

# Phase II Monitoring of Multivariate Multiple Linear Regression Profiles

M. Eyvazian,<sup>a</sup> R. Noorossana,<sup>a</sup> Abbas Saghaei<sup>b</sup> and Amirhossein Amiri<sup>c,\*†</sup>

In certain cases, the quality of a process or a product can be effectively characterized by two or more multiple linear regression profiles in which response variables are correlated. This structure can be modeled as multivariate multiple linear regression profiles. When linear profiles are monitored separately, then correlation between response variables is ignored and misleading results could be expected. To overcome this problem, the use of methods that consider the multivariate structure between response variables is inevitable. In this paper, we propose four methods to monitor this structure in Phase II. The performance of the methods is compared through simulation studies in terms of the average run length criterion. Furthermore, a method based on likelihood ratio approach is developed to determine the location of shifts and a numerical simulation is used to evaluate the performance of the proposed method. Finally, the use of the methods is illustrated by a numerical example. Copyright © 2010 John Wiley & Sons, Ltd.

**Keywords:** average run length (ARL); change point; multivariate multiple linear regression profiles; Phase II; profile monitoring; statistical process control

## 1. Introduction

In some practical applications, a relationship between a response variable and one or more explanatory variables characterizes the quality of some processes or products. This relationship which is referred to as a 'profile' can be represented by a simple linear regression, a multiple linear or polynomial regression or even by a more complicated model such as nonlinear regression. Several applications of profiles monitoring have been introduced in the literature; See for example, Kang and Albin<sup>1</sup>, Mahmoud and Woodall<sup>2</sup>, Woodall *et al.*<sup>3</sup>, Wang and Tsung<sup>4</sup>, Montgomery<sup>5</sup>, Woodall<sup>6</sup>, Zou *et al.*<sup>7</sup>, and Amiri *et al.*<sup>8</sup>. Many authors have studied Phase I and Phase II monitoring of simple linear profiles; See for example, Kang and Albin<sup>1</sup>, Kim *et al.*<sup>9</sup>, Zhang *et al.*<sup>10</sup>, Saghaei *et al.*<sup>11</sup>, and Mahmoud *et al.*<sup>12</sup> for Phase II and Kang and Albin<sup>1</sup>, Mahmoud and Woodall<sup>2</sup>, and Mahmoud *et al.*<sup>13</sup> for Phase I monitoring of simple linear profiles. In Phase I, process stability is checked whereas the main purpose of Phase II is detecting shifts as quickly as possible. Noorossana *et al.*<sup>14</sup>, Soleimani *et al.*<sup>15</sup>, and Noorossana *et al.*<sup>16,17</sup> have investigated violations of independency and normality assumptions for simple linear profiles. Phase I multiple linear and polynomial profiles monitoring have also been studied by authors including Mahmoud<sup>18</sup>, Jensen *et al.*<sup>19</sup>, and Kazemzadeh *et al.*<sup>20</sup>. Zou *et al.*<sup>7</sup> proposed a multivariate exponentially weighted moving average (MEWMA) control chart to monitor general linear profiles in Phase II. Kazemzadeh *et al.*<sup>21</sup> proposed a transformation based on orthogonal polynomial model and used a EWMA-based monitoring scheme in Phase II. Jin and Shi<sup>22</sup>, Walker and Wright<sup>23</sup>, Ding *et al.*<sup>24</sup>, Williams *et al.*<sup>25</sup>, Moguerza *et al.*<sup>26</sup>, Vaghefi *et al.*<sup>27</sup>, and Jensen and Birch<sup>28</sup> proposed methods for monitoring nonlinear profiles.

In some applications, the quality of a process or a product is better characterized by two or more correlated profiles. Parker *et al.*<sup>29</sup> discussed a case in which the relationships between six response variables and six explanatory variables are investigated in a calibration process at NASA Langley Research Center. In the literature, this structure is referred to as multivariate profiles. In this case, there is a relationship between each response variable and one or more explanatory variables where the response variables are correlated. Noorossana *et al.*<sup>30</sup> investigated multivariate simple linear regression profiles in Phase II where each response variable is modeled as a function of one explanatory variable by a simple linear regression. They proposed some control schemes to monitor multivariate simple linear regression profiles in Phase II.

<sup>a</sup>Industrial Engineering Department, Iran University of Science and Technology, Tehran, Iran

<sup>b</sup>Industrial Engineering Department, Islamic Azad University—Science and Research Branch, Tehran, Iran

<sup>c</sup>Industrial Engineering Department, Faculty of Engineering, Shahed University, Tehran, Iran

\*Correspondence to: Amirhossein Amiri, Industrial Engineering Department, Faculty of Engineering, Shahed University, Tehran, Iran.

†E-mail: amiri@shahed.ac.ir

Contract/grant sponsor: Iran National Science Foundation for Dr Noorossana's research

Sometimes, more than one explanatory variable is needed to model the quality of a process or a product in multivariate structure. This case, which is referred to as multivariate multiple linear regression profiles, consists of several correlated response variables where each response is modeled by a multiple linear regression. Noorossana *et al.*<sup>31</sup> proposed four methods including likelihood ratio, Wilk's lambda,  $T^2$ , and principal components analysis to monitor multivariate multiple linear regression profiles in Phase I.

In this paper, we especially focus on Phase II analysis and investigate issues related to a multivariate multiple linear structure. We propose the use of four statistical control procedures to monitor multivariate multiple linear regression profiles in Phase II. The methods are compared through simulation runs via ARL under step shifts. In addition, to find the location of a shift, we extended the change point method by Zou *et al.*<sup>7</sup> to the multivariate case. The model and its assumptions are discussed in the next section. Four Phase II methods are developed in Section 3. In Section 4, a change point method is developed to determine the location of shifts. In Sections 5 and 6, the performance of the monitoring schemes and change point estimator is evaluated through simulation studies, respectively. A numerical example is illustrated in Section 7 and finally our concluding remarks are given in the last section.

## 2. Model and assumptions

Let us assume that for the  $k$ th random sample collected over time, we have  $n$  observations given as  $(x_{1i}, x_{2i}, \dots, x_{qi}, y_{1ik}, y_{2ik}, \dots, y_{pik})$ ,  $i = 1, 2, \dots, n$ , where  $p$  is the number of response variables and  $q$  is the number of explanatory variables. When process is in statistical control, the model that relates the response variables with explanatory variables is a multivariate multiple linear regression and can be given as follows:

$$\mathbf{Y}_k = \mathbf{X}\mathbf{B} + \mathbf{E}_k \quad (1)$$

or equivalently,

$$\begin{bmatrix} y_{11k} & y_{12k} & \dots & y_{1pk} \\ y_{21k} & y_{22k} & \dots & y_{2pk} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1k} & y_{n2k} & \dots & y_{npk} \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1q} \\ 1 & x_{21} & \dots & x_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \dots & x_{nq} \end{bmatrix} \begin{bmatrix} \beta_{01} & \beta_{02} & \dots & \beta_{0p} \\ \beta_{11} & \beta_{12} & \dots & \beta_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{q1} & \beta_{q2} & \dots & \beta_{qp} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11k} & \varepsilon_{12k} & \dots & \varepsilon_{1pk} \\ \varepsilon_{21k} & \varepsilon_{22k} & \dots & \varepsilon_{2pk} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{n1k} & \varepsilon_{n2k} & \dots & \varepsilon_{npk} \end{bmatrix}, \quad (2)$$

where  $\mathbf{Y}_k$  is an  $n \times p$  matrix of response variables for the  $k$ th sample,  $\mathbf{X}$  is an  $n \times (q+1)$  matrix of explanatory variables,  $\mathbf{B}$  is a  $(q+1) \times p$  matrix of regression parameters, and  $\mathbf{E}_k$  is an  $n \times p$  matrix of error terms. It is assumed that the vector of error terms has a  $p$ -variate normal distribution with mean vector  $\mathbf{0}$  and  $p \times p$  covariance matrix equals to

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pp} \end{bmatrix}, \quad (3)$$

where  $\sigma_{hj}$  denotes the covariance between error vector terms of  $h$ th and  $j$ th response variables at each observation.

For the  $k$ th random sample, the ordinary least square (OLS) estimator of the matrix  $\mathbf{B}$  is given by (See Rencher<sup>32</sup>, p. 337 for more information),

$$\hat{\mathbf{B}}_k = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}_k \quad (4)$$

## 3. Proposed methods for Phase II

Four methods for Phase II monitoring of multivariate multiple linear regression profiles are developed in this section. The first method is an extension of the MEWMA method proposed by Noorossana *et al.*<sup>30</sup>. The second method is a modified version of the first method that helps to solve the problem of high dimensionality. The third method is an extension of the likelihood ratio method proposed by Zhang *et al.*<sup>10</sup>. Finally, the use of a combined MEWMA and chi-squared control charts procedure first proposed by Noorossana *et al.*<sup>30</sup> is considered as the fourth method.

