3 Results

In the analysis of particular VaR models we utilize time series of S&P 500 US equity index. The data were obtained from www.finance.yahoo.com with daily frequency over about last 18 years. Value at Risk (VaR) is estimated for one day horizon assuming each of the four models defined above (HS, GI, NIG, AGG) on a mowing window basis assuming 40 different estimation intervals running from 50 days to approximately 8 years (2,000 days) and the probability levels as follows: \((0.5, 0.4, 0.3, 0.2, 0.1, 0.15, 0.05, 0.025, 0.01, 0.005, 0.001, 0.0001, 0.0003, 0.0005)\). True losses and one day VaR estimates are apparent from Figure 1.

![Figure 1: Comparison of VaR estimations and losses over time](image)

In order to evaluate the performance of particular VaR models and their various implementation due to different estimation period, we calculate the efficiency score via CCR model, see [3, 4, 5, 7] and references therein.

The integrated efficiency results are provided in Figure 2. Apparently, GI and AGG models are treated as inefficient except 2 and 3 cases, respectively. Moreover, NIG model slightly overcomes HS model.

![Figure 2: Integrated efficiency of VaR models](image)

4 Conclusion

In this paper we suggested alternative approach to evaluation of the performance of the backtesting procedure of VaR models – Data Envelopment Approach. The approach allows us to consider several probability levels as well as time costs.

Subsequent study of complex approaches to efficiency evaluation of risk models can be very useful, especially taking into account assumed introducing of cVaR approach into Basel III methodology.

Acknowledgements

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References


Among various statistical and data mining discriminant analysis proposed so far for group classification, linear programming discriminant analysis have recently attracted the researchers’ interest. This study introduces fuzzy multi-group discriminant linear programming (fuzzy MDLP) for classification problems. MDLP is less complex compared to other methods and does not suffer from local optima, and fuzzy MDLP overcomes the uncertainty inherently exists during collecting data. The model determines fuzzy boundaries for the groups and finds fuzzy membership grades for the customers, which outperforms the conventional classification methods.

Keywords Customer Classification; Multi-group Linear Programming; Fuzzy Linear Programming.

1 Introduction

The applications of classification methods are wide-ranging; specially managers use classification techniques to make decisions in different business operation areas. A classification procedure is then some formal method for repeatedly making such judgments in new situations [3]. Mangasarian was the first who used LP method for classification problems. LP has some advantages over other approaches which can be enumerated as follow: First, there is no assumption about the functional form and hence it is distribution free. Second, they are less sensitive to outliers. Third, they do not need large datasets. Nonetheless, linear programming methods also have a disadvantage, which is the lengthy computation. However, immense increase in computing power and drop in computing cost overcome the disadvantage and made LP practical [1].

This research uses Multi-Group Discriminant Linear Programming (MDLP) to classify the customers of an Internet Service Provider company based on their real demographic data including age, gender, education, income, purpose, in order to present the strength and accuracy of classification models, specially for MDLP. Then, a fuzzy LP is developed to overcome the ambiguous boundaries of the groups. This helps companies to classify their customers with their current rudimentary databases.

2 Multi-group Discriminant LP

The model proposed by Lam and Moy has recently attracted the interest of researchers [1]. They proved that it is more powerful in terms of hit-rate criterion and its stability rather than statistical methods. Their model is as follows:

Suppose there are \(n\) observations distributed in \(m\) groups so that \(n = n_1 + n_2 + \cdots + n_m\) in which \(n_k\) is the number of observations in group \(k\). If \(x_{ij}\) is value of the \(j\)-th variable (attribute) for the \(i\)-th observation in the sample (where there are \(q\) variables), for each pair of groups \((u, v, u = 1, \cdots, m-1, v = u + 1, \cdots, m)\), the minimization of the sum of deviations model is as follows (phase 1):

\[
\begin{align*}
\min & \sum_{i \in G_u \cup G_v} d_i \\
\text{s.t.} & \sum_{j=1}^{q} w_j (x_{ij} - \bar{x}_j(u)) + d_i \geq 0, \quad \forall i \in G_u \\
& \sum_{j=1}^{q} w_j (x_{ij} - \bar{x}_j(v)) - d_i \leq 0, \quad \forall i \in G_v \\
& \sum_{j=1}^{q} w_j (\bar{x}_j(u) - \bar{x}_j(k)) \geq 1
\end{align*}
\]
in which $d_i$ is the deviation of the individual observations from cut-off scores $(c)$, $w_j$ is the weight of variable $j$ and $\bar{x}_j(k)$ is the mean of the $j$-th variable in group $k$ ($k = 1, 2, \ldots, m$).

The calculated $w_j$ values are used to obtain the values of classification scores in each group $G_k$ as:

$$S_i = \sum_{j=1}^{q} w_j x_{ij}, \quad \forall i \in G_k$$

Then the cut-off scores $(C_{uv})$, which indicate the separating boundary between the groups, are calculated by the following LP model (phase 2):

$$\min \sum_{u=1}^{m-1} \sum_{v=u+1}^{m} \left( \sum_{i \in G_u} d_{iuv} + \sum_{i \in G_v} d_{iuv} \right)$$

s.t.

$$S_{iuv} + d_{iuv} \geq C_{uv}, \quad \forall u, v, i \in G_u$$

$$S_{iuv} - d_{iuv} \leq C_{uv}, \quad \forall u, v, i \in G_v$$

3 Fuzzy Multi-group Discriminant Linear Programming (FMDLP)

Now, it is more realistic, that the cut-off points are not exact, resulted from in-exact data. If the acceptable tolerance for the cut-off point $C_{uv}$ between groups $u$ and $v$ is $P_{uv}$ (usually a fraction of $C_{uv}$, based on experts’ view), to develop the fuzzy LP, which determines the fuzzy boundaries between every pair of groups with tolerance $P_{uv}$, whose objective value is not bigger than $Z_{min}$ (with tolerance $\rho_z$), we have bellow fuzzy inequalities:

$$S_{iuv} + d_{iuv} \gtrless C_{uv} - P_{uv}$$

$$S_{iuv} - d_{iuv} \lessgtr C_{uv} + P_{uv}$$

$$\sum_{u} \sum_{v} \sum_{i} d_{iuv} \lessgtr Z_{min} + \rho_z$$

Then,

$$\lambda_{iu} = \frac{(d_{iuv} - C_{uv}) - (-S_{iuv} - P_{uv})}{P_{uv}}$$

$$\lambda_{iv} = \frac{(d_{iuv} + C_{uv}) - (S_{iuv} - P_{uv})}{P_{uv}}$$

$$\lambda_z = \frac{(Z_{min} - \rho_z) - (\sum_{u} \sum_{v} \sum_{i} d_{iuv})}{P_{uv}}$$

