p}$ can be obtained by using equation (7).

$$\text{cov}(\overline{x}_k, \overline{x}_l) = \frac{\sigma_{x_k, x_l}}{n^2}(n + (n-1)\phi_k\phi_l + (n-2)\phi_k^2\phi_l^2 + ... + \phi_k^{n-1}\phi_l^{n-1}), \tag{7}$$

where ϕ_h and ϕ_l, respectively, are the auto-correlation coefficients related to the h^{th} and l^{th} quality characteristics.

The UCL of the proposed statistic in equation (5) is determined by simulation such that the IC ARL of the control chart equals to 200 is obtained. The value of C_{2k}, corresponding to the positive values of ϕ_k , is placed in the (0,1] interval.

In table 1, the values of C_{2k} are given for $n = \{1, 2, ..., 7\}, \phi = \{0.1, 0.2, ..., 0.7\}$. According to the results of the table, it is clear that the value of C_{2k} approaches to one by increasing n or decreasing ϕ . As C_{2k} values are decreasing, the power of control chart to detect the shift in the mean of process reduces for auto-correlated data. To reduce the auto-correlated data effect on the performance of the T^2 control chart, samples with non-neighboring items are used. In other words, one jump or two jumps or more before the next sampling can remove the influence of auto-correlation and the performance of T^2 control chart. A process with p quality characteristics discussing the AR(1) auto-correlation between observations within a sample with no measurement errors is shown schematically in figure 1. Note that the p quality characteristics are also correlated.

Table 1. The values of C_{2k} for different values of n and f

	C_{2k}			
Φ	n=1	n=2	n=3	n=4
0.0	1.0000	1.0000	1.0000	1.0000
0.1	1.0000	0.9535	0.9366	0.9283
0.2	1.0000	0.9129	0.8793	0.8626
0.3	1.0000	0.8771	0.8276	0.8023
0.4	1.0000	0.8452	0.7809	0.7470
0.5	1.0000	0.8165	0.7385	0.6963
0.6	1.0000	0.7906	0.7001	0.6498
0.7	1.0000	0.7670	0.6652	0.6073
	n=5	n=6	n=7	n=10
0.0	1.0000	1.0000	1.0000	1.0000
0.1	0.9234	0.9202	0.9179	0.9120
0.2	0.8528	0.8464	0.8419	0.8214
0.3	0.7874	0.7777	0.7710	0.7213
0.4	0.7267	0.7135	0.7042	0.6108
0.5	0.6704	0.6532	0.6411	0.4934
0.6	0.6182	0.5967	0.5814	0.3771
0.7	0.5699	0.5440	0.5250	0.2672

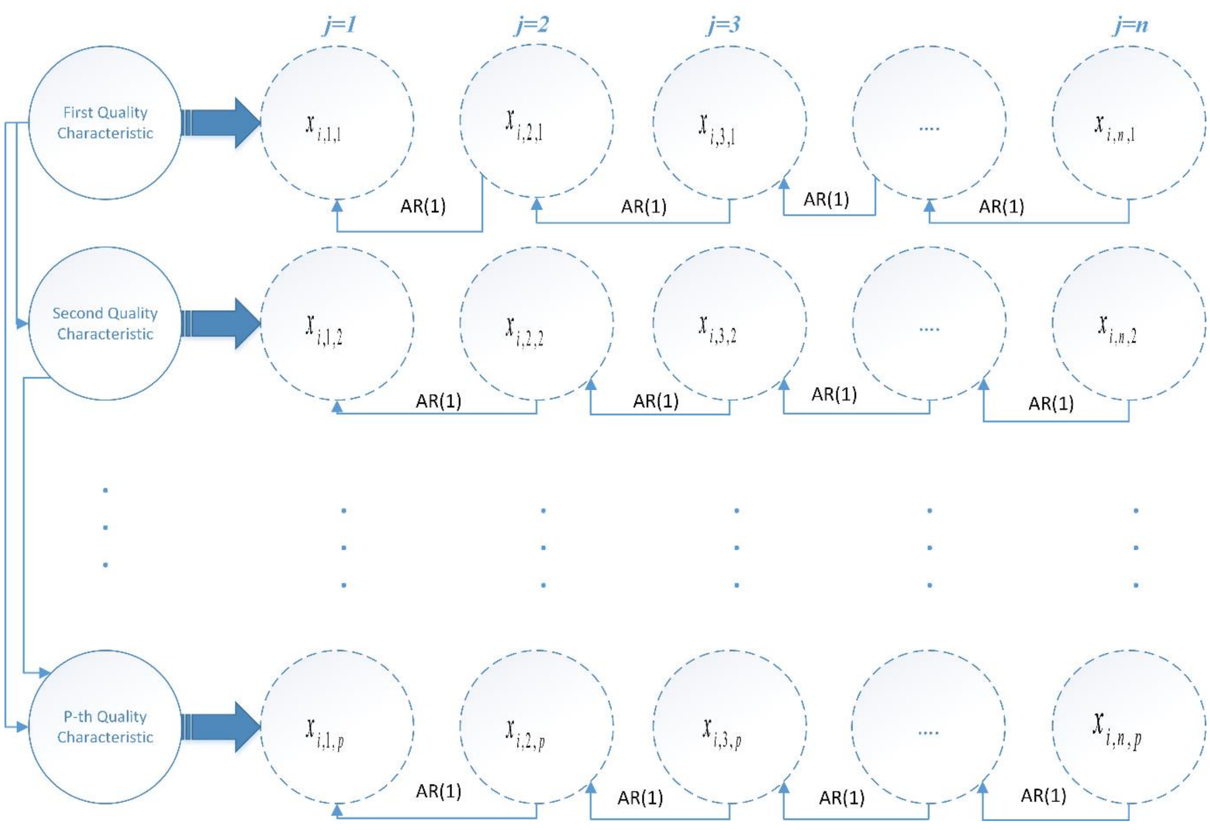

Fig 1. A p-variate normal process discussing AR(1) auto-correlation within each sample

The auto-regressive model with the jump strategy is given as:

$$\mathrm{x}_{i,j,k} - \mu_{0,k} = \phi_k^{s+1}\left(\mathrm{x}_{i,j-s-1,k} - \mu_{0,k}\right) + \varepsilon'_{ijk}, i = 1,2,..., j = 1,...,n, k = 1,2,...,p, s = 1,2,... \quad (8)$$

The error term ε'_{ijk}, is equal to $\varepsilon'_{ijk} = \varepsilon_{ijk} + \phi_k \varepsilon_{i(j-1)k} + \cdots + \phi_k^s \varepsilon_{i(j-s)k}$. As a result, the observations of the k^{th} quality characteristic $\{x_{i,1,k}, x_{i,s+2,k}, x_{i,2s+3,k}, x_{i,3s+4,k}, ...\}$ is proportional to the AR (1) model with the parameter ϕ^{s+1}. While the number of jumps increases in sampling, the values of C_{2k} close to one, and as a result, the auto-correlation of the data decreases and the performance of the control chart improves.