### 3.1. MEWMA method

We first rewrite the matrix  $\hat{\mathbf{B}}_k$  as a  $((q+1)p) \times 1$  multivariate normal random vector and denote it by  $\hat{\beta}_k$  as follows:

$$\hat{\beta}_k = (\hat{\beta}_{01k}, \hat{\beta}_{11k}, \dots, \hat{\beta}_{q1k}, \hat{\beta}_{02k}, \hat{\beta}_{12k}, \dots, \hat{\beta}_{q2k}, \dots, \hat{\beta}_{0pk}, \hat{\beta}_{1pk}, \dots, \hat{\beta}_{qpk})^T \quad (5)$$

When the process is in-control, the expected value and covariance matrix for  $\hat{\beta}_k$  is given as follows:

$$E(\hat{\beta}_k) = (\beta_{01}, \beta_{11}, \dots, \beta_{q1}, \beta_{02}, \beta_{12}, \dots, \beta_{q2}, \dots, \beta_{0p}, \beta_{1p}, \dots, \beta_{qp})^T, \quad (6)$$

$$\Sigma_{\hat{\beta}_k} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1p} \\ \Sigma_{21} & \Sigma_{22} & \dots & \Sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{p1} & \Sigma_{p2} & \dots & \Sigma_{pp} \end{pmatrix}, \quad (7)$$

where  $(\beta_{01}, \beta_{11}, \dots, \beta_{q1}, \beta_{02}, \beta_{12}, \dots, \beta_{q2}, \dots, \beta_{0p}, \beta_{1p}, \dots, \beta_{qp})^T$  is denoted by  $\beta$ . It can be shown that  $\Sigma_{hj}$  is a  $(q+1) \times (q+1)$  matrix equal to  $[\mathbf{X}^T \mathbf{X}]^{-1} \sigma_{hj}$ , where  $\sigma_{hj}$  denotes the  $hj$ th element of the covariance matrix  $\Sigma$  in Equation (3) (see Appendix A for proof). We use MEWMA control chart proposed by Lowry *et al.*<sup>33</sup> to monitor the regression parameters. The statistic for this scheme is given by

$$T_{Z_k}^2 = \mathbf{z}_k^T \Sigma_Z^{-1} \mathbf{z}_k, \quad (8)$$

where

$$\mathbf{z}_k = \lambda(\hat{\beta}_k - \beta) + (1 - \lambda)\mathbf{z}_{(k-1)}, \quad (9)$$

and the asymptotic covariance matrix

$$\Sigma_Z = \frac{\lambda}{2 - \lambda} \Sigma_{\hat{\beta}_k}. \quad (10)$$

The parameter  $\lambda(0 < \lambda \leq 1)$  is the smoothing parameter,  $\mathbf{z}_k$  is a multivariate normal random vector, and  $\mathbf{z}_0$  is  $((q+1)p) \times 1$  vector of zeros. The upper control limit for the statistic in Equation (8) is chosen to achieve a specified in-control ARL.

### 3.2. MEWMA method in low-dimensional space

The previous method has a dimensionality problem since it includes  $((q+1)p)$  regression parameters. Mahmoud<sup>18</sup> discussed that the performance of the  $T^2$ -based methods deteriorates as the dimension of coefficient vector increases. He proposed a method to reduce the number of profile parameters in Phase I monitoring of multiple linear regression. In the proposed method, one monitors the parameters of a simple linear regression including intercept, slope, and SD regardless of the number of explanatory variables in a multiple linear regression model. We used this idea to overcome the problem of high dimensionality in the previous method and extended the method by Mahmoud<sup>18</sup> to multivariate structure in Phase II. This changes the structure of multivariate multiple linear regression to multivariate simple linear regression. Hence, the dimension decreases from  $(q+1)p$  to  $2p$ . In this method, one should first fit the following model to each profile data set at time  $k$ :

$$\begin{bmatrix} y_{1jk} \\ y_{2jk} \\ \vdots \\ y_{njk} \end{bmatrix} = \begin{bmatrix} 1 & u_{1j} \\ 1 & u_{2j} \\ \vdots & \vdots \\ 1 & u_{nj} \end{bmatrix} \begin{bmatrix} A_{0jk} \\ A_{1jk} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1jk} \\ \varepsilon_{2jk} \\ \vdots \\ \varepsilon_{njk} \end{bmatrix}, \quad j = 1, 2, \dots, p, \quad (11)$$

where  $u_{ij} = E(y_{ij})$  and is equal to  $\beta_{0j} + \beta_{1j}x_{1i} + \dots + \beta_{qj}x_{qi}$  for the  $j$ th response. Using the OLS estimators,  $A_{0jk}$  and  $A_{1jk}$  are estimated for all responses ( $j = 1, 2, \dots, p$ ). It should be noted that the OLS estimators of  $A_{0jk}$  and  $A_{1jk}$  are linear functions of the corresponding response values. Since response variables are correlated, the estimators for the parameters in different profiles are correlated and should not be monitored separately.

After fitting the model in Equation (11) to each profile, the estimated parameters based on each sample are written as

$$\hat{\mathbf{A}}_k = (\hat{A}_{01k}, \hat{A}_{11k}, \hat{A}_{02k}, \hat{A}_{12k}, \dots, \hat{A}_{0pk}, \hat{A}_{1pk})^T. \quad (12)$$

When the process is in-control, the expected value and variance–covariance matrix for  $\hat{\mathbf{A}}_k$  are as follows, respectively:

$$E(\hat{\mathbf{A}}_k) = (A_{01}, A_{11}, A_{02}, A_{12}, \dots, A_{0p}, A_{1p})^T = (0, 1, 0, 1, \dots, 0, 1)^T. \quad (13)$$

$$\Sigma_{\hat{\mathbf{A}}_k} = \begin{pmatrix} \Sigma'_{11} & \Sigma'_{12} & \dots & \Sigma'_{1p} \\ \Sigma'_{21} & \Sigma'_{22} & \dots & \Sigma'_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma'_{p1} & \Sigma'_{p2} & \dots & \Sigma'_{pp} \end{pmatrix}, \quad (14)$$

where  $(A_{01}, A_{11}, A_{02}, A_{12}, \dots, A_{0p}, A_{1p})^T = (0, 1, 0, 1, \dots, 0, 1)^T$  is denoted by  $\mathbf{A}$ .

It can be shown that  $\Sigma'_{hj}$  is a  $2 \times 2$  matrix and the elements of  $\Sigma'_{hj}$  are given in Appendix B.

Finally, similar to the previous method, the MEWMA procedure proposed by Lowry *et al.*<sup>33</sup> is used to monitor the multivariate linear regression structure over time.

### 3.3. Likelihood ratio method

This method is the extension of the method by Zhang *et al.*<sup>10</sup> to the multivariate multiple linear regression profile case. It can be shown (see Appendix C) that the likelihood ratio statistic is given as

$$LRT_k = n \log |\Sigma| - n \log |\hat{\Sigma}_k| + C_k - np, \quad (15)$$

where  $C_k$  is equal to  $\sum_{i=1}^n (\mathbf{y}_{ik} - \mathbf{x}_i \mathbf{B}) \Sigma^{-1} (\mathbf{y}_{ik} - \mathbf{x}_i \mathbf{B})^T$  in which  $(\mathbf{y}_{ik} - \mathbf{x}_i \mathbf{B})$  is the  $i$ th row of matrix  $(\mathbf{Y}_k - \mathbf{X} \mathbf{B})$  and  $\hat{\Sigma}_k$  is the maximum likelihood estimator (MLE) of  $\Sigma$  for the  $k$ th sample defined as

$$\hat{\Sigma}_k = \frac{(\mathbf{Y}_k - \mathbf{X} \hat{\mathbf{B}}_k)^T (\mathbf{Y}_k - \mathbf{X} \hat{\mathbf{B}}_k)}{n}, \quad (16)$$

$\hat{\mathbf{B}}_k$  in the above equation is the MLE of  $\mathbf{B}$  for the  $k$ th sample, which is the same as OLS estimator given in Equation (4).

Similar to the method by Zhang *et al.*<sup>10</sup>, we replace  $\hat{\mathbf{B}}_k$  in Equation (16) by its corresponding exponentially weighted moving average statistic given by

$$E \hat{\mathbf{B}}_k = \lambda \hat{\mathbf{B}}_k + (1 - \lambda) E \hat{\mathbf{B}}_{(k-1)}, \quad (17)$$

and define the statistic  $S_k$  as

$$S_k = \frac{(\mathbf{Y}_k - \mathbf{X}(E \hat{\mathbf{B}}_k))^T (\mathbf{Y}_k - \mathbf{X}(E \hat{\mathbf{B}}_k))}{n}, \quad (18)$$

where  $E \hat{\mathbf{B}}_0 = \mathbf{B}$ .