In which $\lambda$s are the boundaries-satisfaction rates. Finally, to maximize the whole satisfaction for all members of all groups, the equivalence LP would be:

$$\min \lambda$$

s.t.

$$d_{iuv} - C_{uv} - \lambda P_{uv} \geq -S_{iuv} - P_{uv}, \quad \forall u, v, i \in G_u$$

$$d_{iuv} + C_{uv} - \lambda P_{uv} \geq S_{iuv} - P_{uv}, \quad \forall u, v, i \in G_u$$

$$\sum_{u} \sum_{v} \sum_{i} d_{iuv} + \lambda P_{uv} \leq Z_{min} + \rho_z,$$

4 Computational Results

To apply the aforementioned classification models in real settings, we use a data-set related to 5271 customers provided by Irangate Internet Service Provider (ISP) Company for a five years period from 2007 to 2012. Recency, Frequency and Monetary (RFM) model is applied to segment the customers using clustering ensemble method, such as K-means. Based on the clustering ensemble concept, three
customer segments identified with different internet usage pattern. Then both MDLP and fuzzy MDLP had been applied to determine the classification boundaries.

In order to verify and measure the accuracy of the classification, typically hold-out method is used in which the data-set is partitioned into two portions: training data and evaluation data (in this study, with the ratio of 65 and 35 percent, respectively) [1]. The first step is training the model and obtaining the weights of variables or predictors in each pair of segments. The predictors in this study are the five demographic variables. The acquired weights indicate the effect of each variable in classifying the customers between two segments. The results were consistence with ISP company experts’ experiences.

The real life data often contaminated, because it is usually include noisy, incomplete and redundant data. To overcome this problem, the Fuzzy MDLP has been developed. The results approved Fuzzy MDLP for classification according to the average performance and the accuracy.

References


A NEW NETWORK DATA ENVIRONMENTAL ANALYSIS MODEL BY BALANCED SCORECARD APPROACH FOR PROJECTS EFFICIENCY EVALUATION

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Purpose of this paper is to provide a framework for evaluating the overall performance of decision-making units by means of a data envelopment analysis (DEA) model with balanced scorecard approach. Many papers have regressed nonparametric estimate of efficiency in the four stages DEA procedure. All of these studies have several problems in the benchmark or projection unit and present the DMU with relative efficiency. Most of these papers do not present an efficient benchmark unit and cannot evaluate the relative efficiency unit. In this paper, we review these studies and then create a four-stage model (based on BSC) that does not have these problems. Our model presents the efficient benchmark unit and also, presents DMU with relative efficiency. In the next step we apply our four stage model in Civil projects promoted by the Omran institute in IRAN.

Keywords Network DEA; BSC; Efficiency Evaluation.

1 Introduction

To maintain viability and continuous improvement, organizations must evaluate their performance, efficiency and productivity. The project organizations are forced to evaluate their projects networks efficiency and effectiveness in order to improve the overall performance of their branches. Industrial and economical units should be evaluated by utilizing scientific methods in order to evaluate and improve the performance and structuring a good position in comparison to other units. Performance is predetermined parameters and measurement sounds like the ability to monitor events and activities in a meaningful way, so performance measurement can be defined as the process of quantifying the effectiveness and efficiency of action (Neely et al, 1995). The main aim of this paper is to study project profitability efficiency, effectiveness and overall performance by utilizing four stages DEA model based on balanced scorecard approach.

They argued that BSC provides managers with the means they need to navigate future competitive success. It included more non-financial measures derived specifically from the organization’s strategy. BSC is one of the most comprehensive and simple performance measurement means that emphasizes both aspects of financial and non-financial, long-term and short-term strategies as well as internal and external business measures. The strongest point of BSC is its ability to illustrate the cause and effect relations between strategies and processes through the four BSC’s perspectives of financial perspective; customer perspective; internal business process perspective, as well as learning and growth perspective. Since it is not possible to determine the efficiency evaluation score by applying BSC, we used the data envelopment analysis model (DEA) to calculate the efficiency score of supply chain performance.

2 Literature review
2.1 Data Envelopment Analysis

DEA was first proposed by Charnes, Cooper and Rhodes (CCR) in 1978. The evolutionary form of CCR model was suggested in 1984 by Banker et al. In subsequent years, several models were developed by a large number of researchers. Orientation, disposability, diversification, and return to scale are different aspects that can be seen in these models.

2.2 Balanced Scorecard

The BSC is a conceptual framework for translating an organization’s strategic objectives into a set of performance measures distributed among four perspectives: financial, customer, internal business processes, and learning and growth.

3 Proposal Model

In this stage, we create two stage DEA model and then modify it into four stages DEA model based on the four perspective of Balanced Scorecard. Consider a two stage model that is shown in Figure 1.

Figure 1: A Two Stage Model

Based upon the properties of PPS and arranging the data sets in matrices $X = (x_j)$, $Z = (z_j)$ and $Y = (y_j)$, we can define the production possibility set of a two stage model that satisfies 1 through 4 expressed below.

Property 1. The observed activities $(x_j, y_j)$ $(j = 1, \ldots, n)$ belong to $P$. According to this property, we have:

$$P_{two-stage} = \left\{ \begin{pmatrix} x_j \\ z_j \\ z_j \\ y_j \end{pmatrix} \right| i = 1, \ldots , n \right\}$$

Property 2. Any semi positive linear combination of activities in $P$ belongs to $P$. According to the convexity axiom, we have:

$$P_{two-stage} = \left\{ \begin{pmatrix} x_j \\ z_j \\ z_j \\ y_j \end{pmatrix} \right| \begin{pmatrix} x_j \\ z_j \\ z_j \\ y_j \end{pmatrix} = \sum_{j=1}^{n} \lambda_j \begin{pmatrix} x_j \\ z_j \\ z_j \\ y_j \end{pmatrix} \text{ and } \sum_{j=1}^{n} \lambda_j = 1 \right\}$$

Property 3. For activities $(x, z)$ in $P$, any semi positive activity $(\bar{x}, \bar{z})$ with $x \leq \bar{x}$ and $\bar{z} \leq z$ is included in $P$ and for an activities $(z, y)$ in $P$, any semi positive activity $(\bar{z}, \bar{y})$ with $z \leq \bar{z}$ and $\bar{y} \leq y$ is included in $P$. With this property we have:

$$P_{two-stage} = \left\{ \begin{pmatrix} x_j \\ z_j \\ z_j \\ y_j \end{pmatrix} \right| x \geq \sum_{j=1}^{n} \lambda_j x_j, \; z \leq \sum_{j=1}^{n} \lambda_j z_j, \; \lambda_j \geq 0, \; \lambda_j \leq 1 \right\}$$

Property 4. If activities $(x, y)$ belong to $P$, then the activities $(tx, ty)$ belong to $P$ for any positive scalar $t$. We call this property the constant return to scale assumption. With accept this assumption the PPS to become changed into:

$$P_{two-stage} = \left\{ \begin{pmatrix} x_j \\ z_j \\ z_j \\ y_j \end{pmatrix} \right| x \geq \sum_{j=1}^{n} \lambda_j x_j, \; z \leq \sum_{j=1}^{n} \lambda_j z_j, \; \lambda_j \geq 0, \; \lambda_j \leq 1 \right\}$$
Now, to build the model with the use of PPS definition, we have:

$$\text{Min } \theta$$

S.to

$$( \theta x \ z \ y ) \in P_{\text{two-stage}}$$

Now, we modify the two stage model onto four stages DEA model based on BSC. Suppose, we can four stages based on BSC perspectives. These stages are shown in Figure 2.