3- Effect of measurement errors on the performance of control chart

The measured values corresponding to the real value of $x_{i,j,k}$ with m replications is as the set of $\{y_{i,j,k,1}, y_{i,j,k,2}, ..., y_{i,j,k,m}\}$, $m \geq 1$. Therefore, the observed values are equal to equation (9):

$$y_{i,j,k,h} = x_{i,j,k} + e_{i,j,k,h}. \quad (9)$$

$e_{i,j,k,h}$ is the independent error term of k^{th} quality characteristic with the mean zero and known variance. A process with p quality characteristics in the existence of measurement errors is schematically shown in figure 2.Note that the p quality characteristics are also correlated and $y_{i,j,k,h}$ is the h^{th} replication for the k^{th} quality characteristic at time i.

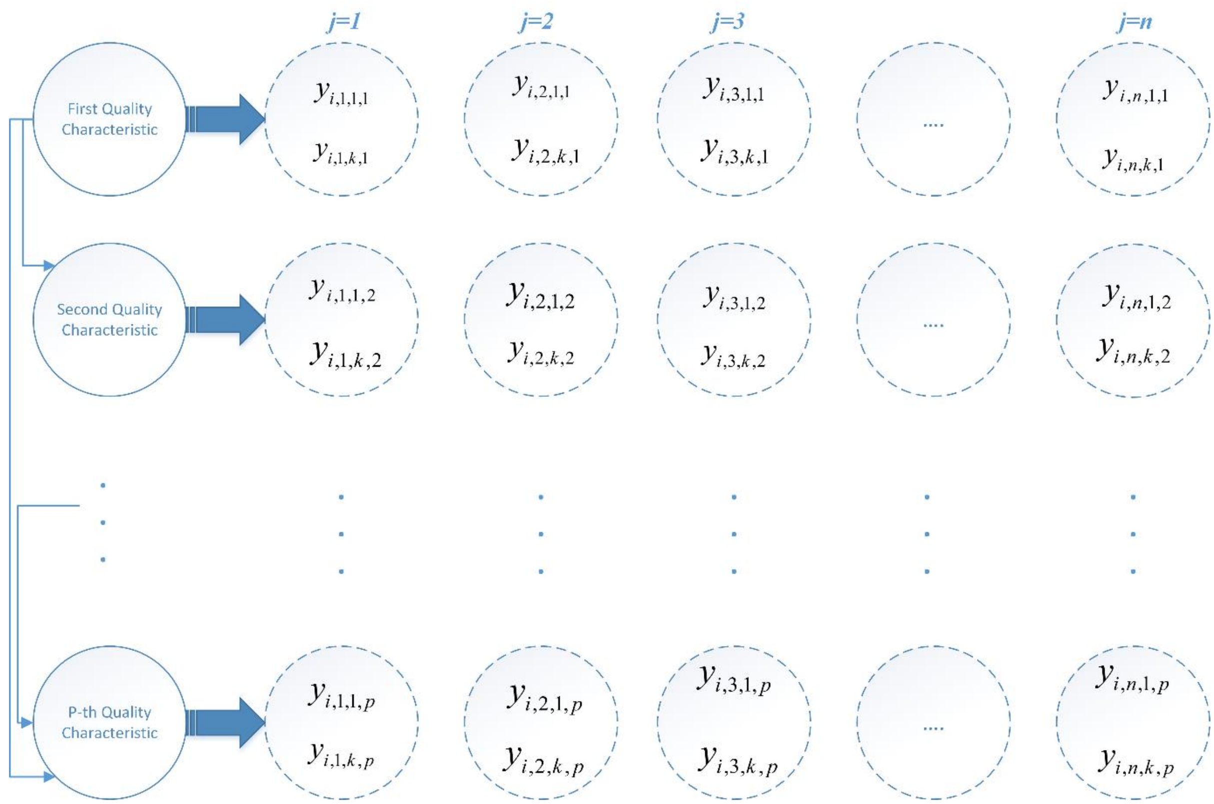

Fig 2. A p-variate normal process discussing measurement errors

The sample mean of observations for k^{th} quality characteristic is as follows:

$$\bar{y}_{i,k} = \frac{1}{mn}\sum_{j=1}^{n}\sum_{h=1}^{m} y_{i,j,k,h} = \frac{1}{n}\left(\sum_{j=1}^{n} x_{i,j,k} + \frac{1}{m}\sum_{j=1}^{n}\sum_{h=1}^{m} e_{i,j,k,h}\right), \tag{10}$$

$i = 1,2,\ldots\ j = 1,2,\ldots,n\ , h = 1,2,\ldots,m, k = 1,2,..,p.$

The standard deviation of the mean sample of the k^{th} quality characteristic at time i is:

$$\sigma(\bar{y}_{i,k}) = \sqrt{\frac{1}{n}\left(\sigma_k^2 + \frac{\sigma_{m_k}^2}{m}\right)} = \frac{\sigma_k}{\sqrt{n}C_{1k}}. \tag{11}$$

In the equation (12), σ_k^2 is the variance of the actual values of k^{th} quality characteristic and $\sigma_{m_k}^2$ is the variance of the of the measurement error term. C_{1k} is calculated as

$$C_{1k} = \sqrt{\frac{m}{m + (\sigma_{m_k}/\sigma_k)^2}} = \sqrt{\frac{m}{m + \gamma_k^2}}. \tag{12}$$

It is assumed that σ_k^2 and $\sigma_{m_k}^2$ for all quality characteristic are the same. In other words, $\sigma_{m_1}^2 = \sigma_{m_2}^2 = \ldots = \sigma_{m_p}^2 = \sigma^2$ and $\sigma_1^2 = \sigma_2^2 = \ldots. = \sigma_p^2 = \sigma^2$.

In equation (12), if measurement errors exist, $\gamma = \sigma_{m_k}/\sigma_k \geq 0$ and as a result, $C_{1k} \in (0,1]$. According to table 2, C_{1k} values are given for $m = \{1,2,3,4,5,10,15\}, \gamma = \{0, 0.1, 0.2, \ldots, 1\}$.