In addition, we substitute the following statistics for  $\hat{\Sigma}_k$  and  $C_k$  in Equation (15), respectively:

$$E S_k = \lambda S_k + (1 - \lambda) E S_{k-1}, \quad (19)$$

$$E C_k = \lambda C_k + (1 - \lambda) E C_{k-1}, \quad (20)$$

where  $E \hat{\Sigma}_0 = \Sigma$  and  $E C_0 = np$ .

Consequently, Equation (15) can be rewritten as follows:

$$ELRT_k = n \log |\Sigma| - n \log |E S_k| + E C_k - np. \quad (21)$$

The upper control limit for the above statistic is determined by simulation to give a specified in-control ARL.

### 3.4. Combined MEWMA and chi-square control chart method

The fourth method is the same as the combined MEWMA and chi-square control charts scheme proposed by Noorossana *et al.*<sup>30</sup>. Assume that for the  $k$ th random sample collected over time,  $\bar{\mathbf{e}}_k = (\bar{e}_{1k}, \bar{e}_{2k}, \dots, \bar{e}_{pk})$  is a  $1 \times p$  vector of average error where  $\bar{e}_{jk} = n^{-1} \sum_{i=1}^n \varepsilon_{ijk}$ ,  $j = 1, 2, \dots, p$  ( $\varepsilon_{ijk}$ 's are defined in Equation (2)). The average error vector  $\bar{\mathbf{e}}_k$  follows a multivariate normal distribution with mean vector zero and known covariance matrix  $\Sigma_{\bar{\mathbf{e}}} = n^{-1} \Sigma$ .

For the  $k$ th sample, we use the MEWMA statistic in Equation (8) where

$$\mathbf{z}_k = \lambda \bar{\mathbf{e}}_k^T + (1 - \lambda) \mathbf{z}_{k-1}, \quad k = 1, 2, \dots, \quad (22)$$

and

$$\Sigma_z = \frac{\lambda}{n(2 - \lambda)} \Sigma. \quad (23)$$

In Equation (22),  $\mathbf{z}_0$  is  $p \times 1$  vector of zeros. The upper control limit for the MEWMA control chart is chosen to give a specified in-control ARL.

Noorossana *et al.*<sup>30</sup> proposed a chi-square statistic in combination to a MEWMA control chart to monitor process variability. We use the same statistic described as follows:

$$\chi_k^2 = \sum_{i=1}^n (\varepsilon_{ik} \Sigma^{-1} \varepsilon_{ik}^T), \quad (24)$$

where  $\varepsilon_{ik}$  is the  $i$ th row of error term matrix in Equation (2). It should be noted that  $\varepsilon_{ik}$  is a multivariate normal random vector with mean vector zero and known covariance matrix  $\Sigma$ . In this case, when the process is in-control,  $\chi_k^2$  is a chi square random variable with  $np$  degrees of freedom. Hence, a  $100(1 - \alpha)$  percentile of the chi-square distribution with  $np$  degrees of freedom is used to construct an upper control limit as  $UCL = \chi_{np, \alpha}^2$ .

## 4. Change point estimator

If an out-of-control signal is detected by the multivariate multiple methods, identifying location of the shift, or what is referred to as change point, and classifying its nature would help quality engineers to identify the source of the assignable cause(s) quickly. In this section, a change point method to estimate the location of a shift in Phase II monitoring of multivariate multiple linear regression profiles is presented. This method is the extension of the change point method proposed by Zou *et al.*<sup>7</sup>

In order to estimate the change point, MLE of the change point is used. When an approach signals at subgroup  $k$ , the value of  $t$  that maximizes the likelihood function is the MLE for the change point. The estimator is computed by

$$\begin{aligned} \hat{t} &= \arg \max(lr_{t,k}), \\ 1 &\leq t < k, \end{aligned} \quad (25)$$

where  $lr_{t,k}$  is the generalized likelihood ratio statistic and is given as follows: (The proof is given in Appendix D.)

$$lr_{t,k} = (k-t)n[\log |\Sigma| - \log |\hat{\Sigma}_{t,k}| - p] + \sum_{j=t+1}^k \sum_{i=1}^n (\mathbf{y}_{ik} - \mathbf{x}_{ik}\mathbf{B})\Sigma^{-1}(\mathbf{y}_{ik} - \mathbf{x}_{ik}\mathbf{B})^T. \quad (26)$$

$\hat{\Sigma}_{t,k}$  is calculated using Equation (16) after pooling samples  $t+1, t+2, \dots$ , and  $k$  as follows:

$$\hat{\Sigma}_{t,k} = \frac{(\mathbf{Y}_{t,k} - \mathbf{X}_{t,k}\hat{\mathbf{B}}_{t,k})^T(\mathbf{Y}_{t,k} - \mathbf{X}_{t,k}\hat{\mathbf{B}}_{t,k})}{n(k-t)}, \quad (27)$$

where  $\mathbf{Y}_{t,k}$  is the  $n(k-t) \times p$  matrix of response variables and  $\mathbf{X}_{t,k}$  is the  $n(k-t) \times (q+1)$  matrix of explanatory variables.  $\hat{\mathbf{B}}_{t,k}$  is the OLS estimator of the matrix  $\mathbf{B}$  and is given by

$$\hat{\mathbf{B}}_{t,k} = (\mathbf{X}_{t,k}^T \mathbf{X}_{t,k})^{-1} \mathbf{X}_{t,k}^T \mathbf{Y}_{t,k}. \quad (28)$$

## 5. ARL comparisons

In this section, we compare the ARL performance of the proposed methods discussed in Section 3. These methods are namely Method A, the MEWMA method, which is an extension of the MEWMA method given by Noorossana *et al.*<sup>30</sup>, Method B, a modified version of Method A to reduce dimension of the parameters vector, Method C, the likelihood ratio method, which is an extension of the method by Zhang *et al.*<sup>10</sup>, and Method D, the combined MEWMA and chi-square method proposed by Noorossana *et al.*<sup>30</sup>. All control chart schemes are designed to have an in-control ARL of approximately 200. In all the EWMA and MEWMA statistics, the value of smoothing constants,  $\lambda$ , is set equal to 0.2 as generally used in the literature. In our simulation studies, each ARL value is estimated using 5000 replications. The underlying multivariate profile model considered in this paper is given by

$$Y_1 = 3 + 2x_1 + x_2 + \varepsilon_1, \quad Y_2 = 2 + x_1 + x_2 + \varepsilon_2. \quad (29)$$

The pair of observations (2,1), (4,2), (6,3), and (8,2) are considered as the value for independent variables  $x_1$  and  $x_2$ . The vector of error terms  $(\varepsilon_1, \varepsilon_2)$  is a bivariate normal random variable with mean vector zero and known covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

where  $\sigma_1^2 = 1, \sigma_2^2 = 1$ . To investigate the effect of correlation between response variables, different values of  $\rho$ , namely,  $\rho = 0.1, 0.5$ , and 0.9 are used in our simulation studies.

For Method A, the upper control limit is set equal to 17.55 to achieve an in-control ARL of roughly 200. In Method B, the upper control limit is set equal to UCL = 13.88 to give an in-control ARL of approximately 200. The upper control limit of Method C is set equal to 3.79 to achieve an in-control ARL of roughly 200. For Method D, the upper control limit for MEWMA chart is set equal to 11.1 to yield an in-control ARL of approximately 400. In addition, for the  $\chi^2$  control chart, UCL is set equal to 23.77 to give an in-control ARL of roughly 400. Therefore, the overall ARL for Method D is approximately equal to 200.

Table I shows the simulated out-of-control ARL values for shifts in  $Y$ -intercept of the first profile ( $\beta_{01}$ ) in units of  $\sigma_1$ . As we can see, all the methods perform relatively well. Method D performs uniformly better than the competing methods for all shifts considered. Method B outperforms Method A. The performance of Method B is also better than Method C except in large shifts when  $\rho$  equals to 0.9 in which the performance of both the methods is roughly the same. Method A is better than Method C except in large shifts when  $\rho$  is equal to 0.9. Note that the performance of each method improves as the value of  $\rho$  increases. Similar results (not reported here) are obtained for shifts in the intercept of the second profile ( $\beta_{02}$ ) in unit of  $\sigma_2$ .