Figure 2: A Four Stage Model

We can define the production possibility set of a four stage model that satisfies 1 through 4 expressed below:

$$P_{\text{four-stage}} = \left\{ \begin{array}{l} x_j \\ z_j \\ z_j \\ y_j \end{array} \right| \left\{ \begin{array}{l} x \geq \sum_{j=1}^{n} \lambda_j x_j , z = \sum_{j=1}^{n} \lambda_j z_j , y \leq \sum_{j=1}^{n} \lambda_j y_j \end{array} \right\}$$

Now, to build the model with the use of PPS definition, we have:

$$\text{Min } \theta$$

S.to

$$( \theta x \ z \ y ) \in P_{\text{four-stage}}$$

4 Results and discussions

In the next step we apply the performance for a sample of the Iranian Project. We used our four stage DEA model to evaluate the Iranian projects performance. In this case study, we have several inputs and outputs for the each stage. The results show that the DMU 1, 2, 4, 7, 8, 10, 11, 12, 14 and 15 are efficient.

5 Conclusions

A general framework to evaluate the overall performance in terms of profitability efficiency and effectiveness by means of four stage DEA model and BSC approach is proposed. There are several studies about Network stage DEA model in the literature of DEA. The most of these models have several problems, such as, they cannot determine the relative efficiency of DMUs, nor can they determine an efficient projection unit. In our model, these problems have been solved. The proposed model can determine the relative efficiency and an efficient projection unit.
6 References


A NEW SUPER-EFFICIENCY MODEL IN THE PRESENCE OF NEGATIVE DATA

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In a recent paper by Vencheh and Esmaeilzadeh, a super-efficiency model based on RDM model is formulated where some input and output data are negative. However, we show that the formulation in the paper is not always feasible or bounded for its purpose and cannot produce the numerical results claimed. Moreover, we demonstrate that the super-efficiency model based on RDM model is unable to provide a super-efficiency measure in some cases. To deal with this drawback we propose a new model, which is always feasible, to rank the DMUs in the presence of negative data.

Keywords DEA; Super-efficiency; Negative data; RDM model

1 Introduction

Data envelopment analysis (DEA) developed by Charnes et al. [2] is a nonparametric method of measuring the efficiency of peer decision making units (DMUs) that have multiple inputs and outputs. One of the prominent methods in order to increase the discrimination power of DEA is the super-efficiency method proposed by Andersen and Petersen [1], which excludes the DMU under evaluation from the reference set. The super-efficiency model under constant returns to scale (CRS) does not suffer from the infeasibility problem while the super-efficiency model under the condition of variable returns to scale (VRS) can encounter infeasibility (Seiford and Zhu [4]). A number of studies have tried to solve the problem of VRS super-efficiency model’s infeasibility. So far many models have been developed to generalize the application of this method. However, in all of these models the non-negativity of data set is assumed. The recently published article by Vencheh and Esmaeilzadeh [5] (hereafter called VE) develops a super-efficiency model based on the RDM model [3] to establish a complete ranking of decision-making units (DMUs). They prove that their proposed model is feasible and bounded. However, this note shows that the VE model suffers from the common infeasibility and unboundedness problems. In this regard, we describe its infeasibility and unboundedness problem by means of a proposition, then we propose a new super-efficiency model. Our proposed super-efficiency model is always feasible.

2 Re-examination of the VE model

Consider a set of n observed DMUs, \( DMU_j, (j = 1, \ldots, n) \) where each observation transforms m inputs, \( x_{ij}, (i = 1, \ldots, m) \), into s outputs, \( y_{rj}, (r = 1, \ldots, s) \). Furthermore, assume that some data can take negative values.
The VE model is developed based on a range directional model (RDM+) proposed by Portela et al. [3]. The purpose of the VE super-efficiency model is to obtain the efficiency scores and the priorities of the
When evaluating \( DMU_o \), the primal VE model and its dual are, respectively, expressed as follows:

\[
\max \quad \delta_o = 1 + \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + u_o \\
\text{s.t.} \quad \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + u_o \leq 0, \quad \forall j, (j \neq o), \\
\sum_{r=1}^{s} u_r P_{ro} - \sum_{i=1}^{m} v_i P_{io} = 1, \\
\sum_{j=1}^{n} \lambda_j = 1, \\
u_r, v_i \geq 0, \quad \forall r, i \quad \text{and} \quad u_o \text{ is free.} 
\]

(2.1)

\[
\min \quad 1 - \beta \\
\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io} - \beta P_{io}^{-}, \quad i = 1, \ldots, m, \\
\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta P_{ro}^{+}, \quad r = 1, \ldots, s, \\
\sum_{j=1}^{n} \lambda_j = 1, \\
\lambda_j \geq 0 (\forall j, j \neq o), \\
\beta \text{ is free,}
\]

(2.2)

where

\[ P_{ro}^{+} = \max_j y_{rj} - y_{ro}, \quad r = 1, \ldots, s, \]
\[ P_{io}^{-} = x_{io} - \min_j x_{ij}, \quad i = 1, \ldots, m. \]

refer to a range of possible improvement of \( DMU_o \) that can never be negative.

Vencheh and Esmaeilzadeh (2013, p.398, Theorem 2) claim that Model 2.1 is feasible and bounded. However, the feasibility and boundedness of Model 2.1 is not correct in general as will be demonstrated in the following proposition.

**Proposition 2.1.** The boundedness of Model 2.1 may not hold if there exists at least one \( i \) or \( r \) such that \( P_{io}^{-} = 0 \) or \( P_{ro}^{+} = 0 \).

**Lemma 2.2.** Models 2.1 and 2.2 are bounded and feasible, respectively, if there do not exist any \( P_{io}^{-} = 0 \) and \( P_{ro}^{+} = 0 \).