Table 2. The values of C_{1k} for different values of the γ and m

	C_{1k}		
γ	*m*=1	*m*=2	*m*=3
0.0	1.0000	1.0000	1.0000
0.1	0.9950	0.9975	0.9983
0.2	0.9806	0.9901	0.9934
0.3	0.9578	0.9782	0.9853
0.4	0.9285	0.9623	0.9744
0.5	0.8944	0.9428	0.9608
0.6	0.8575	0.9206	0.9449
0.7	0.8192	0.8962	0.9271
0.8	0.7809	0.8704	0.9078
0.9	0.7433	0.8436	0.8874
1.0	0.7071	0.8165	0.8660
γ	*m*=5	*m*=10	*m*=15
0.0	1.0000	1.0000	1.0000
0.1	0.9990	0.9995	0.9997
0.2	0.9960	0.9980	0.9987
0.3	0.9911	0.9955	0.9970
0.4	0.9844	0.9921	0.9947
0.5	0.9759	0.9877	0.9918
0.6	0.9658	0.9825	0.9882
0.7	0.9543	0.9764	0.9841
0.8	0.9416	0.9695	0.9793
0.9	0.9277	0.9618	0.9740

It is clear from the results of the table 2 that the value of C_{1k} increases by increasing *m* and decreasing γ. The performance of the control chart increases by increasing *m* (replications). The proposed statistic for monitoring the mean process vector is as

$$T_i^2 = \left(\bar{\mathbf{y}}_i - \boldsymbol{\mu}_0\right)^T \Sigma_{\bar{\mathbf{y}}}^{-1} \left(\bar{\mathbf{y}}_i - \boldsymbol{\mu}_0\right), \tag{13}$$

where $\bar{\mathbf{y}}_i = (\bar{y}_{i1}, \bar{y}_{i2}, ..., \bar{y}_{ip})^T$, $\boldsymbol{\mu_0} = (\mu_1, \mu_2, ..., \mu_p)$.

Also, the variance-covariance matrix of vector $\bar{\mathbf{y}}$ is equal to equation (14):

$$\Sigma_{\bar{\mathbf{y}}} = \begin{pmatrix} \sigma^2_{\bar{y}_1} & \sigma_{\bar{y}_1\bar{y}_2} & \cdots & \sigma_{\bar{y}_1\bar{y}_p} \\ \sigma_{\bar{y}_2\bar{y}_1} & \sigma^2_{\bar{y}_2} & \cdots & \sigma_{\bar{y}_2\bar{y}_p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\bar{y}_p\bar{y}_1} & \sigma_{\bar{y}_p\bar{y}_2} & \cdots & \sigma^2_{\bar{y}_p} \end{pmatrix}. \tag{14}$$

The values of $\sigma_{\bar{y}_1\bar{y}_2}, ..., \sigma_{\bar{y}_p\bar{y}_p}$ can be calculated as

$$\operatorname{cov}\left(\bar{\mathbf{y}}_h, \bar{\mathbf{y}}_l\right) = \frac{1}{n}\left[\rho \frac{\sigma_h \sigma_l}{\sqrt{n} C_{1k}} + \frac{\sigma_{m_{l,h}}}{m}\right]. \tag{15}$$

Also, the UCL of the proposed statistic in Equation (5) is determined by simulation such that the IC-ARL of the control chart equals to 200 is obtained. Therefore, if $\sigma_{m_k} > \sigma_k$, the control chart performance in detecting the shift in the mean of the process is decreased. In order to neutralize the effect of the measurement errors, the multiple measurements strategy is used, and in Section 5.1 it is shown that the more the replications (*m*), the better performance of the suggested control chart.

4- Effect of both measurement errors and auto-correlation on the control chart

We discussed the effect of both auto-correlation and measurement errors on the performance of Hotelling's T^2 control chart. For this goal, we consider the model which is represented as follows:

$$\begin{cases} x_{i,j,k} - \mu_{0,k} = \phi(x_{i,(j-1),k} - \mu_{0,k}) + \varepsilon_{i,j,k}, i = 1, 2, 3, ..., \ j = 1, 2, ..., n, k = 1, 2, ..., p. \\ y_{i,j,k,h} = x_{i,j,k} + e_{i,j,k,h}. \end{cases} \tag{16}$$

In the above model, we suppose that there is the AR(1) model between observations within a sample for each quality characteristic.

In this condition, the standard deviation of the sample mean for k^{th} quality characteristic at time i , $\sigma\left(\bar{y}_{ik}\right)$ is equal to:

$$\sigma\left(\bar{y}_{ik}\right) = \sqrt{\frac{1}{n}\left(\frac{\sigma_k^2}{C_{2k}^2} + \frac{\sigma_m^2}{m}\right)} = \frac{\sigma_k}{\sqrt{n}} \sqrt{\left(\frac{1}{C_{2k}^2} + \frac{\gamma_m^2}{m}\right)} = \frac{\sigma_k}{\sqrt{n} C_{3k}}, i = 1, 2, ..., k = 1, 2, ..., p, \tag{17}$$

where C_{3k} is computed as follows:

$$C_{3k} = \left(\frac{1}{C_{1k}^2} + \frac{1}{C_{2k}^2} - 1\right)^{-\frac{1}{2}}. \tag{18}$$

And C_{1k} and C_{2k} are defined in equations (12) and (4), respectively.

If the measurement system is exact then $C_{1k} = 1$, and if the data are uncorrelated, then $C_{2k} = 1$, hence, the value of C_{3k} is equal to one. Also, C_{1k} and C_{2k}, $C_{3k} \in (0,1]$. As the C_{3k} value decreases, the performance of the T^2 control chart in detecting shifts in the mean of process deteriorates.
In table 3, the values obtained for C_{3k} are given for, $n = \{3,5\}$, $m = \{1,2,4\}$, $\gamma = \{0, 0.1, 0.3, 0.5, 1\}$, $\phi = \{0, 0.2, 0.5, 0.7\}$. It is clear that the values of C_{3k} increases by increasing m or n and by decreasing γ or ϕ. A process with p quality characteristics discussing the AR(1) model auto-correlation structure between observations within a sample with measurement errors is shown schematically in Figure 3. Note that the p quality characteristics are also correlated.

Table 3. The values of C_{3k} for different values of n, γ and m

		$n=3$				$n=5$			
		$\Phi=0.0$	$\Phi=0.2$	$\Phi=0.5$	$\Phi=0.7$	$\Phi=0.0$	$\Phi=0.2$	$\Phi=0.5$	$\Phi=0.7$
γ	C_{1k}	C_{2k} =1.0000	C_{2k} =0.8793	C_{2k} =0.7385	C_{2k} =0.6652	C_{2k} =1.0000	C_{2k} =0.8528	C_{2k} =0.6704	C_{2k} =0.5699
m=1									
0.0	1.0000	1.0000	0.8793	0.7385	0.665	1.0000	0.8528	0.6704	0.5699
0.1	0.9950	0.9950	0.8759	0.7365	0.6637	0.9950	0.8497	0.6689	0.5690
0.3	0.9578	0.9578	0.8502	0.7211	0.6523	0.9578	0.8262	0.6572	0.5618
0.5	0.8944	0.8944	0.8050	0.6928	0.6312	0.8944	0.7845	0.6356	0.5481
1.0	0.7071	0.7071	0.6603	0.5941	0.5538	0.7071	0.6489	0.5568	0.4952
m=2									
0.0	1.0000	1.0000	0.8793	0.7385	0.6652	1.0000	0.8528	0.6704	0.6599
0.1	0.9975	0.9975	0.8776	0.7375	0.6645	0.9975	0.8512	0.6696	0.5694
0.3	0.9782	0.9782	0.8644	0.7296	0.6587	0.9782	0.8392	0.6637	0.5698
0.5	0.9428	0.9428	0.8397	0.7145	0.6475	0.9428	0.8165	0.6523	0.5587
1.0	0.8165	0.8165	0.7467	0.6546	0.9019	0.8165	0.7303	0.6058	0.5286
m=4									
0.0	1.0000	1.0000	0.8793	0.7385	0.6652	1.0000	0.8528	0.6704	0.5699
0.1	0.9988	0.9988	0.8785	0.7380	0.6648	0.9988	0.8520	0.6700	0.5697
0.3	0.9889	0.9889	0.8718	0.7341	0.6619	0.9889	0.8459	0.6670	0.5679
0.5	0.9701	0.9701	0.8588	0.7243	0.6662	0.9701	0.8340	0.6612	0.5642
1.0	0.8944	0.8944	0.8050	0.6928	0.6312	0.8944	0.7845	0.6356	0.5481