The simulated out-of-control ARL values for shifts in the slope of the first profile from  $\beta_{11}$  to  $\beta_{11} + \lambda_1\sigma_1$  are shown in Table II. Similarly, all methods perform well. Methods D and B have a better performance compared with other methods for this type of

**Table I.** The simulated out-of-control ARL values under the shifts from  $\beta_{01}$  to  $\beta_{01} + \lambda_0 \sigma_1$  (in-control ARL=200)

$\rho=0.1$	$\lambda_0$									
	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Method A	77.11	21.82	10.41	6.61	4.88	3.84	3.24	2.82	2.49	2.26
Method B	67.02	18.48	9.17	5.90	4.35	3.49	2.95	2.60	2.30	2.12
Method C	89.74	28.83	13.83	8.61	6.22	4.83	3.90	3.23	2.75	2.36
Method D	64.71	17.60	8.46	5.46	3.97	3.16	2.61	2.17	1.83	1.59
$\rho=0.5$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Method A	63.06	16.96	8.33	5.45	4.09	3.31	2.82	2.45	2.20	2.04
Method B	53.47	14.78	7.34	4.83	3.72	3.03	2.57	2.26	2.06	1.93
Method C	74.33	21.82	10.77	7.02	5.05	3.96	3.19	2.65	2.25	1.97
Method D	51.63	13.43	6.76	4.46	3.29	2.62	2.16	1.80	1.53	1.31
$\rho=0.9$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Method A	17.29	5.43	3.34	2.45	2.06	1.87	1.59	1.24	1.06	1.01
Method B	14.97	4.91	3.04	2.27	1.94	1.68	1.32	1.08	1.01	1.00
Method C	22.21	7.08	4.01	2.68	2.00	1.55	1.20	1.04	1.00	1.00
Method D	13.66	4.53	2.65	1.82	1.34	1.09	1.02	1.00	1.00	1.00

**Table II.** The simulated out-of-control ARL values under the shifts from  $\beta_{11}$  to  $\beta_{11} + \lambda_1 \sigma_1$  (in-control ARL=200)

$\rho=0.1$	$\lambda_1$									
	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
Method A	119.42	46.59	21.24	12.15	8.36	6.38	5.09	4.29	3.75	3.31
Method B	108.24	39.23	18.59	10.83	7.46	5.73	4.66	3.93	3.40	3.04
Method C	129.81	58.04	27.27	15.94	10.92	8.20	6.58	5.52	4.63	3.98
Method D	116.18	43.08	20.21	11.65	7.86	5.87	4.69	3.93	3.35	2.90
$\rho=0.5$	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
Method A	106.37	35.51	16.12	9.53	6.70	5.22	4.28	3.63	3.21	2.87
Method B	91.63	30.34	14.01	8.54	6.10	4.69	3.91	3.36	2.94	2.63
Method C	118.05	44.86	21.12	12.70	8.82	6.76	5.41	4.48	3.84	3.31
Method D	97.42	33.68	15.40	9.05	6.27	4.80	3.90	3.23	2.76	2.40
$\rho=0.9$	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
Method A	35.67	9.66	5.26	3.66	2.89	2.39	2.10	1.96	1.81	1.63
Method B	31.10	8.74	4.87	3.41	2.68	2.26	2.00	1.85	1.65	1.41
Method C	44.39	12.85	6.89	4.56	3.39	2.59	2.09	1.75	1.48	1.24
Method D	34.40	9.06	4.91	3.26	2.41	1.84	1.44	1.19	1.07	1.02

shift. Method B performs better than Method D for small shifts. However, for large shifts, the performance of Method D is better than Method B. Method A performs better than Method C except for large values of  $\rho$  and  $\lambda_1$ . As the value of  $\rho$  increases, the performance of all methods improves. Similar results are obtained for shifts in  $\beta_{21}, \beta_{12},$  and  $\beta_{22}$  which, due to similarity, are not reported here.

The simulation results show that the chi square control chart used in Method D not only performs well in detecting the SD shifts, but also helps to detect large shifts in the coefficients of the regression lines. This is why Method D performs well in detecting small and large shifts in the coefficients of the regression lines.

Table III shows the out-of-control simulated ARL values for shifts in the SD of the first profile from  $\sigma_1$  to  $\gamma\sigma_1$ . Methods C and D are uniformly better than Methods A and B. Method C performs better than Method D for small shifts. However, the performance of Method D is slightly better than Method C for large shifts.

The poor performance of Methods A and B in detecting shifts in the elements of variance-covariance matrix is due to the fact that the MEWMA charts of these methods are constructed based on the vector of the process mean. The performance of Methods A and B in reacting to SD shifts will improve if the chi-squared control chart described in Method D, for example, is used in conjunction with the MEWMA chart of Method A or Method B.

The performance of all methods gets better as the value of  $\rho$  increases. Similar results (not reported here) are obtained for shifts in  $\sigma_2$ .

**Table III.** The simulated out-of-control ARL values under the shifts from  $\sigma_1$  to  $\gamma\sigma_1$  (in-control ARL=200)

$\rho=0.1$	$\gamma$									
	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Method A	65.45	16.86	6.21	3.27	2.02	1.49	1.24	1.14	1.07	1.04
Method B	75.45	20.17	8.04	4.01	2.52	1.85	1.47	1.30	1.19	1.13
Method C	44.26	7.53	3.16	1.86	1.36	1.15	1.05	1.03	1.01	1.00
Method D	51.89	8.00	2.48	1.43	1.15	1.06	1.02	1.01	1.01	1.00
$\rho=0.5$	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Method A	63.29	15.48	5.62	2.89	1.82	1.38	1.19	1.10	1.05	1.04
Method B	71.71	18.73	7.20	3.59	2.27	1.69	1.39	1.24	1.16	1.10
Method C	38.75	6.84	2.82	1.71	1.26	1.10	1.04	1.02	1.01	1.01
Method D	48.22	7.10	2.22	1.34	1.11	1.04	1.01	1.01	1.00	1.00
$\rho=0.9$	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Method A	42.57	7.67	2.74	1.54	1.20	1.08	1.03	1.02	1.01	1.01
Method B	50.76	9.86	3.58	1.93	1.42	1.22	1.11	1.07	1.05	1.03
Method C	16.84	3.50	1.60	1.17	1.05	1.01	1.01	1.00	1.00	1.00
Method D	28.68	3.02	1.33	1.07	1.02	1.01	1.00	1.00	1.00	1.00

**Table IV.** The simulated out-of-control ARL values under simultaneous shifts from  $\beta_{01}$  to  $\beta_{01} + \lambda_0\sigma_1$  and  $\beta_{02}$  to  $\beta_{02} + \delta_0\sigma_2$  (in-control ARL=200)

		$\delta_0$					
		0.2	0.4	0.6	0.8	1	
$\lambda_0$	0.2	Method A	63.05	22.15	10.19	6.25	4.58
		Method B	50.26	18.64	8.82	5.52	4.03
		Method C	76.01	28.91	13.47	8.09	5.81
		Method D	50.58	17.89	8.24	5.13	3.73
	0.4	Method A	22.36	16.89	10.15	6.72	4.82
		Method B	18.28	14.36	8.77	5.80	4.31
		Method C	29.13	22.07	13.28	8.63	6.17
		Method D	18.13	13.56	8.31	5.44	3.97
	0.6	Method A	10.19	10.18	8.33	6.23	4.82
		Method B	8.91	8.79	7.23	5.54	4.31
		Method C	13.30	13.34	10.71	8.15	6.15
		Method D	8.33	8.18	6.74	5.16	3.97
0.8	Method A	6.25	6.55	6.25	5.45	4.58	
	Method B	5.52	5.85	5.49	4.83	4.07	
	Method C	8.18	8.68	8.11	7.02	5.73	
	Method D	5.18	5.43	5.19	4.41	3.75	
1	Method A	4.56	4.87	4.86	4.55	4.09	
	Method B	4.07	4.28	4.29	4.09	3.67	
	Method C	5.80	6.23	6.25	5.78	5.09	
	Method D	3.72	3.99	3.95	3.75	3.31	

Tables IV–VI show some simultaneous shifts in the parameters of investigated profiles. As shown in Tables IV and V, for simultaneous shifts in the coefficients of regression lines, Methods B and D work better than the other methods and Method A performs better than Method C. Table VI shows that in the case of simultaneous shifts in the SDs, Method D performs slightly better than Method C, and both Methods C and D perform better than Methods A and B.

To investigate the effect of sample size on the performance of the proposed methods, we also did simulation runs with  $n=10$  and  $n=20$ . The result of the simulations studies (not reported here) show that as the sample size increases, the performance of all methods except Method A in detecting shifts in the SD improves.