### 3 Proposed super-efficiency model

Infeasibility of model 2.2 occurs when a VRS efficient DMU under evaluation cannot reach the frontier formed by the rest of DMUs, towards ideal point. In practical terms, rather than asking, for a given efficient DMU, either how much increase in inputs is possible, or how much reduction in outputs is possible, while still retaining its efficient status, our proposed model describes the minimum movement in both directions needed to reach the frontier generated by the remaining DMUs in the pass towards ideal point. Viewed another way, in the case of infeasibility, our model derives the minimum change needed to project a data
Consider the following model for $DMU_o$:

$$\begin{align*}
\min & \quad 1 - \beta + M \cdot \left( \sum_{r=1}^{s} s_r + \sum_{i=1}^{m} t_i \right) \\
\text{s.t.} & \quad \sum_{j=1, j \neq o}^{n} \lambda_j x_{ij} \leq x_{io} - \beta P_{io}^+ + t_i, \quad i = 1, \ldots, m, \\
& \quad \sum_{j=1, j \neq o}^{n} \lambda_j y_{rj} \geq y_{ro} + \beta P_{ro}^+ - s_r, \quad r = 1, \ldots, s, \\
& \quad \sum_{j=1, j \neq o}^{n} \lambda_j = 1, \\
& \quad \lambda_j \geq 0 \quad (\forall j, j \neq o), \\
& \quad s_r, t_i \geq 0 \quad (\forall r, i), \\
& \quad \beta \text{ is free.}
\end{align*}$$

(3.1)

where $M$ is a user-defined sufficiently large positive number.

**Proposition 3.1.** Model 2.2 is infeasible if and only if some $s_r^+ > 0$ or $t_i^+ > 0$.

**Lemma 3.2.** When model 2.2 is feasible, model 3.1 is equivalent to model 2.2.

### 4 Conclusion

Conventional DEA models have been introduced to deal with non-negative data. In real world applications of DEA, there are some situations that we faced with input/output variables, including both positive and negative values. So far, there are many approaches to deal with negative data in DEA literature, but an appropriate model which can rank the DMUs in the presence of negative data is not provided. In this paper we proposed a new super-efficiency based on RDM model which is always feasible and able to rank DMUs.

### References


A MODIFIED SBM-NDEA APPROACH FOR THE EFFICIENCY IMPROVEMENT IN BANK BRANCHES

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In this study an envelopment form of NDEA model is used for efficiency measurement of bank branches. So, a slack based NDEA model introduced by Tone and Tsutsui (SBM-NDEA) is nominated for its mathematical model. But according to the new categorization of the efficiency measurement factors introduced in this paper and also regarding some previous reviews on SBM-NDEA model, the model will be modified to include desired properties.

Keywords Data envelopment analysis; network DEA; SBM-NDEA model; bank branches.

1 Introduction

Suppose that there exist \( j = 1, \ldots, J \) DMUs (or branches). Each DMU has \( k = 1, \ldots, K \) processes. Assume that \( N_k , M_k , \tilde{N}_k \) and \( \tilde{M}_k \) denote to the sets of main inputs, intermediate inputs, main outputs and intermediate outputs of process \( k \) respectively. Also suppose that the amount of \( n \)-th main inputs and \( m \)-th main outputs in process \( k \) is defined by \( x_{kn}^k \) and \( y_{km}^k \) respectively and \( z_{qj}^{(k,h)} \) is the amount of intermediate factor \( q \) produced in process \( k \) and used in process \( h \) in DMU\( j \). According to these notations, Tone and Tsutsui’\'s model [1] is as below:

\[
\theta_0^* = \min \sum_{k=1}^{K} \omega_k \theta_{ko} = \frac{\sum_{k=1}^{K} \omega_k \left[ 1 - \frac{1}{n_k} \left( \sum_{n=1}^{n_k} s_n \right) \right]}{\sum_{k=1}^{K} \omega_k \left[ 1 + \frac{1}{m_k} \left( \sum_{m=1}^{m_k} s_m \right) \right]}
\]

subject to:

\[
\begin{align*}
& x_{no}^k = \sum_j \lambda_{kj} x_{nj}^k + s_n^- \\
& y_{mo} = \sum_j \lambda_{kj} y_{mj}^k - s_m^+ \\
& \sum_j \lambda_{kj} = 1 \\
& \sum_j \lambda_{kj} z_{qj}^{(k,h)} = \sum_j \lambda_{kj} z_{qj}^{(h,k)} \quad \forall (k, h)
\end{align*}
\]

In further studies [2,3] Tone and Tsutsui’\'s model and its definition of overall efficiency and process efficiency measures were reviewed. Fukuyama and Mirdehghan [2] and also Chen et al., [3] noted that overall efficiency cannot be regarded as the weighted average of process efficiencies as mentioned in Tone and Tsutsui [1].

2 Proposing approach

In order to overcome the above shortfalls, the overall efficiency in the objective function will be modified. Moreover intermediate factors are contributed in for their related process depending on the categories which will be introduced for them within a more detailed categorization which is suggested here. The modifications are briefly mentioned in the table.
3 Case study: Bank branches evaluation

In our proposed model all contributing factors in each process either main or intermediate factors are considered in the constraints. “General and administrative expenses” is an inseparable shared factor. The modification ensures that its changes in its contributing processes be identical.

Generally, in this new approach, our ultimate goal is to increase main outputs and reduction of main inputs totally in DMU not in processes individually. Therefore only the main inputs and outputs are appeared in the objective function. Meanwhile, intermediate factors can be increased, decreased or remain constant depending on their categorization:

1. In ordinary intermediate factors, the model itself recognizes that how should these factors be changed in line with the optimization of the overall efficiency. So their corresponding slack variables are considered to be free in sign (Such as daily cash flow in this study).

2. In some network structures, intermediate factors naturally cannot be decreased (increased) or we do not tend to decrease (increase) them. Such factors are called increasing (decreasing) intermediate factors. In this study, the attracted “deposits” are increasing intermediate factor. Their corresponding slacks are non-negative to insure that the targets are increased.

3. In this study, the commitment cost is a decreasing intermediate factor. So, its corresponding change amount in “deposit attraction process” and also in “deposit allocation process” is considered to be non-positive. This insures that the considered target for this factor is reduced.

4 Conclusions

According to the bank branches network structure defined in this paper, a more detailed categorization of factor types in a network structure was introduced and some modifications in SBM-NDEA model were represented. These modifications constitute some revisions in the overall efficiency measure and contribution of intermediate factors in constraints according to their categorization.