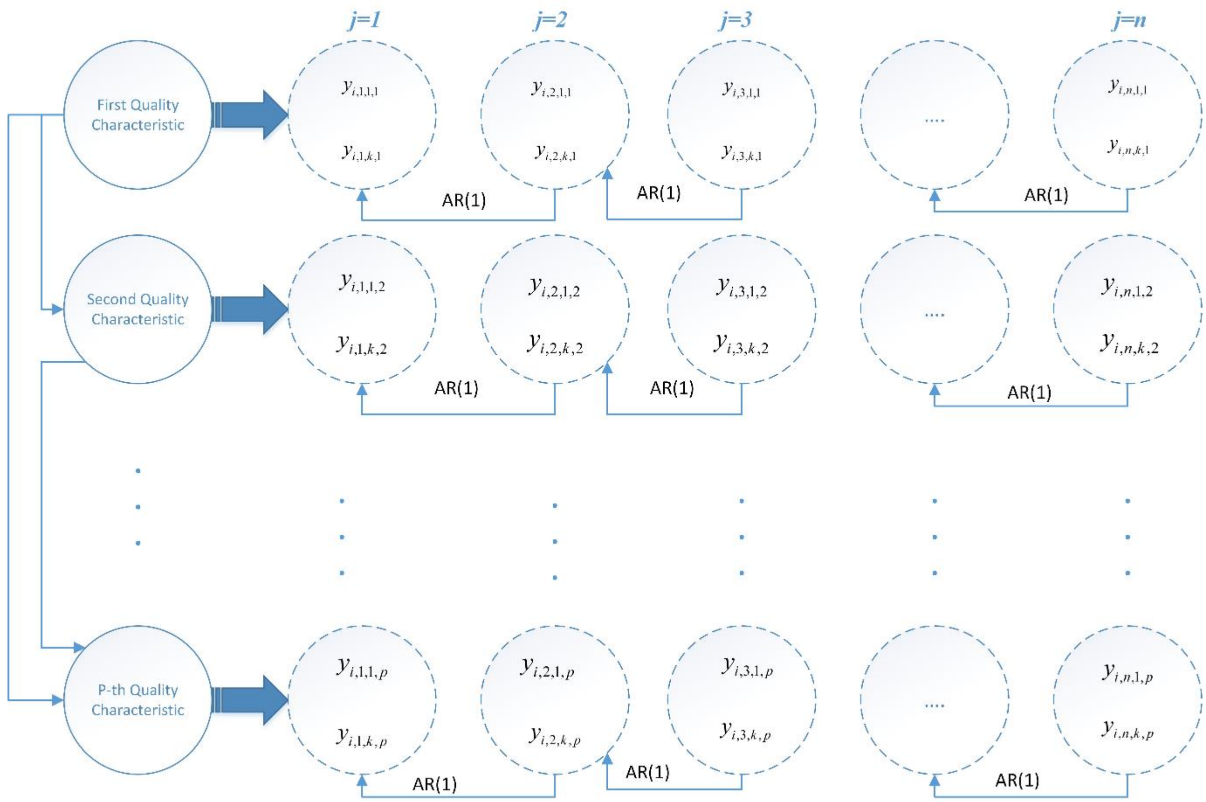

Fig 3. A p-variate normal process discussing AR(1) auto-correlation within each sample and measurement errors

The proposed statistic for monitoring the process mean vector is:

$$T_i^2 = \left(\bar{\mathbf{y}}_i - \boldsymbol{\mu}_\mathbf{0}\right)^T \boldsymbol{\Sigma}_{\bar{\mathbf{y}}}^{-1} \left(\bar{\mathbf{y}}_i - \boldsymbol{\mu}_\mathbf{0}\right). \tag{19}$$

The UCL of the Hotelling's T^2 statistic in Equation (19) is determined by simulation such that the ICARL of the control chart equals to 200 is obtained. Also, the covariance between sample means of h^{th} and l^{th} quality characteristics at time i is calculated as follows: (Refer to Appendix A for proof).

$$\text{cov}(\bar{y}_h, \bar{y}_l) = \frac{\sigma_{x_h,x_l}}{n^2}(n + (n-1)\phi_h\phi_l + (n-2)\phi_h^2\phi_l^2 + (n-3)\phi_h^3\phi_l^3 + ... + (n-(n-1))\phi_h^{n-1}\phi_l^{n-1}) + \frac{n}{m}\sigma_m^2. \tag{20}$$

5- Numerical example

In this example, the quality of a product is influenced by both auto-correlation and measurement errors. To appraise the effect of the measurement errors on the performance of the chi-square control chart for monitoring auto-correlated normal processes, it is assumed that a sample of five observations and bivariate normal process with a known mean vector and covariance matrix according to the first-order auto regressive model and auto-correlation coefficients equal to 0, 0.1 and 0.5 are generated according to the AR(1) model. The error vector $\boldsymbol{\varepsilon}_j$ (j = 1,2, ..., 5) for each sample, which is a bivariate normal random variable, is generated with the mean vector $\begin{bmatrix}0 & 0\end{bmatrix}$ and with the known variance-covariance matrix $\begin{bmatrix}0.54 & 0.09\\ 0.09 & 0.54\end{bmatrix}$. The mean vector $\boldsymbol{\mu}_0$ is equal to $\begin{bmatrix}4.3 & 4.2\end{bmatrix}$ and the covariance matrix is equal to $\begin{bmatrix}0.9 & 0.1\\ 0.1 & 0.8\end{bmatrix}$ and the number of simulation runs is 5000.