In order to assess the effect of an increase in the values of  $p$  and  $q$ , we repeat our simulations with different values of  $p$  and  $q$ . Our simulation study shows that as the value of  $p$  and  $q$  increases, the performance of all methods gets worse in detecting

**Table V.** The simulated out-of-control ARL values under simultaneous shifts from  $\beta_{11}$  to  $\beta_{11} + \lambda_1 \sigma_1$  and  $\beta_{12}$  to  $\beta_{12} + \delta_1 \sigma_2$  (in-control ARL=200)

			$\delta_1$				
			0.025	0.05	0.075	0.1	0.125
$\lambda_1$	0.025	Method A	104.41	47.05	20.81	11.48	7.68
		Method B	86.39	38.04	17.51	9.98	6.92
		Method C	115.05	57.19	26.34	15.17	10.19
		Method D	102.87	44.26	19.89	10.77	7.27
	0.05	Method A	47.01	36.49	21.04	12.28	8.37
		Method B	38.18	29.33	17.61	10.80	7.42
		Method C	57.47	43.81	26.39	16.29	10.93
		Method D	44.81	34.26	19.33	11.77	7.94
	0.075	Method A	20.79	20.10	16.13	11.54	8.34
		Method B	17.45	17.01	13.80	10.11	7.37
		Method C	26.43	26.43	21.05	15.15	10.93
		Method D	19.61	19.83	15.11	10.83	7.80
	0.1	Method A	11.70	12.25	11.44	9.72	7.75
		Method B	10.00	10.61	10.02	8.41	6.89
		Method C	14.97	16.25	15.02	12.81	10.16
		Method D	10.95	11.65	10.77	8.95	7.20
	0.125	Method A	7.76	8.31	8.39	7.66	6.73
		Method B	6.80	7.33	7.29	6.79	6.06
		Method C	10.06	10.94	10.96	10.22	8.96
		Method D	7.21	7.89	7.87	7.23	6.31

**Table VI.** The simulated out-of-control ARL values under simultaneous shifts from  $\sigma_1$  to  $\gamma_1 \sigma_1$  and  $\sigma_2$  to  $\gamma_2 \sigma_2$  (in-control ARL=200)

			$\gamma_2$				
			1.2	1.4	1.6	1.8	2
$\gamma_1$	1.2	Method A	34.94	9.76	4.38	2.45	1.64
		Method B	39.68	12.10	5.42	3.03	2.03
		Method C	23.88	5.42	2.50	1.58	1.21
		Method D	23.03	4.04	1.66	1.17	1.05
	1.4	Method A	1.49	1.32	1.19	1.10	1.06
		Method B	1.80	1.56	1.37	1.24	1.15
		Method C	1.14	1.09	1.04	1.02	1.01
		Method D	1.03	1.02	1.00	1.00	1.00
	1.6	Method A	1.04	1.02	1.01	1.01	1.00
		Method B	1.12	1.09	1.07	1.04	1.03
		Method C	1.00	1.00	1.00	1.00	1.00
		Method D	1.00	1.00	1.00	1.00	1.00
	1.8	Method A	1.00	1.00	1.00	1.00	1.00
		Method B	1.03	1.02	1.01	1.01	1.01
		Method C	1.00	1.00	1.00	1.00	1.00
		Method D	1.00	1.00	1.00	1.00	1.00
	2	Method A	1.00	1.00	1.00	1.00	1.00
		Method B	1.01	1.01	1.00	1.00	1.00
		Method C	1.00	1.00	1.00	1.00	1.00
		Method D	1.00	1.00	1.00	1.00	1.00

**Table VII.** The simulated out-of-control ARL values for  $p=3, q=3, n=4$  (in-control ARL=200)

Shifts from $\beta_{01}$ to $\beta_{01} + \lambda_0 \sigma_1$										
$\lambda_0$										
$\rho=0.5$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Method A	74.75	19.99	9.23	6.04	4.49	3.66	3.10	2.71	2.40	2.18
Method B	58.68	15.54	7.66	5.09	3.85	3.12	2.65	2.32	2.11	1.99
Method C	86.73	28.30	14.74	9.98	7.55	5.98	4.93	4.15	3.54	3.06
Method D	52.86	13.76	6.78	4.52	3.36	2.68	2.18	1.82	1.53	1.32
Shifts from $\beta_{11}$ to $\beta_{11} + \lambda_1 \sigma_1$										
$\lambda_1$										
$\rho=0.5$	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
Method A	118.19	43.43	19.23	11.03	7.59	5.82	4.80	4.08	3.57	3.18
Method B	98.40	31.61	14.69	8.89	6.23	4.84	4.02	3.44	3.03	2.69
Method C	125.29	54.75	26.74	16.97	12.26	9.62	7.95	6.76	5.81	5.08
Method D	102.33	35.08	15.45	9.01	6.29	4.85	3.92	3.28	2.81	2.41
Shifts from $\sigma_1$ to $\gamma \sigma_1$										
$\gamma$										
$\rho=0.5$	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Method A	68.22	15.11	5.54	2.83	1.71	1.30	1.12	1.06	1.02	1.01
Method B	83.69	21.65	8.07	3.93	2.45	1.76	1.44	1.27	1.18	1.11
Method C	50.47	10.36	4.42	2.47	1.64	1.29	1.13	1.05	1.04	1.01
Method D	58.35	8.49	2.37	1.38	1.12	1.05	1.02	1.01	1.00	1.00

shifts in the regression coefficients. However, the increase in the values of  $p$  and  $q$  impacts the performance of Method A more negatively than Methods B and D. This is due to the dimensionality problem of Method A. In Methods D and B a  $p \times 1$  vector and  $2p \times 1$  vector is involved, respectively, whereas Method A deals with vectors of  $p(q+1) \times 1$  dimension. Therefore, as  $q$  increases, the performance of Method A deteriorates. In fact, Methods B and D fit the reduction mission, which is explained in detail in Section 3.2, much better than Method A and this is why they led to better results in the simulation studies than Method A when  $q$  increases.

Table VII shows the out-of-control simulated ARL values for  $p=3$  and  $q=3$ . As shown in this table, the performance of Method A deteriorates relatively faster than Methods B and D in detecting shifts in the regression coefficients.

## 6. Performance of the change point method

To evaluate the effectiveness of the change point estimator, again we use the example in Equation (29) to estimate the mean and SD of  $\hat{t}$ . However, in this section we assume that  $n=10$  with the ordered pairs of (2,1), (4,4), (6,3), (8,2), (10,5), (12,6), (14,8), (16,5), (18,4), and (20,7) for the explanatory variables. We generated 10 000 replications with  $t=25$  and used Method D to detect the shifts. Any replications in which a signal occurs before the  $t+1$  were discarded. Table VIII shows the mean and SD of  $\hat{t}$  for the shifts in parameters of first profile. It also gives the observed frequencies with which the estimators are within a given number of samples around the actual change point, i.e. the probabilities  $P(\hat{t}=t), P(|\hat{t}-t| \leq 1), P(|\hat{t}-t| \leq 3), P(|\hat{t}-t| \leq 5),$  and  $P(|\hat{t}-t| \leq 10)$  denoted by **P0, P1, P3, P5, and P10**.

The results show that the change point method performs well in estimating the location of shifts and its performance improves as the magnitude of the shift increases.

## 7. Illustrative example

As discussed in the introduction section, Parker *et al.*<sup>29</sup> introduced a calibration case at NASA Langley Research Center in which the relationships between six response variables and six explanatory variables are investigated. In this case, there are six forces and moments as explanatory variables and six electrical measurements of forces and moments as response variables. The relationships between response variables and explanatory variables can be modeled as multivariate multiple linear profiles. In this section,

**Table VIII.** The mean, standard deviation (SD) and precision of change point estimate

$\lambda_0$	Mean	SD	P0	P1	P3	P5	P10
Shifts from $\beta_{01}$ to $\beta_{01} + \lambda_0\sigma_1$							
0.2	30.20	12.53	0.09	0.20	0.35	0.46	0.67
0.4	24.87	5.17	0.28	0.48	0.71	0.83	0.94
0.6	24.55	3.35	0.49	0.73	0.90	0.95	0.97
0.8	24.70	2.28	0.68	0.87	0.96	0.98	0.99
1	24.82	1.48	0.82	0.95	0.98	0.99	1.00
Shifts from $\beta_{11}$ to $\beta_{11} + \lambda_1\sigma_1$							
$\lambda_1$	Mean	SD	P0	P1	P3	P5	P10
0.025	25.89	6.81	0.20	0.37	0.57	0.71	0.88
0.05	24.62	3.08	0.54	0.77	0.92	0.96	0.98
0.075	24.86	1.47	0.78	0.94	0.98	0.99	1.00
0.1	24.91	0.96	0.91	0.98	0.99	1.00	1.00
0.125	24.95	0.83	0.97	0.99	1.00	1.00	1.00
Shifts from $\sigma_1$ to $\gamma\sigma_1$							
$\gamma$	Mean	SD	P0	P1	P3	P5	P10
0.2	27.05	9.01	0.16	0.32	0.51	0.63	0.81
0.4	24.59	3.13	0.54	0.78	0.92	0.96	0.98
0.6	24.72	1.78	0.83	0.94	0.98	0.98	0.99
0.8	24.92	0.73	0.95	0.99	1.00	1.00	1.00
1	24.99	0.16	0.99	1.00	1.00	1.00	1.00

we apply the proposed methods to monitor the performance of the underlying process in Phase II. Suppose that the in-control model between response variables and explanatory variables are given as