References


Figure 1: modifications in Tone and Tsutsui’s model
In conventional data envelopment analysis (DEA) models, the role of each variable (i.e. input/output) is
certain. Nevertheless, in real world, there are situations that a measure can play both input and output
roles. In this paper, these measures are defined as adaptable measures. At first, the efficiencies of
decision making units (DMUs) from two viewpoints, optimistic and pessimistic, are evaluated where
adaptable variables present. Then, the efficiencies are integrated by using the geometric average. A
numerical example is used to illustrate the approach.

Keywords DEA; efficiency; adaptable factor; input/output.

1 Introduction

Data envelopment analysis (DEA) technique is a non-parametric method to evaluate the performance
decision making units (DMUs) that use multiple inputs to produce multiple outputs. In traditional DEA
models, the status of variables from input and/or output viewpoints is certain and the performance is
calculated from optimistic viewpoint. Nevertheless, there are occasions in real applications that a factor
can play both input and output roles. These factors are defined as adaptable measures in this study. Cook
and Zhu [3] proposed a mixed integer linear program where a measure can play either input or output
roles. Afterwards, in DEA literature, papers have been presented that deal with the subject (see[1], [2],
[4]). Moreover, Amirteimoori et al. [2] introduced a DEA approach for evaluating the efficiency of DMUs
when recyclable outputs exist in a production process. They defined recyclable outputs as products that
can be used as inputs again. Actually, recyclable outputs are a special case of adaptable variables. They
evaluate the efficiency from optimistic point of view. It seems, the model overestimates the efficiency
because of considering only optimistic perspective. Wang et al. [5] determined the performance of
DMUs from two different viewpoints, optimistic and pessimistic.

In the current paper, the efficiencies of DMUs with adaptable variables are evaluated from optimistic
and pessimistic point of views. After calculating the efficiencies from two different viewpoints, the overall
efficiency is calculated by using geometric average. Moreover, adaptable variables are split into input
and output variables.

In sum, models of Wang et al. [5] are extended for situations that adaptable factors present.

2 The Proposed Approach

Assume, there are \( n \) DMUs with \( m \) inputs \( x_{ij} \) \((i = 1, \ldots, m)\), \( s \) outputs \( y_{rj} \) \((r = 1, \ldots, s)\) and \( k \) adaptable
variable \( w_{kj} \) \((k = 1, \ldots, K)\). \( d_k \) indicates the portion of the adaptable variable as output and \( 1 - d_k \)
shows the portion of the adaptable measure as input. From optimistic point of view, the efficiency can
be calculated as follows:

\[
\begin{align*}
\text{Max} & \quad e_{\text{best}} = \frac{\sum_{r=1}^{s} u_r y_{r0} + \sum_{k=1}^{K} w_k d_k \tilde{z}_{k0}}{
\sum_{r=1}^{s} v_i x_{i0} + \sum_{k=1}^{K} w_k (1 - d_k) \tilde{z}_{k0}} \\
\text{s.t.} & \quad \sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{K} w_k d_k \tilde{z}_{kj} + \sum_{r=1}^{s} u_r y_{rj} + \sum_{k=1}^{K} w_k (1 - d_k) \tilde{z}_{kj} \leq 1, \\
& \quad v_i, u_r, w_k \geq \varepsilon, \\
& \quad 0 \leq d_k \leq 1.
\end{align*}
\]
From pessimistic perspective, the following model is suggested for evaluating the performance $DMU_o$:

\[
\begin{align*}
\text{Min } & e_{o}^{\text{worst}} = \frac{\sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{K} w_k d_k z_{ko}}{\sum_{r=1}^{s} v_r x_{rio} + \sum_{k=1}^{K} w_k (1 - d_k) z_{ko}} \\ 
s.t. & \sum_{r=1}^{s} v_r x_{rio} + \sum_{k=1}^{K} w_k (1 - d_k) z_{ko} \geq 1, \\
& 0 \leq d_k \leq 1.
\end{align*}
\] (2.2)

By using Charnes and Cooper transformation and the change of variables $w_k d_k = \lambda_k$, the aforementioned models can be transformed to the following linear programming problems:

\[
\begin{align*}
\text{Max } & e_{o}^{\text{best}} = \sum_{r=1}^{s} \bar{u}_r y_{ro} + \sum_{k=1}^{K} \bar{\lambda}_k z_{ko} \\
\text{s.t. } & \sum_{i=1}^{m} \bar{v}_i x_{rio} + \sum_{k=1}^{K} \bar{w}_k z_{ko} = \sum_{k=1}^{K} \bar{\lambda}_k z_{ko} = 1 \\
& \sum_{r=1}^{s} \bar{u}_r y_{ro} + \sum_{k=1}^{K} \bar{\lambda}_k z_{ko} = 0, \\
& \bar{v}_i, \bar{u}_r, \bar{w}_k \geq \varepsilon, \forall i, r, k, \\
& 0 \leq \bar{\lambda}_k \leq \bar{w}_k, \ k = 1, ..., K.
\end{align*}
\] (2.3)

\[
\begin{align*}
\text{Min } & e_{o}^{\text{worst}} = \sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{K} \bar{\lambda}_k z_{ko} \\
\text{s.t. } & \sum_{i=1}^{m} v_i x_{rio} + \sum_{k=1}^{K} w_k z_{ko} = \sum_{k=1}^{K} \bar{\lambda}_k z_{ko} = 1 \\
& \sum_{r=1}^{s} u_r y_{ro} + \sum_{k=1}^{K} \bar{\lambda}_k z_{ko} = 0, \\
& v_i, u_r, w_k \geq \varepsilon, \forall i, r, k, \\
& 0 \leq \bar{\lambda}_k \leq \bar{w}_k, \ k = 1, ..., K.
\end{align*}
\] (2.4)

The overall efficiency is calculated by using geometric average of efficiencies that are obtained from optimistic and pessimistic perspectives. It means

\[ e_{\text{Overall}} = \sqrt{e_{\text{best}} e_{\text{worst}}}. \] (2.5)

Moreover, $d_{j}^{\text{Overall}}$ and $\tilde{d}_{j}^{\text{Overall}}$ are defined as

\[ d_{j}^{\text{Overall}} = \frac{(d_{j}^{\text{best}} + d_{j}^{\text{worst}})}{2}. \] (2.6)

and

\[ \tilde{d}_{j}^{\text{Overall}} = \frac{(1 - d_{j}^{\text{best}}) + (1 - d_{j}^{\text{worst}})}{2}. \] (2.7)

$d_{j}^{\text{Overall}}$ indicates the overall portion of adaptable measures as output while $\tilde{d}_{j}^{\text{Overall}}$ determines it as input.