5-1- Performance evaluation of chi-square control chart in the presence of auto-correlation

We assumed that for each quality characteristic, five consecutive samples are generated at each sampling time and the replication is equal to 1. Also, the auto-correlation coefficients of 0.1 and 0.5 are discussed for quality characteristics. Assume that for i (i=1,2,...), the sequence of observations $\left\{x_{i,1,k}, x_{i,2,k}, ..., x_{i,n,k}\right\}$ according to the AR(1) model is generated. According to Table 2, by increasing auto-correlation coefficients, the covariance value between quality characteristics also increases. The OC ARLs for the shifts in the mean vector of quality characteristics in units of standard deviation are given in table 4. Based on the results shown in table 4, the statistical performance of control chart is affected by auto-correlated data. To counteract this effect, the jump strategy has been proposed to improve the performance of the chart. To evaluate the performance of the proposed method, for monitoring multivariate normal processes, random vectors are generated based on jump strategy. The algorithm of jump strategy is such that samples are generated with non-neighboring items; in other words, in the sampling process, non-neighboring items should be used, that is, one can ignore the sampled data, namely, one jump or two jumps or more before the next sampling to eliminate the auto-correlation effect. In this case, the observed values are generated according to the AR(1) model using jump strategy. Through 5000 simulation runs, the UCL of the proposed control chart is specified by simulation such that the IC ARL is obtained equal to 200. In order to appraise the performance of the suggested control chart in the detection of OC situation, OC ARLs of control chart for different shifts in the mean of quality characteristics are reported in table 4. According to Table 4, it can be seen that the proposed control chart can detect the OC states satisfactory. Also, the proposed control chart significantly ameliorates the power of the control chart in detecting the OC shifts.

Table 4. The values of OC ARL simulated under changes in the mean of the first quality characteristic and the second quality characteristic for p=2, taking into account the auto-correlation structure (ARL_0 =200)

		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Changes in the mean of the first quality characteristic	*m*=2, *p*=2, *n*=5, UCL=7.48	200	160.33	99.60	60.34	29.34	19.71	11.14	7.6	5.03	3.40	2.52
	m=2, *Φ*=0.1, UCL=7.58	200	165.69	117.15	68.49	38.42	23.72	14.43	9.58	6.36	4.64	3.25
	m=2, *Φ* =0.5, UCL=9.57	200	190.79	158.44	113.08	85.54	58.34	41.43	28.32	22.15	14.36	11.71
	One jump , UCL=12.58	200	188.23	156.94	112.29	80.47	54.09	38.36	26.22	19.58	13.68	10.87
	3 jumps , UCL=9.337	200	173.50	118.62	77.32	48.30	28.97	17.94	11.72	7.58	5.32	3.91
Changes in the mean of the second quality characteristic	*m*=2, *p*=2, *n*=5, UCL=7.48	200	154.15	91.65	51.80	25.71	15.08	8.99	5.71	4.04	2.94	2.13
	m=2, *Φ* =0.1 , UCL=7.58	200	163.26	100.47	57.45	33.57	19.11	12.30	7.79	5.18	3.55	2.76
	m=2, *Φ* =0.5, UCL=9.57	200	182.30	151.69	104.79	75.08	50.07	35.68	24.93	18.05	12.48	9.01
	One jump , UCL=12.58	200	180.27	137.66	101.23	69.35	47.81	31.97	21.86	16.43	11.06	8.41
	3 jumps , UCL=9.337	200	161.89	117.40	63.18	38.44	23.05	14.35	9.29	6.56	4.39	3.89
Changes in the mean of the first and second quality characteristics	*m*=2, *p*=2, *n*=5, *UCL*=7.48	200	133.28	59.53	26.79	13.34	7.18	4.18	2.79	2.04	1.50	1.29
	m=2, *Φ* =0.1 ,UCL= 7.58	200	147.26	72.55	34.61	18.14	9.12	5.32	3.70	2.50	1.84	1.47
	m=2, *Φ* =0.5, UCL=9.57	200	175.33	127.40	84.17	52.13	32.18	21.25	13.31	9.23	6.29	4.99
	One jump , UCL=12.58	200	171.35	124.04	83.41	51.98	30.83	18.53	21.02	8.06	6.04	4.34
	3 jumps, UCL=9.337	200	145.86	82.58	41.35	22.02	11.72	7.24	4.51	3.10	2.30	1.71

5-2- Performance evaluation of chi-square control chart in the presence of measurement errors

We supposed that at each sampling time five consecutive samples with 2, 4 or 15 replications on the product is performed. x_{ijk} is the real values of quality characteristics, generated by the AR(1) model. Also, the measured values corresponding to x_{ijk} with 2, 4 or 15 replications as the set of $\{y_{i,j,k,1}, y_{i,j,k,2},..., y_{i,j,k,m}\}$ are considered. Therefore, the observed values $y_{i,j,k,h}$ are generated according to equation (9). The residual values $e_{i,j,k,h}$, for each replication, which follows a bivariate normal random variable, generated with the mean vector $\begin{bmatrix}0 & 0\end{bmatrix}$ and known variance-covariance matrix $\begin{bmatrix}0.4 & 0.1\\ 0.1 & 0.3\end{bmatrix}$.

According to table 5, the UCL of the suggested control chart based on two quality characteristics and without the effect of the measurement errors is equal to 7.48 to attain the IC ARL of 200. Also, for the control chart that is influenced by the measurement errors, with a sample of 5 consecutive observations and two replications, the UCL ($m = 2$, $p = 2$, $n = 5$) is 7.98. The OC ARL for different shifts in the mean of quality characteristics is given in table 5. The auto-correlation coefficient is considered zero in this case; in other words, the effect of auto-correlation is not considered in this subsection. The results show that the multiple measurements strategy significantly improves the power of the control chart in detecting the OC state.

Table 5. The values of OC ARL simulated under changes in the mean of the first quality characteristic and the second quality characteristic for the existence of measurement error (ARL_0 = 200)

		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Changes in the mean of the first quality characteristic	No measurement errors *m*=2, *n*=5, *UCL*=7.48	200	160.33	99.60	60.34	34.92	19.71	11.41	7.60	5.03	3.40	2.52
	m=2, *n*=5, *UCL*=7.98	200	172.20	118.73	71.95	44.76	27.08	16.91	10.50	7.14	4.80	3.52
	m=4, *n*=5, *UCL*=7.811	200	170.25	114.66	67.15	39.32	23.91	14.08	9.23	6.16	4.16	3.15
	m=15, *n*=5, *UCL*=7.611	200	168.12	110.93	63.51	34.46	21.55	12.35	7.81	4.92	3.72	2.65
Changes in the mean of the second quality characteristic	No measurement errors *m*=2, *n*=5, *UCL*=7.48	200	154.15	91.65	51.80	25.71	15.08	8.99	5.71	4.04	2.94	2.13
	m=2, *n*=5, *UCL*=7.98	200	166.53	106.81	59.73	34.39	19.93	13.03	7.95	5.59	3.80	2.84
	m=4, *n*=5, *UCL*=7.811	200	163.22	101.32	57.25	32.52	18.87	11.42	7.05	4.89	3.34	2.51
	m=15, *n*=5, *UCL*=7.611	200	160.91	98.91	52.71	29.98	16.43	9.57	6.21	4.31	3.30	2.19
Changes in the mean of the first and second quality characteristics	No measurement errors *m*=2, *n*=5, *UCL*=7.48	200	133.28	64.53	26.79	13.34	7.18	4.18	2.79	2.04	1.50	1.29
	m=2, *n*=5, *UCL*=7.98	200	149.42	74.07	39.61	19.35	11.50	6.47	4.08	2.71	2.08	1.65
	m=4, *n*=5, *UCL*=7.811	200	145.62	72.12	35.37	17.52	9.31	5.31	3.59	2.42	1.83	1.47
	m=15, *n*=5, *UCL*=7.611	200	139.65	70.83	30.00	14.69	7.69	4.66	3.09	2.16	1.59	1.30