$$\begin{aligned}
 Y_1 &= -0.05 + 10x_1 - 0.01x_2 - 0.03x_3 + 0.26x_4 + 0x_5 + 0.03x_6 + \varepsilon_1, \\
 Y_2 &= 0.48 + 0.24x_1 + 21.01x_2 - 0.09x_3 + 0.03x_4 - 0.12x_5 + 0.01x_6 + \varepsilon_2, \\
 Y_3 &= 0.37 + 0.09x_1 + 0.01x_2 + 6.81x_3 + 0.04x_4 + 0.02x_5 - 0.03x_6 + \varepsilon_3, \\
 Y_4 &= 0.04 + 0x_1 + 0x_2 + 0x_3 + 10.53x_4 - 0.47x_5 + 0.21x_6 + \varepsilon_4, \\
 Y_5 &= 0.09 - 0.021x_1 + 0x_2 + 0.01x_3 + 0.02x_4 + 7x_5 - 0.34x_6 + \varepsilon_5, \\
 Y_6 &= 0.09 + 0.04x_1 + 0x_2 - 0.01x_3 + 0.18x_4 - 0.34x_5 + 11.46x_6 + \varepsilon_6,
 \end{aligned}
 \tag{30}$$

and the covariance matrix for the response variables, which is computed by using historical data set in Phase I, is known and given as

$$\Sigma = \begin{bmatrix} 99 & 14 & 17 & 22 & 18 & 15 \\ 14 & 94 & 20 & 24 & 18 & 15 \\ 17 & 20 & 91 & 27 & 11 & 22 \\ 22 & 24 & 27 & 104 & 20 & 21 \\ 18 & 18 & 11 & 20 & 101 & 19 \\ 15 & 15 & 22 & 21 & 19 & 90 \end{bmatrix}.
 \tag{31}$$

The values for the explanatory variables are given in Appendix E. We generate 15 samples with a sustained shift in the coefficient of the second multiple regression profile from  $\beta_{12}=0.24$  to  $\beta_{12}=0.5$  in sample 11 and compute the statistics for the proposed methods. The control charts related to the proposed methods are presented in Figures 1–3. The upper control limits for the control charts of the MEWMA in Methods A and B as well as LRT method (Method C) are chosen by simulation to achieve in-control ARL roughly equal to 200. In Method D, to obtain the overall in-control ARL of 200, we set the in-control ARL for each control chart roughly equal to 400. Hence, the upper control limit for MEWMA control chart based on the in-control ARL of approximately 400 is equal to 19.5. The upper control limit for chi-square control chart is equal to 139.5 using  $UCL = \chi_{96,0.0025}^2$ . Figures 1–4 show how each control chart reacts in detecting a shift in one of the coefficients of the second multiple regression profile.

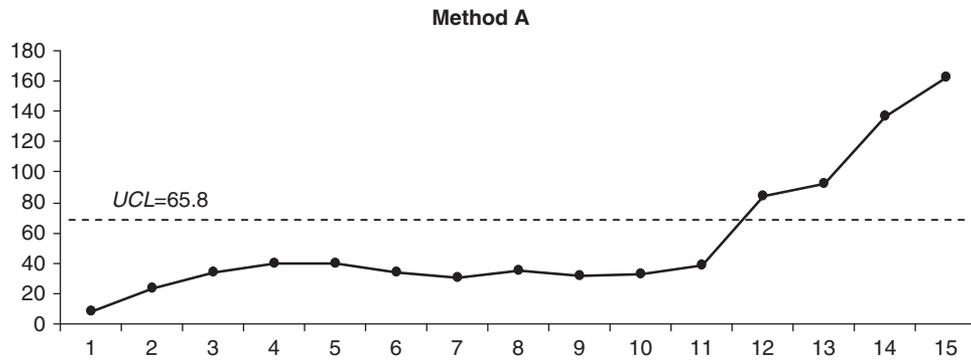


Figure 1. Performance of Method A for simulated samples with shift from  $\beta_{12}=0.24$  to  $\beta_{12}=0.5$  at sample 11

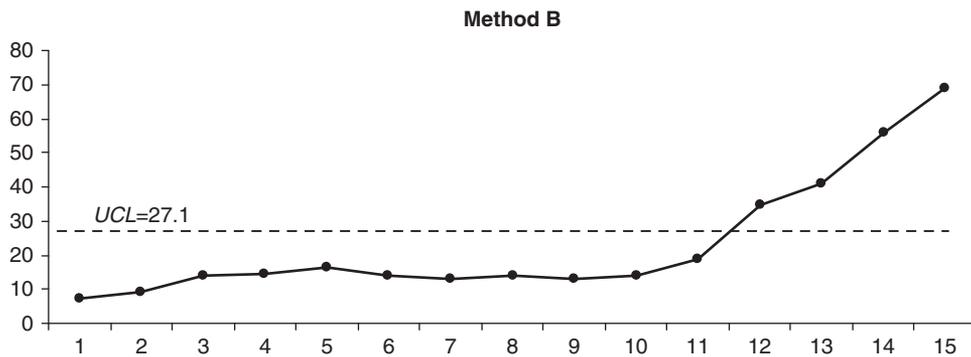


Figure 2. Performance of Method B for simulated samples with shift from  $\beta_{12}=0.24$  to  $\beta_{12}=0.5$  at sample 11

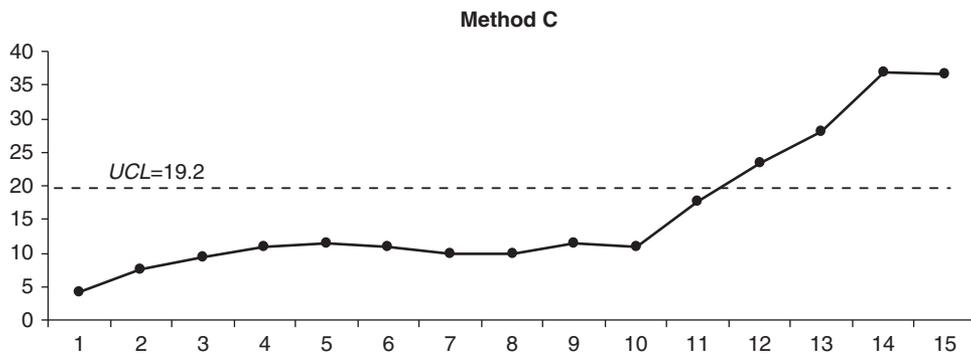


Figure 3. Performance of Method B for simulated samples with shift from  $\beta_{12}=0.24$  to  $\beta_{12}=0.5$  at sample 11

## 8. Conclusions

In this paper, we proposed four methods to monitor multivariate multiple linear regression profiles in Phase II. MEWMA control scheme (Method A) was the first proposed method. However, due to the dimensionality problem of this method, a second method referred to as Method B was proposed which by transforming the multivariate multiple linear structure into multivariate simple linear regression helped to reduce the number of parameters that one needs to monitor. Our simulation studies showed superior performance for the latter method. The third method proposed was a likelihood ratio method referred to as Method C and the fourth method was the combined MEWMA and chi-square control charts scheme (Method D) proposed by Noorossana *et al.*<sup>30</sup>. The performance of each method was investigated through simulation studies via ARL criterion. The results showed that Methods D and B perform better than Methods A and C in detecting shifts in the regression coefficients, whereas in detecting shift in the elements of variance-covariance matrix, Methods C and D perform better than Methods A and B. In addition, a change point method based on the likelihood ratio approach was developed to determine the location of shifts. Again the performance of the change point estimator was evaluated through simulation studies. We also showed the use of the methods under a simulated data set based on a calibration case at NASA Langley Research Center. The illustrative example showed the appropriate performance of the proposed methods in detecting shifts in the parameters of multiple linear regression profiles.