**Example 2.1.** Consider 10 DMUs with two inputs, one output and one adaptable variable. Data can be seen in Table 1. Models (2.1) and (2.2) are calculated. The results are indicated in Table 2. The forth column of Table 2 shows the geometric average of optimistic and pessimistic efficiencies. Portions of adaptable measures are given in Table 3. The overall portion of $d_{j}$ that is obtained by arithmetic average of $d_{j}^{\text{best}}$ and $d_{j}^{\text{worst}}$ can be found in the second column of Table 4. Third column of Table 4 shows the overall portion of $1 - d_{j}$.
Table 1: Data of an Example.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Input 1</th>
<th>Input 2</th>
<th>Output 1</th>
<th>Adaptable variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>35</td>
<td>60</td>
<td>35</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td>85</td>
<td>50</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>50</td>
<td>140</td>
<td>95</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>65</td>
<td>170</td>
<td>180</td>
</tr>
<tr>
<td>5</td>
<td>45</td>
<td>95</td>
<td>50</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>100</td>
<td>140</td>
<td>65</td>
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<tr>
<td>7</td>
<td>15</td>
<td>60</td>
<td>55</td>
<td>25</td>
</tr>
<tr>
<td>8</td>
<td>20</td>
<td>70</td>
<td>60</td>
<td>50</td>
</tr>
<tr>
<td>9</td>
<td>55</td>
<td>140</td>
<td>90</td>
<td>65</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>180</td>
<td>50</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 2: Efficiencies.

<table>
<thead>
<tr>
<th>DMU</th>
<th>Optimistic</th>
<th>Pessimistic</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1.000000</td>
<td>1.001778</td>
<td>1.000889</td>
</tr>
<tr>
<td>3</td>
<td>1.000000</td>
<td>1.049532</td>
<td>1.024467</td>
</tr>
<tr>
<td>4</td>
<td>1.000000</td>
<td>1.000000</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0.989399</td>
<td>1.010000</td>
<td>0.999646</td>
</tr>
<tr>
<td>6</td>
<td>1.000000</td>
<td>1.042167</td>
<td>1.020866</td>
</tr>
<tr>
<td>7</td>
<td>0.998120</td>
<td>1.013833</td>
<td>1.005946</td>
</tr>
<tr>
<td>8</td>
<td>0.993579</td>
<td>1.006859</td>
<td>1.000197</td>
</tr>
<tr>
<td>9</td>
<td>0.985053</td>
<td>1.013532</td>
<td>0.999191</td>
</tr>
<tr>
<td>10</td>
<td>0.979399</td>
<td>1.000000</td>
<td>0.989646</td>
</tr>
</tbody>
</table>

Table 3: Portions of Adaptable Variable.

<table>
<thead>
<tr>
<th>DMU</th>
<th>(d^\text{best}_j)</th>
<th>(1 - d^\text{best}_j)</th>
<th>(d^\text{worst}_j)</th>
<th>(1 - d^\text{worst}_j)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
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<td>1</td>
<td>0.5025</td>
<td>0.4975</td>
</tr>
<tr>
<td>3</td>
<td>0.4954</td>
<td>0.5046</td>
<td>0.4767</td>
<td>0.5233</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>0.5</td>
<td>0.4544</td>
<td>0.5456</td>
</tr>
<tr>
<td>5</td>
<td>0.4986</td>
<td>0.5014</td>
<td>0.4928</td>
<td>0.5072</td>
</tr>
<tr>
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<td>0.5031</td>
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<td>0.4897</td>
</tr>
<tr>
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<td>0.5061</td>
<td>0.5039</td>
<td>0.4961</td>
</tr>
<tr>
<td>8</td>
<td>0.4991</td>
<td>0.5009</td>
<td>0.4881</td>
<td>0.5119</td>
</tr>
<tr>
<td>9</td>
<td>0.4990</td>
<td>0.5010</td>
<td>0.4843</td>
<td>0.5157</td>
</tr>
<tr>
<td>10</td>
<td>0.4986</td>
<td>0.5014</td>
<td>0.4928</td>
<td>0.5072</td>
</tr>
</tbody>
</table>

References


<table>
<thead>
<tr>
<th>DMU</th>
<th>$d^\text{overall}$</th>
<th>$d^\text{overall}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.2513</td>
<td>0.7487</td>
</tr>
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<td>3</td>
<td>0.4861</td>
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</tr>
<tr>
<td>4</td>
<td>0.4772</td>
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<td>6</td>
<td>0.5036</td>
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</tr>
<tr>
<td>7</td>
<td>0.4989</td>
<td>0.5011</td>
</tr>
<tr>
<td>8</td>
<td>0.4936</td>
<td>0.5064</td>
</tr>
<tr>
<td>9</td>
<td>0.4916</td>
<td>0.5084</td>
</tr>
<tr>
<td>10</td>
<td>0.4957</td>
<td>0.5043</td>
</tr>
</tbody>
</table>


A Complete Ranking of DMUs Based on $\alpha$-Efficiency in DEA with Imprecise Data

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The imprecise inputs and outputs in Data Envelopment Analysis (IDEA) is a topic that has attracted attention of many researchers. Our view of the ambiguity in the data focuses on fuzzy relations. We introduce a fuzzy monotonicity assumption and construct a fuzzy production possibility set (FPPS) with varying degrees of feasibility. Using the tolerance approach a nonsymmetric fuzzy linear programming model and subsequently a parametric DEA model are constructed. By applying this model, it will be seen that, for a specific and small tolerance of constraints, the efficiency scores are made more various. Hence, we have a complete ranking of DMUs only based on the $\alpha$-efficiency scores.

Keywords Imprecise data envelopment analysis; Fuzzy relation; Production possibility set; Ranking

1 Introduction

Data Envelopment Analysis (DEA) is a mathematical programming technique which is used to compute relative efficiency, rank, return to scale, benchmarks and other applications of Decision Making Units (DMUs). The conventional DEA methods require accurate measurement of both the inputs and outputs. But in real-world problems the observed data are sometimes imprecise. However, for the first time, imprecise data envelopment analysis (IDEA) was introduced by Cooper and et al.\cite{1} and various fuzzy methods were prepared for dealing with it. Most of the Researches often expressed this problem with bounded intervals, fuzzy numbers and Statistical data, but our view of the ambiguity in the data focuses on fuzzy relations that have paid less attention to. In DEA, the production possibility set (PPS) is formed from observed input-output data based on some assumptions: Convexity, Monotonicity, Inclusion of observations, Constant returns to scale and Minimum extrapolation. We believe that the Monotonicity assumption should be replaced by a fuzzy Monotonicity assumption. The basic reason for this idea is to envelop those points which are placed in the vicinity and outside the PPS, because of the impreciseness or vagueness of input-output data. We incorporate the fuzzy Monotonicity assumption by fuzzy relations in PPS. Also we use the tolerance approach which was proposed by Sengupta \cite{2} for the first time. At first, we obtain a non symmetric fuzzy LP model and then, a parametric LP model is constructed by applying the Verdegay’s approach \cite{4}. The proposed method and Sengupta-method both incorporate uncertainty into the DEA models by defining tolerance levels on constraint violations. Sengupta introduced fuzziness in the objective function and the constraints of the conventional DEA model but did not provide an application roadmap of his proposed framework Triantis \cite{3}.