5-3- Performance evaluation of chi-square control chart for both auto-correlation and measurement errors

We supposed that for each quality characteristic, five consecutive samples are taken at each sampling time and 2 or 15 measurements are done on the product. Also, the auto-correlation coefficients of 0.1 and 0.5 are considered for quality characteristics. Moreover, the consecutive observations $\{x_{i,1,k}, x_{i,2,k}, ..., x_{i,n,k}\}$ are generated according to the AR(1) model, and the observed values $y_{i,j,k,h}$ are generated according to Equation (9). As noted in subsections 5.1 and 5.2, to neutralize the effect of the measurement errors and auto-correlation, multiple measurements and jump strategy are used, respectively. According to table 6, the UCL of the suggested control chart based on two quality characteristics and without the effect of both measurement errors and auto-correlation is equal to 7.48 to attain the IC ARL of 200. Also, for the control chart that is influenced by both measurement errors and auto-correlation, with a sample of 5 consecutive observations and two replications and Φ =0.5 , the UCL ($m = 2, p = 2, n = 5$) is 9.57. The OC ARL for different shifts in the mean of quality characteristics is given in table 6.

Generally, the existence of the auto-correlation and measurement errors causes the poor performance of the control charts and increases OC ARL. To diminish the effect of auto-correlation and measurement errors on the performance of the proposed control chart, jump strategy and multiple measurements methods are used, respectively. Results show the acceptable performance of the multiple measurements and jumps methods in decreasing the negative effect of measurement errors and auto-correlation, because the control charts can detect changes faster by applying the abovementioned strategies.

Table 6. The values of OC ARL under changes in the mean of the first quality characteristic and the second quality characteristic for p=2 , n=5 taking into account the existence of measurement errors and auto-correlation structure(ARL_0 = 200)

		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Changes in the mean of the first quality characteristic	No measurement errors m=2, UCL=7.48	200	160.33	99.60	60.34	34.29	19.71	11.41	7.60	5.03	3.40	2.52
	Φ=0.1, UCL=8	200	178.91	126.92	80.33	52.47	36.08	20.26	12.63	8.82	5.90	4.48
	Φ =0.5, UCL= 9.57	200	193.63	158.96	125.32	89.27	63.92	46.06	33.98	25.20	18.32	13.44
	m=15, Φ =0.5, UCL=9.6	200	188.51	152.68	122.32	86.26	60.45	44.22	31.35	21.92	16.85	12.45
	3 jumps Φ =0.5 , UCL=9.545	200	176.15	123.26	87.50	55.78	35.12	23.19	15.71	10.26	7.18	5.24
	m=15, 3 jumps Φ =0.5 , UCL=9.4	200	172.21	117.48	80.13	51.03	30.16	18.93	12.42	8.10	5.79	4.16
Changes in the mean of the second quality characteristic	No measurement errors m=2, UCL=7.48	200	154.15	91.65	51.80	25.71	15.08	8.99	5.71	4.04	2.94	2.13
	Φ=0.1, UCL=8	200	160.82	111.51	69.38	40.31	24.74	15.37	10.37	7.17	5.02	3.57
	Φ =0.5, UCL= 9.57	200	183.22	152.79	105.28	78.20	55.10	40.51	27.06	19.52	14.54	10.37
	m=15, Φ =0.5, UCL=9.6	200	178.53	147.56	98.22	73.32	51.65	37.87	25.32	16.21	12.72	9.32
	3 jumps, Φ =0.5 , UCL=9.545	200	160.84	114.17	71.74	44.07	29.66	18.05	11.52	7.74	5.65	4.27
	m=15, 3 jumps, Φ =0.5 , UCL=9.4	200	158.95	109.15	68.79	37.95	23.58	14.83	9.58	6.59	4.62	3.54
Changes in the mean of the first and second quality characteristics	No measurement errors m=2, UCL=7.48	200	132.28	59.53	26.79	13.34	7.18	4.18	2.79	2.04	1.50	1.29
	Φ=0.1, UCL=8	200	149.32	93.90	44.22	23.15	14.57	8.37	5.15	3.44	2.40	1.99
	Φ =0.5, UCL= 9.57	200	180.54	123.78	85.19	55.07	32.90	22.47	15.10	10.15	7.50	5.26
	m=15, Φ =0.5, UCL=9.6	200	176.45	119.28	80.84	51.25	29.78	19.28	13.32	9.28	6.29	4.54
	3 jumps, Φ =0.5 , UCL=9.545	200	155.44	99.40	53.71	30.39	17.06	10.18	6.20	4.29	3.14	2.21
	m=15, 3 jumps, Φ =0.5 , UCL=9.4	200	148.98	92.95	42.55	22.81	12.45	7.29	4.69	3.34	2.26	1.83

5-4- Managerial Insights

As the results of numerical example showed, the existence of the auto-correlation and measurement errors lead to poor performance of the Hotelling's T^2 control chart in terms of ARL_1. Hence, it is important to address these effects through suitable strategies such as multiple measurements for measurement errors and jump strategy for auto-correlation.

For this aim, the proposed Hotelling's T^2 control chart should be applied in practice to overcome the negative effects of both measurement errors and auto-correlation, through the following steps:

1. Estimate the model by which the process works.
2. Obtain a control limit using simulation.
3. Plot the Hotelling's T^2 statistics on the control chart.
4. Implement the proposed strategies (Multiple measurements and jump strategy).

6- Case study

For industrial application, we adopt the case presented by Aparisi (1996). In this case, a mechanical part with three main quality characteristics is available as it is shown in figure 4. The quality characteristics are the width X_1, the inner diameter X_2 and the length X_3. Beside the in-control mean vector $\boldsymbol{\mu}_0$ and the variance-covariance matrix $\boldsymbol{\Sigma}_0$ of the quality characteristics which are used in Aparisi (1996), we also assume the following variance-covariance matrix of measurements errors $\boldsymbol{\Sigma}_m$:

$$\boldsymbol{\mu}_0 = (7,3,15), \boldsymbol{\Sigma}_0 = \begin{bmatrix} 0.2 & 0.054 & 0.162 \\ 0.054 & 0.09 & 0.042 \\ 0.162 & 0.042 & 0.31 \end{bmatrix}, \boldsymbol{\Sigma}_m = \begin{bmatrix} 0.16 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}.$$