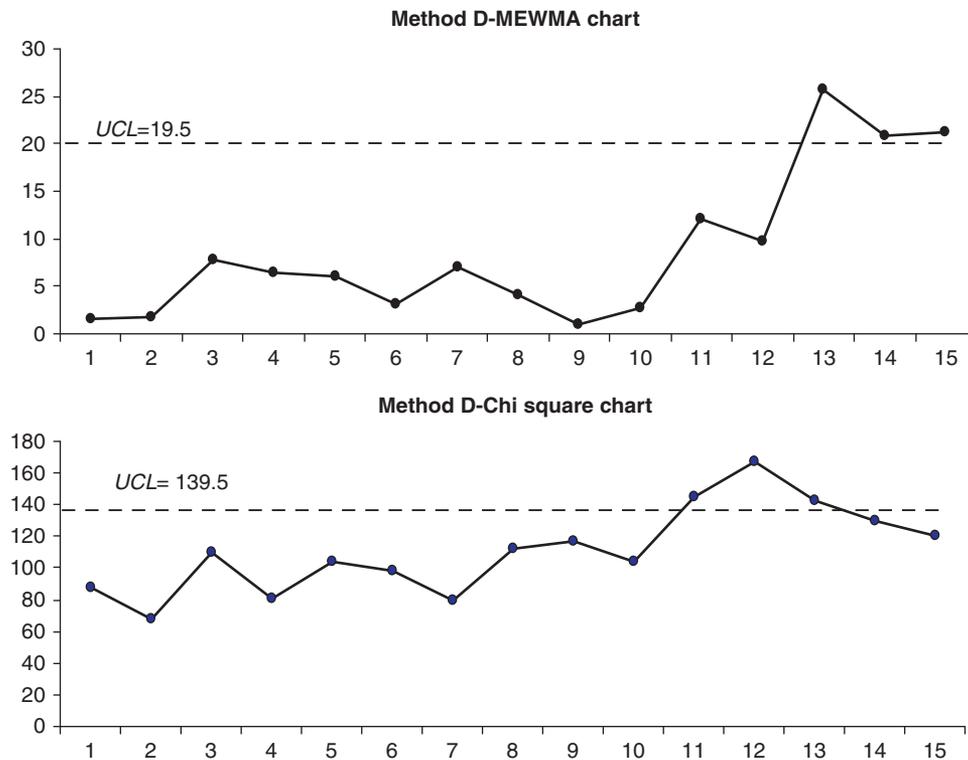


Figure 4. Performance of Method C for simulated samples with shift from  $\beta_{12}=0.24$  to  $\beta_{12}=0.5$  at sample 11

For future research in this area, one can evaluate the worst-case ARL or signal resistance of the competing charts as suggested by the anonymous reviewer.

## Acknowledgements

The authors are grateful to Peter A. Parker, Senior Research Scientist at NASA Langley Research Center and Mahmoud A. Mahmoud from the Cairo University for providing the initial data set used in the example. The authors also appreciate the precious and constructive comments by the anonymous referee and the editor, which significantly improved the content of the paper. Dr Noorossana's research is supported partially by a grant from Iran National Science Foundation.

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## Appendix A: Covariance matrix of the regression parameters estimators

Assume that  $\hat{\beta}_{kj}$  is  $(\hat{\beta}_{0jk}, \hat{\beta}_{1jk}, \dots, \hat{\beta}_{qjk})^T$  and  $\mathbf{y}_{kj}$  is the  $n \times 1$  vector of  $j$ th response variable for the  $k$ th sample. Then  $\hat{\beta}_k$  can be written as follows:

$$\hat{\beta}_k = (\hat{\beta}_{k1}^T, \hat{\beta}_{k2}^T, \dots, \hat{\beta}_{kp}^T)^T, \quad (A1)$$

and covariance matrix of  $\hat{\beta}_k$  is equal to

$$\Sigma_{\hat{\beta}_k} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1p} \\ \Sigma_{21} & \Sigma_{22} & \dots & \Sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{p1} & \Sigma_{p2} & \dots & \Sigma_{pp} \end{pmatrix}, \quad (A2)$$

where  $\Sigma_{hj}$  is  $\text{cov}(\hat{\beta}_{kh}, \hat{\beta}_{kj})$  and is equal to

$$\begin{aligned} \text{cov}(\hat{\beta}_{kh}, \hat{\beta}_{kj}) &= \text{cov}[(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}_{kh}, (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}_{kj}] \\ &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \text{cov}(\mathbf{y}_{kh}, \mathbf{y}_{kj}) [(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T]^T \\ &= (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T (\sigma_{hj} \mathbf{I}) \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} \\ &= \sigma_{hj} (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{I} \mathbf{X} (\mathbf{X}^T \mathbf{X})^{-1} = \sigma_{hj} (\mathbf{X}^T \mathbf{X})^{-1} (\mathbf{X}^T \mathbf{X}) (\mathbf{X}^T \mathbf{X})^{-1}. \end{aligned}$$

Consequently,

$$\text{cov}(\hat{\beta}_{kh}, \hat{\beta}_{kj}) = \sigma_{hj} \mathbf{I} (\mathbf{X}^T \mathbf{X})^{-1} = \sigma_{hj} (\mathbf{X}^T \mathbf{X})^{-1}. \quad (A3)$$

## Appendix B: Elements of covariance matrix for the regression parameters estimators in method B

When the process is in control, the elements of  $\Sigma_{\hat{A}_k}$  in Equation (14) are given by

$$\Sigma'_{hj} = \begin{bmatrix} \text{cov}(\hat{A}_{0hk}, \hat{A}_{0jk}) & \text{cov}(\hat{A}_{0hk}, \hat{A}_{1jk}) \\ \text{cov}(\hat{A}_{1hk}, \hat{A}_{0jk}) & \text{cov}(\hat{A}_{1hk}, \hat{A}_{1jk}) \end{bmatrix}, \quad (B1)$$

where

$$\begin{aligned} \text{cov}(\hat{A}_{0hk}, \hat{A}_{0jk}) &= \text{cov}(\bar{y}_{.hk} - \hat{A}_{1hk}\bar{u}_h, \bar{y}_{.jk} - \hat{A}_{1jk}\bar{u}_j) \\ &= \text{cov}\left(\sum_{i=1}^n \left(\frac{1}{n} + \frac{(u_{ih} - \bar{u}_h)\bar{u}_h}{S_{u_h u_h}}\right) y_{ihk}, \sum_{i=1}^n \left(\frac{1}{n} + \frac{(u_{ij} - \bar{u}_j)\bar{u}_j}{S_{u_j u_j}}\right) y_{ijk}\right) \\ &= \sum_{i=1}^n \left(\frac{1}{n} + \frac{(u_{ih} - \bar{u}_h)\bar{u}_h}{S_{u_h u_h}}\right) \left(\frac{1}{n} + \frac{(u_{ij} - \bar{u}_j)\bar{u}_j}{S_{u_j u_j}}\right) \text{cov}(y_{ihk}, y_{ijk}) \\ &= \sigma_{hj} \sum_{i=1}^n \left(\frac{1}{n} + \frac{(u_{ih} - \bar{u}_h)\bar{u}_h}{S_{u_h u_h}}\right) \left(\frac{1}{n} + \frac{(u_{ij} - \bar{u}_j)\bar{u}_j}{S_{u_j u_j}}\right), \end{aligned} \quad (B2)$$

$$\begin{aligned} \text{cov}(\hat{A}_{1hk}, \hat{A}_{1jk}) &= \text{cov}\left(\frac{S_{u_h y}}{S_{u_h u_h}}, \frac{S_{u_j y}}{S_{u_j u_j}}\right) \\ &= \frac{1}{S_{u_h u_h} S_{u_j u_j}} \text{cov}\left(\sum_{i=1}^n (u_{ih} - \bar{u}_h) y_{ihk}, \sum_{i=1}^n (u_{ij} - \bar{u}_j) y_{ijk}\right) \\ &= \frac{\sigma_{hj}}{S_{u_h u_h} S_{u_j u_j}} \sum_{i=1}^n (u_{ih} - \bar{u}_h)(u_{ij} - \bar{u}_j) = \frac{S_{u_h u_j} \sigma_{hj}}{S_{u_h u_h} S_{u_j u_j}}, \end{aligned} \quad (B3)$$

$$\begin{aligned} \text{cov}(\hat{A}_{0hk}, \hat{A}_{1jk}) &= \text{cov}(\bar{y}_{.hk} - \hat{A}_{1hk}\bar{u}_h, \hat{A}_{1jk}) = \text{cov}(\bar{y}_{.hk}, \hat{A}_{1jk}) - \bar{u}_h \text{cov}(\hat{A}_{1hk}, \hat{A}_{1jk}) \\ &= \text{cov}(\bar{y}_{.hk}, \hat{A}_{1jk}) - \frac{S_{u_h u_j} \sigma_{hj}}{S_{u_h u_h} S_{u_j u_j}} \bar{u}_h = \text{cov}\left(\sum_{i=1}^n \frac{1}{n} y_{ihk}, \frac{1}{S_{u_j u_j}} \sum_{i=1}^n (u_{ij} - \bar{u}_j) y_{ijk}\right) - \frac{S_{u_h u_j} \sigma_{hj}}{S_{u_h u_h} S_{u_j u_j}} \bar{u}_h \\ &= \frac{\sigma_{hj}}{n S_{u_j u_j}} \sum_{i=1}^n (u_{ij} - \bar{u}_j) - \frac{S_{u_h u_j} \sigma_{hj}}{S_{u_h u_h} S_{u_j u_j}} \bar{u}_h = 0 - \frac{S_{u_h u_j} \sigma_{hj}}{S_{u_h u_h} S_{u_j u_j}} \bar{u}_h, \end{aligned} \quad (B4)$$

and equivalently

$$\text{cov}(\hat{A}_{1hk}, \hat{A}_{0jk}) = -\frac{S_{u_h u_j} \sigma_{hj}}{S_{u_h u_h} S_{u_j u_j}} \bar{u}_j \quad (B5)$$