2 Definitions

A production plan is a specific combination of inputs and outputs such as $(\bar{x}, \bar{y})$ where $\bar{y}$ can be produced by consuming $\bar{x}$ which may be possible or impossible. The set of all possible production plane called production possibility set (PPS), here it is denoted by T. Suppose that there are $n$ DMUs where each DMU consumes different amount of $m$ inputs to produce different amount of $s$ outputs. Let $x_j \in \mathbb{R}^m$ and $y_j \in \mathbb{R}^s$ show the input and output vectors corresponding to DMU $j$, respectively. A virtual DMU is given by $(x(\lambda), y(\lambda))$ where:
x(\lambda) = \sum_{j=1}^{n} \lambda_j x_j, \quad Y(\lambda) = \sum_{j=1}^{n} \lambda_j y_j, \quad \lambda \in S$

S is a technology set. The PPS with the CRS assumption for \((x_j, y_j)_{j=1}^{n}\) is:

\[ T = \{(x, y) \mid x \geq x(\lambda), y \leq y(\lambda), \lambda \in S = \lambda \in \mathbb{R}^n : \lambda_j \geq 0, \quad \forall j\} \]

The CCR model for evaluating \((x_o, y_o)\) is as follows:

\[ \theta^o = \min \theta \quad \text{S.t.} \quad (\theta x_o, y_o) \in T \]

**Definition 2.1.** \((\alpha\text{-cut})\). Let \(A\) be a fuzzy set in \(X\) and \(\alpha \in [0, 1]\). The \(\alpha\text{-cut of the fuzzy set } A \) is the crisp set \(A_{\alpha}\) given by \(A_{\alpha} = \{x \in X : \mu_A(x) \geq \alpha\}\).

**Definition 2.2.** \((\text{Binary fuzzy relation})\). A binary fuzzy relation \(R(X, Y)\) on \(X \times Y\) is defined as \(R(X, Y) = ((x, y), \mu_R(x, y)) : (x, y) \in X \times Y\) where \(\mu_R : X \times Y \to [0, 1]\) is a grade of membership function. If \(X = Y\) then \(R(X, X)\) is called a binary fuzzy relation on \(X\).

### 3 Basics idea and the proposed model

In general, a production plan is possible if it dominates a virtual DMU based on a preference order i.e. \((X, Y)\) is possible whenever provided the following constraints hold for at least one \(\lambda\) in \(S\):

\[ X \geq X(\lambda) \quad \text{and} \quad Y \leq Y(\lambda) \quad (3.1) \]

So, the possibility of \((x, y)\) depends to the observed input-output data indirectly. The observed values of the input and output data in real-world problems are sometimes imprecise. Therefore, it may not be reasonable to require that the constraints define in terms of a crisp preference order. For dealing this issue, Sengupta proposed to express each constraint \(i\) of by a fuzzy constraint.

**Definition 3.1.** \((\text{Fuzzy Monotonicity})\). If \((x, y) \in T\), \(x \geq x\) and \(y \geq y\), then \((x, y) \in T\). Where \(\geq\) is a fuzzy relation and called “fuzzy greater than or equal to”.

**Definition 3.2.** \((\text{Fuzzy greater than or equal to})\). Let \(Y \subseteq \mathbb{R}^+\) be a set and \(y_1, y_2 \in Y\). The fuzzy relation \(\geq\) on \(Y\) is defined with membership function \(\mu_\geq\) as follows where \(\beta\) is the maximum acceptable tolerance, as determined by the decision maker[5].

\[ \mu_\geq(z, y) = \begin{cases} 1, & z \geq y; \\ 1 + \frac{z - y}{\beta}, & y - \beta \leq z \leq y; \\ 0, & z \leq y - \beta. \end{cases} \quad (3.2) \]

**Proposition 3.3.** For \(\alpha \in [0, 1]\), \(\mu_\geq(z, y) \geq \alpha\) if and only if \(z \geq y - (1 - \alpha)\beta\).

The proof of the proposition is easy and so is omitted.

**Definition 3.4.** \((\text{Fuzzy Pareto order})\). Let \(X = (x_1, x_2, ..., x_n), Y = (y_1, y_2, ..., y_n) \in \mathbb{R}^{n+}\). In Fuzzy Parato preference, \(X \geq Y\) if and only if \(x_i \geq y_i\) for \(i = 1, 2, ..., n\). If \(\mu_{\geq}\) denotes the membership function of \(x_i \geq y_i\) for \(i = 1, 2, ..., n\), then \(\mu(X, Y) = \min_i \mu_{\geq}(x_i, y_i)\).

Based on this definition, we proposed the fuzzy parato on the PPS input-output system:

\[ (x, y) \geq (w, z) \iff w \geq x, y \geq z \quad (x, y), (w, z) \in T. \]

Considering fuzzy Monotonicity with the technology set S, the PPS for \((X_j, Y_j)_{j=1}^{n}\) is:

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\( \mathcal{P} = \{(x, y) | x \succeq x(\lambda), y \preceq y(\lambda), \lambda \in \Sigma \} \)

Let \( I, O \) denote the indices sets of inputs and outputs, respectively.

**Definition 3.5.** (The degree of feasibility). We define the degree of feasibility for any \((x, y) \in T\) as follows:

\[
\mu(x, y) = \sup_{\lambda \in \Sigma} \min \{ \min_{i} \mu_{i}(x_{i}(\lambda)), \min_{o} \mu_{r}(Y_{r}(\lambda), y_{r}) \}
\]

\((x, y) \in \mathcal{P}\) with the grade of membership \( \mu(x, y) \).

Obviously, \((x, y) \in \mathcal{P}\) when the minimum value, in the above expression, will be positive at least for a \( \lambda \in \Sigma \).

We define \( \mathcal{P}^{\alpha} \) for \( \alpha \in [0, 1] \) as follows:

\[
\mathcal{P}^{\alpha} = \{(X, Y) | \mu_{i}(x_{i}, X_{i}(\lambda)) \geq \alpha, i \in I, \mu_{r}(Y_{r}, y_{r}) \geq \alpha, r \in O, \text{ for all } \lambda \in \Sigma \} \quad (3.3)
\]

**Lemma 3.6.** For each \( \alpha \in [0, 1] \), \( \mathcal{P}^{\alpha} \subseteq \mathcal{P}_{\alpha} \), where \( \mathcal{P}_{\alpha} \) is an \( \alpha \)-cut of \( \tilde{P} \).