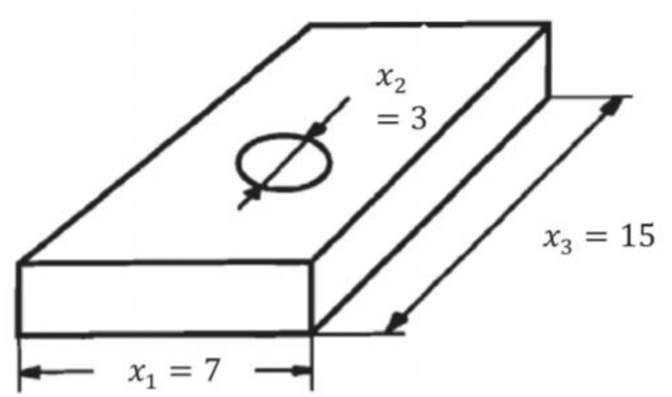

Fig 4. A mechanical part with three quality characteristics (Aparisi , 1996)

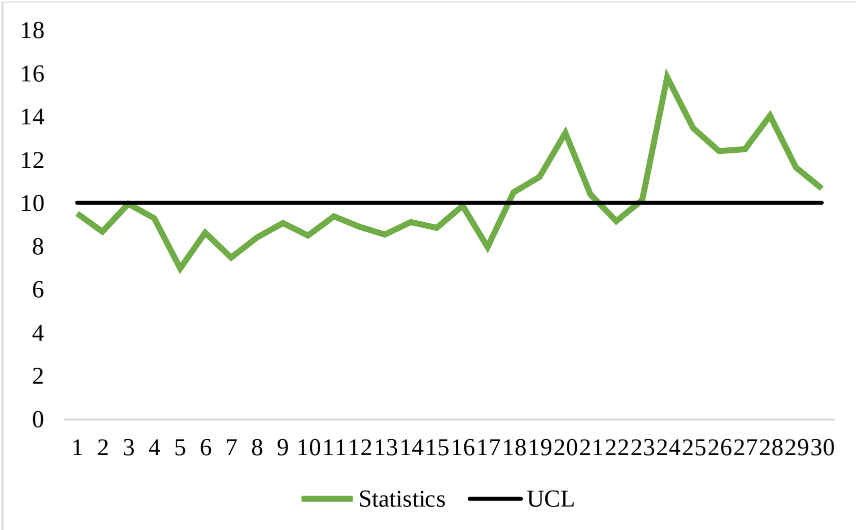

Fig 5. The proposed Hotteling's T^2 control chart

Based on the following in-control model and through simulation runs, UCL is set equal to 10.04 to achieve in-control ARL of roughly 200 where $e_{i,j,k,h}$ is the independent error term of k^{th} quality characteristic with the mean zero and known variance and AR(1) model between observations within a sample for each quality characteristic is assumed. Note that $\phi = 0.5$ is considered.

$$\begin{cases} x_{i,j,k} - \mu_{0,k} = \phi(x_{i,(j-1),k} - \mu_{0,k}) + \varepsilon_{i,j,k}, i = 1,2,3,..., j = 1,2,...,n, k = 1,2,..., p, \\ y_{i,j,k,h} = x_{i,j,k} + e_{i,j,k,h}. \end{cases}$$

After generating 15 in-control samples from the above in-control model, a shift equals to 0.5 is imposed in the process mean values of all three quality characteristics from sample #16. As it can be seen in figure 5, the proposed Hotteling's T^2 chart was able to find an out-of-control alarm in sample #17. After finding an out-of-control situation, quality engineers should investigate the source of variation, detect the assignable cause and eliminate it.

7- Conclusion and a suggestion for future research

In this paper, we considered a multivariate normal process and evaluated the effect of the auto-correlation and measurement errors on the IC and OC ARL performance of Hotelling's T^2 control charts. In particular, the first-order autoregressive time series model was used to model the auto-correlation structure. Also, to reduce the effect of measurement errors, multiple measurements method was used. Moreover, the jump strategy was applied to address the auto-correlation effect. Then, a numerical example was presented to appraise the performance of control chart influenced by the measurement errors and auto-correlation structure. The results of simulation studies, in terms of ARL criterion, demonstrated that the performance of the Hotelling's T^2 control chart is affected by measurement errors as well as auto-correlation structure. The multiple measurements as well as jumping strategy were used to improve the negative effects of measurement errors and auto-correlation structure on the performance of the Hotelling's T^2 control chart, respectively. Future research might be proposing an adaptive multivariate control charts by discussing measurement errors and auto-correlation simultaneously.

References

Abbasi, S. A. (2010). On the performance of EWMA chart in the presence of two-component measurement error. *Quality Engineering*, 22(3): 199-213.

Alwan, L. C. & Radson, D. (1992). Time-series investigation of subsample mean charts. *IIE Transactions*, 24(5): 66-80.

Aparisi, F. (1996). Hotelling's T^2 control chart with adaptive sample sizes. *International Journal of Production Research*, *34*(10), 2853–2862.

Chattinnawat, W., & Bilen, C. (2017). Performance analysis of Hotelling's T^2 under multivariate inspection errors. *Quality Technology & Quantitative Management*, *14*(3), 249-268.

Costa, A. F. B. and P. Castagliola (2011). Effect of measurement error and autocorrelation on the $\bar{X}$ chart *Journal of Applied Statistics*, 38(4): 661-673.

Ding, G. and L. Zeng (2015). On the effect of measurement errors in regression-adjusted monitoring of multistage manufacturing processes. *Journal of Manufacturing Systems*, 36: 263-273.

Franco, B. C., Castagliola, P., Celano, G. & Costa, A. F. B. (2014). A new sampling strategy to reduce the effect of autocorrelation on a control chart. *Journal of Applied Statistics,* 41(7): 1408-1421.

Goswami, A. & Dutta, H. N. (2014). Studies on EWMA Control Chart in Presence of Autocorrelation. *International Journal of Scientific & Engineering Research*, 5(1): 871-877.
Kalgonda, A. & KulkarniU, S. (2004). Multivariate quality control chart for autocorrelated processes. *Journal of Applied Statistics,* 31(3): 317-327.

Kazemzadeh, R., Noorossana, R. & Amiri, A. (2010). Phase II monitoring of autocorrelated polynomial profiles in AR(1) processes. *Scientia Iranica. Transaction E, Industrial Engineering,* 17(1): 12-24.

Khedmati, M. & Niaki, S. T. A. (2016). Phase II monitoring of general linear profiles in the presence of between-profile autocorrelation. *Quality and Reliability Engineering International,* 32(2): 443-452.

Leoni, R. C., Costa, A. F. B., Franco, B. C. & Machado, M. A. G. (2015). The skipping strategy to reduce the effect of the autocorrelation on the T^2 chart's performance. *The International Journal of Advanced Manufacturing Technology,* 80(9-12): 1547-1559.

Linna, K. W. and W. H. Woodall (2001). Effect of measurement error on Shewhart control $\bar{X}$ charts. *Journal of Quality Technology*, 33(2): 213-222.

Linna, K. W., et al. (2001). The performance of multivariate control charts in the presence of measurement error." *Journal of Quality Technology*, 33(3): 349-355.

Maravelakis, P., et al. (2004). EWMA chart and measurement error. *Journal of Applied Statistics*, 31(4): 445-455.

Maravelakis, P. E. (2012). Measurement error effect on the CUSUM control chart. *Journal of Applied Statistics*, 39(2): 323-336.