## Appendix C: The expression of $LRT_k$

If the process is in control, the logarithm of the likelihood function based on the model in Equation (1) is as follows:

$$l_0 = -\frac{n}{2} [p \log(2\pi) + \log |\Sigma|] - \frac{1}{2} \sum_{i=1}^n (\mathbf{y}_{ik} - \mathbf{x}_i \mathbf{B}) \Sigma^{-1} (\mathbf{y}_{ik} - \mathbf{x}_i \mathbf{B})^T. \quad (C1)$$

On the other hand, if the process is out of control, the logarithm of the likelihood function is as follows:

$$l_1 = -\frac{n}{2} [p \log(2\pi) + \log |\hat{\Sigma}|] - \frac{1}{2} \sum_{i=1}^n (\mathbf{y}_{ik} - \mathbf{x}_i \hat{\mathbf{B}}) \hat{\Sigma}^{-1} (\mathbf{y}_{ik} - \mathbf{x}_i \hat{\mathbf{B}})^T \quad (C2)$$

leading to

$$l_1 = -\frac{n}{2} [p \log(2\pi) + \log |\hat{\Sigma}|] - \frac{1}{2} \sum_{i=1}^n (\mathbf{y}_{ik} - \mathbf{x}_i \hat{\mathbf{B}}) \left[ \frac{(\mathbf{Y}_k - \mathbf{X} \hat{\mathbf{B}}_k)^T (\mathbf{Y}_k - \mathbf{X} \hat{\mathbf{B}}_k)}{n} \right]^{-1} (\mathbf{y}_{ik} - \mathbf{x}_i \hat{\mathbf{B}})^T, \quad (C3)$$

where  $(\mathbf{y}_{ik} - \mathbf{x}_i \hat{\mathbf{B}})$  is the  $i$ th row of the matrix  $(\mathbf{Y}_k - \mathbf{X} \hat{\mathbf{B}}_k)$ .

In each  $n \times p$  matrix such as  $\mathbf{Z}$ , it can be shown that the following equation is true:

$$\sum_{i=1}^n z_i [\mathbf{Z}^T \mathbf{Z}]^{-1} z_i^T = p, \quad (C4)$$

where  $z_i$  is the  $i^{\text{th}}$  row of the matrix  $\mathbf{Z}$ .

Replacing Equation (C4) into Equation (C3), we obtain

$$l_1 = -\frac{n}{2}[p \log(2\pi) + \log |\hat{\Sigma}|] - \frac{np}{2}. \quad (C5)$$

Consequently,  $LRT_k$  is given as follows:

$$LRT_k = -2(l_0 - l_1) = n \log |\Sigma| - n \log |\hat{\Sigma}| + \sum_{i=1}^n (\mathbf{y}_i - \mathbf{x}_i \mathbf{B}) \Sigma^{-1} (\mathbf{y}_i - \mathbf{x}_i \mathbf{B})^T - np. \quad (C6)$$

## Appendix D: The change point statistic

After  $k$  samples are collected, if the process is in control, the corresponding logarithm of the likelihood function is given as

$$l_0 = -\frac{kn}{2}[p \log(2\pi) + \log |\Sigma|] - \frac{1}{2} \sum_{j=1}^k \sum_{i=1}^n (\mathbf{y}_{ij} - \mathbf{x}_j \mathbf{B}) \Sigma^{-1} (\mathbf{y}_{ij} - \mathbf{x}_j \mathbf{B})^T. \quad (D1)$$

If a step shift occurs after sample  $t$ , the maximum value of the logarithm of the likelihood function is given by

$$l_1 = -\frac{tn}{2}[p \log(2\pi) + \log |\Sigma|] - \frac{1}{2} \sum_{j=1}^t \sum_{i=1}^n (\mathbf{y}_{ij} - \mathbf{x}_j \mathbf{B}) \Sigma^{-1} (\mathbf{y}_{ij} - \mathbf{x}_j \mathbf{B})^T - \frac{(k-t)n}{2}[p \log(2\pi) + \log |\hat{\Sigma}_{t,k}|] - \frac{1}{2} \sum_{j=t+1}^k \sum_{i=1}^n (\mathbf{y}_{ij} - \mathbf{x}_j \hat{\mathbf{B}}_{t,k}) \hat{\Sigma}_{t,k}^{-1} (\mathbf{y}_{ij} - \mathbf{x}_j \hat{\mathbf{B}}_{t,k})^T, \quad (D2)$$

where  $\hat{\mathbf{B}}_{t,k}$  and  $\hat{\Sigma}_{t,k}$  are calculated using Equations (3) and (15) after pooling samples  $t+1, t+2, \dots$ , and  $k$ .

Using Equation (C4), it can be shown that  $l_1$  is equal to:

$$l_1 = -\frac{tn}{2}[p \log(2\pi) + \log |\Sigma|] - \frac{1}{2} \sum_{j=1}^t \sum_{i=1}^n (\mathbf{y}_{ij} - \mathbf{x}_j \mathbf{B}) \Sigma^{-1} (\mathbf{y}_{ij} - \mathbf{x}_j \mathbf{B})^T - \frac{(k-t)n}{2}[p \log(2\pi) + \log |\hat{\Sigma}_{t,k}|] - \frac{(k-t)np}{2}. \quad (D3)$$

Consequently,  $lr_{t,k}$  is given by

$$lr_{t,k} = -2(l_0 - l_1) = (k-t)n[\log |\Sigma| - \log |\hat{\Sigma}_{t,k}| - p] + \sum_{j=t+1}^k \sum_{i=1}^n (\mathbf{y}_{ik} - \mathbf{x}_{ik} \mathbf{B}) \Sigma^{-1} (\mathbf{y}_{ik} - \mathbf{x}_{ik} \mathbf{B})^T. \quad (D4)$$

## Appendix E: X-values for the example in Section 6

The values of explanatory variables for each sample used in the example of Section 6 are given in Table E1.

Table E1. Values of the explanatory variables						
Observation	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
1	0.0	0.0	0.0	0.0	0.0	0.0
2	-68.5	19.2	-106.5	43.5	54.0	26.9
3	-62.1	19.2	-96.5	37.8	-49.7	-39.5
4	-61.7	-21.8	-97.0	39.6	46.9	38.7
5	-68.4	-19.3	-107.4	42.1	-54.2	-27.0
6	68.5	-19.3	106.6	-43.5	-53.8	-26.7
7	61.1	-22.2	94.9	-37.2	47.9	39.5
8	62.1	20.8	97.6	-39.8	-47.4	-38.7
9	0.0	0.0	0.0	0.0	0.0	0.0
10	-68.4	19.3	107.3	42.0	54.5	-27.2
11	-60.5	22.4	95.1	38.8	-48.9	40.3
12	-61.1	-22.3	95.0	37.2	47.6	-39.3
13	-68.5	-19.0	106.5	43.4	-54.2	27.1
14	68.7	-19.1	-107.8	-42.2	-53.4	26.5
15	61.5	-21.6	-96.7	-39.4	47.7	-39.2
16	61.6	22.4	-95.7	-37.5	-46.2	38.5

*Authors' biographies*

**M. Eyvazian** is a PhD candidate in Industrial Engineering at the Iran University of Science and Technology. His primary research interests include applied probability and engineering statistics and its applications in statistical quality control. He has been teaching Probability Theory and Engineering Statistics in the industrial engineering departments at the Iran University of Science and Technology and the Khajeh Nasir Toosi University of Technology.

**R. Noorossana** is Professor of Statistics at the Iran University of Science and Technology. He received his BS in Engineering from the Louisiana State University in 1983 and his MS and PhD in Engineering Management and Statistics from the University of Louisiana in 1986 and 1990, respectively. His primary research interests include statistical process control, process optimization, Six Sigma, and statistical analysis. He is the editor of the *Journal of Industrial Engineering International* and serves on the editorial review board of many journals. He is a member of the founding committee for the Iranian Society for Quality. He is also a senior member of the American Society for Quality, Iranian Society for Quality, Iranian Statistical Association, and Industrial Engineering Society.

**Abbas Saghaei** is a faculty member of the Industrial Engineering Department at the Islamic Azad University, Science and Research branch. He received his PhD in Industrial Engineering from the Iran University of Science and Technology. He holds a BS and an MS in Industrial Engineering. His research interests include statistical process control, Design of experiments, time series analysis and Six Sigma. He is a member of the board of the Iranian Quality Management Society. He is a senior member as well as a certified Six Sigma Black Belt of the American Society for Quality.

**Amirhossein Amiri** is an Assistant Professor at the Shahed University. He holds a BS, MS, and PhD in Industrial Engineering from the Khajeh Nasir Toosi University of Technology, Iran University of Science and Technology, and Tarbiat Modares University, respectively. He is a member of Iranian Statistical Association. His research interests are statistical quality control, profile monitoring, and Six Sigma.