Noorossana, R. and Y. Zerehsaz (2015). Effect of measurement error on phase II monitoring of simple linear profiles. *The International Journal of Advanced Manufacturing Technology*, 79(9): 2031-2040.

Rakhmawati, D. Y., Kim, K. J., & Sumiati. (2020). Performance comparison of generalized confidence interval and modified sampling distribution approaches for assessing one-sided capability indices with gauge measurement errors. Published online in *Communications in Statistics-Theory and Methods*, doi: 10.1080/03610926.2020.1752729.

Sabahno, H., Amiri, A. & Castagliola, P. (2019a). Optimal performance of the variable sample sizes Hotelling's T^2 control chart in the presence of measurement errors. *Quality Technology & Quantitative Management*, 16(5): 588-612.

Sabahno, H., Amiri, A. & Castagliola, P. (2019b). Performance of the variable parameters $\bar{X}$ control chart in presence of measurement errors. *Journal of Testing and Evaluation*, 47(1): 480-497.

Sabahno, H., Amiri, A. & Castagliola, P. (2018). Evaluating the effect of measurement errors on the performance of the variable sampling intervals Hotelling's T^2 control charts. *Quality and Reliability Engineering International*, 34 (8): 1785-1799.

Soleimani, P. & Noorossana, R. (2014). Monitoring multivariate simple linear profiles in the presence of between profile autocorrelation. *Communications in Statistics-Theory and Methods,* 43(3): 530-546.

Soleimani, P., Noorossana, R. & Niaki, S. (2013). Monitoring autocorrelated multivariate simple linear profiles. *The International Journal of Advanced Manufacturing Technology,* 67(5-8): 1857-1865.

Soleimani, P., Noorossana, R. & Amiri, A. (2009). Simple linear profiles monitoring in the presence of within profile autocorrelation. *Computers & Industrial Engineering,* 57: 1015-1021.

Shishebori, D., & Zeinal Hamadani, A. (2010). The effect of gauge measurement capability and dependency measure of process variables on the MCp. *Journal of Industrial and Systems Engineering*, *4*(1), 59-76.

Xiaohong, L. and W. Zhaojun (2009). The CUSUM control chart for the autocorrelated data with measurement error. *Chinese Journal of Applied Probability and Statistics*, 25(5): 461-474.

Appendix A: The proof for equation (20)

Here we will provide the detailed proof of Equation (20). Let us consider auto-correlation and measurement errors simultaneously. So we have;

$$\begin{cases} x_{i,j,k} - \mu_{0,k} = \phi(x_{i,(j-1),k} - \mu_{0,k}) + \varepsilon_{i,j,k}, i = 1,2,3,..., j = 1,2,...,n, k = 1,2,..., p. \\ y_{i,j,k,h} = x_{i,j,k} + e_{i,j,k,h}. \end{cases}$$

The sample mean for h^{th} quality characteristic at time i, $\bar{y}_{ih}$, is equal to:

$$\bar{y}_{ih} = \frac{1}{mn}\sum_{j=1}^{n}\sum_{k=1}^{m} y_{ijkh} = \frac{1}{n}(\sum_{j=1}^{n} x_{ijh} + \frac{1}{m}\sum_{j=1}^{n}\sum_{k=1}^{m} e_{ijkh}), h = 1,2,..., p.$$

Also, sample means of h^{th} quality characteristic is

$$\bar{y}_{h} = \sum_{i}\bar{y}_{ih} = \frac{1}{n}(\sum_{i}\sum_{j=1}^{n} x_{ijh} + \frac{1}{m}\sum_{i}\sum_{j=1}^{n}\sum_{k=1}^{m} e_{ijkh}).$$

The covariance between sample means of h^{th} and l^{th} quality characteristics is as follows:

$$\text{cov}(\bar{y}_h, \bar{y}_l) = \text{cov}(\frac{1}{n}(\sum_{i}\sum_{j=1}^{n} x_{ijh} + \frac{1}{m}\sum_{i}\sum_{j=1}^{n}\sum_{k=1}^{m} e_{ijkh}), \frac{1}{n}(\sum_{i}\sum_{j=1}^{n} x_{ijl} + \frac{1}{m}\sum_{i}\sum_{j=1}^{n}\sum_{k=1}^{m} e_{ijkl}))$$

$$= \frac{1}{n^2}\text{cov}(\sum_{i}\sum_{j=1}^{n} x_{ijh}, \sum_{i}\sum_{j=1}^{n} x_{ijl}) + \frac{1}{m^2}\text{cov}(\sum_{i}\sum_{j=1}^{n}\sum_{k=1}^{m} e_{ijkh}, \sum_{i}\sum_{j=1}^{n}\sum_{k=1}^{m} e_{ijkl}).$$

Now, the covariance between sample means of h^{th} and l^{th} quality characteristics is calculated in two parts.

In the first part, the covariance between the actual value of the j^{th} observation in h^{th} and l^{th} quality characteristics at time i, x_{ijh} and x_{ijl}, is calculated which are modeled by the AR(1) model and the auto-correlation parameters are ϕ_h and ϕ_l, respectively. Hence, the calculations are such that in n states, the lag of the observations is equal to zero. In this case, the covariance between them is equal to $n\sigma_{x_h,x_l}\phi_h^0\phi_l^0$. In $(n-1)$ states, the lag of the observations is equal to one, in this case, the covariance between them is equal to $(n-1)\sigma_{x_h,x_l}\phi_h\phi_l$ and so on. In $(n-(n-1))$ state the lag of the observations is equal to $(n-1)$ where in this case, the covariance between them is equal to $(n-(n-1))\sigma_{x_h,x_l}\phi_h^{n-1}\phi_l^{n-1}$. As a result, sum of the above expressions is the value calculated from the first part, which leads to

$$\frac{\sigma_{x_h,x_l}}{n^2}(n + (n-1)\phi_h\phi_l + (n-2)\phi_h^2\phi_l^2 + (n-3)\phi_h^3\phi_l^3 + ... + (n-(n-1))\phi_h^{n-1}\phi_l^{n-1}).$$

In the second part, the covariance of e_{ijkh} and e_{ijkl} is calculated. Since the summation is on k and j, there are mn states that each of which has a covariance equal to σ_m^2. Then, as a result the covariance of the second part is equal to

$$\frac{1}{m^2} \times mn \operatorname{cov}(e_{ijkh}, e_{ijkl}) = \frac{1}{m^2} \times mn \times \sigma_m^2 = \frac{n}{m} \sigma_m^2.$$

Hence, the covariance in Equation (20) can be calculated by:

$$\operatorname{cov}(\bar{y}_h, \bar{y}_l) = \frac{\sigma_{x_h, x_l}}{n^2} (n + (n-1)\phi_h \phi_l + (n-2)\phi_h^2 \phi_l^2 + (n-3)\phi_h^3 \phi_l^3 + ... + (n-(n-1))\phi_h^{n-1} \phi_l^{n-1}) + \frac{n}{m} \sigma_m^2.$$