**Proof.** It is a direct result of the definition 6.

**Lemma 3.7.** Let \( \alpha \in [0, 1] \). If \( \Sigma \) is compact, then \( \mathcal{P}^{\alpha} = \mathcal{P}_{\alpha} \).

**Proof.** Assume that \((X, Y) \in \mathcal{P}_{\alpha}\). Hence, \( \sup_{\lambda \in \Sigma} D(\lambda) \geq \alpha \). We have from mathematical analysis that \( D(\lambda) \) is continues. In addition, \( \Sigma \) is compact, then there is a \( \lambda^{*} \in \Sigma \) such that \( D(\lambda^{*}) \geq \alpha \). Regarding (3.3), \((X, Y) \in \mathcal{P}^{\alpha}\). This together Lemma complete the proof.

By selecting \( \mathcal{P}^{\alpha}, \alpha \in [0, 1] \), as the PPS, the CCR model will be transformed to the following model that we call it \( \alpha \)-CCR model:

\[
\theta^{\alpha}_{(\theta, q)}(\alpha) = \min_{\lambda \in \Sigma} \theta \quad \text{S.t} \quad (\theta x_{\alpha}, y_{\alpha}) \in \mathcal{P}^{\alpha}
\]

**References**


A NOVEL FEEDBACK NEURAL NETWORK FOR SOLVING DEA PROBLEMS

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Data Envelopment Analysis (DEA) is one of the non-parametric approaches for evaluating efficiency. This paper presents a feedback neural network model for solving DEA models. By applying a suitable Lyapunov function, it is shown that the proposed neural network is Lyapunov stable and convergent to an exact optimal solution of DEA models. A numerical example is provided to show the applicability of the proposed method.

Keywords Data envelopment analysis; Neural network; Linear programming; Lagrangian function.

1 Introduction

In recent decades Data Envelopment Analysis, which was first proposed by Charnes et al. [1], has gained interests of many researchers for assessing relative efficiency and productivity of multiple inputs and outputs decision making units (DMUs). Since the computing time needed to solve a DEA problem greatly depends on its dimension and structure, traditional algorithms cannot satisfy real-time requirement and may not be efficient. Unlike traditional algorithms, artificial neural networks have massively paralleled distributed computation, fast convergence and robust solution. Therefore, artificial neural networks can be considered as a promising approach to solve the large-scale DEA problem in real time [4].

Neural networks for solving mathematical programming problems were first introduced in the 1980s by Hopfield and Tank [2]. The essence of neural network approach for optimization is to construct a nonnegative energy function and establish a dynamic system which is a representation of an artificial neural network. The dynamic system is usually in the form of first order ordinary differential equations. The main feature of these neural networks is that its equilibrium point coincides with the solution of the underlying optimization problem.

Motivated by the above discussions, in this paper, we focus on neural network approach to the DEA problem. Our neural network as a solution tools will be aimed to measure the efficiency of large datasets. The rest of this paper is organized as follows. In Section 2, the DEA problem is first stated and some needed preliminaries are given. Then a feedback neural network is constructed to solve the DEA problem. In Section 3, stability criteria and global convergence are analyzed. In Section 4, A numerical DEA example is discussed to demonstrate the effectiveness of the proposed neural network. Finally, Section 5 concludes this paper.

2 Neural network DEA model

In this section, the DEA-CCR model is first stated and then preliminary information is introduced to facilitate later discussions. Also, we present a feedback neural network based on the Karush-Kuhn-Tucker (KKT) optimality conditions for solving DEA problem.

2.1 Problem formulation and preliminaries

Suppose that for a set of observed DMUs, \{DMU_\ j : j = 1, \ldots, n\}, associated with inputs vector \( X_j = (x_{i1}, x_{i2}, \ldots, x_{im}) \) and outputs vector \( Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj}) \). The DMU under evaluation is represented
by DMU. Charnes et al. [1] introduced the DEA-CCR model as a linear programming problem that can be defined as follows:

\[
\begin{align*}
\text{minimize} & \quad \xi^o = \sum_{i=1}^{m} v_i x_{io} \\
\text{subject to} & \quad \sum_{r=1}^{s} u_r y_{ro} = 1, \\
& \quad \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \geq 0, \quad j = 1, 2, \ldots, n, \\
& \quad v_i \geq 0, \quad i = 1, 2, \ldots, m, \quad \forall i, \\
& \quad u_r \geq 0, \quad r = 1, 2, \ldots, s, \quad \forall r.
\end{align*}
\]

(2.1)

where

- \(\xi^o\) the technical efficiency score for DMU \(o\),
- \(x_{ij}\) the amount of the \(i\)th input at DMU \(j\),
- \(y_{rj}\) the amount of the \(r\)th output from DMU \(j\),
- \(v_i\) the weight assigned to the \(i\)th input,
- \(u_r\) the weight assigned to the \(r\)th output.

Let

\[
\begin{align*}
f(w) &= \sum_{i=1}^{m} v_i x_{io}, \\
g_j(w) &= \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij}, \quad j = 1, 2, \ldots, n, \\
h(w) &= \sum_{r=1}^{s} u_r y_{ro} - 1,
\end{align*}
\]

where \(w = (v, u) = (v_1, v_2, \ldots, v_m)\) and \(u = (u_1, u_2, \ldots, u_s)\). Consider the Lagrangian function of (2.1) similar to [5] as follows:

\[
L(w, v, \mu) = f(w) + \frac{1}{2} \sum_{j=1}^{n} v_j^2 g_j(w) + \mu h(w)
\]

(2.2)

where \(v \in \mathbb{R}^n\) and \(\mu \in \mathbb{R}\) are Lagrangian multipliers. It is well known that \(w^* \in \mathbb{R}^{m+s}\) is an optimal solution of (2.1) if and only if there exist \(v^* \in \mathbb{R}^n\) and \(\mu^* \in \mathbb{R}\) such that \((w^*^T, v^*^T, \mu^*)^T\) satisfies the following KKT conditions

\[
\begin{align*}
& v^* \geq 0, \quad g(w^*) \leq 0, \quad v^*^T g(w^*) = 0, \\
& \nabla f(w^*) + \nabla g(w^*)^T v^* + \nabla h(w^*)^T \mu^* = 0, \\
& h(w^*) = 0,
\end{align*}
\]

(2.3)

where \(g(w) = (g_1(w), \ldots, g_n(w))^T\). \(w^*\) is called a KKT point of (2.1).

**Lemma 2.1.** \(w^*\) is an optimal solution of (2.1) if and only if \(w^*\) is a KKT point of (2.1).

### 2.2 Feedback neural network model

Now we propose a recurrent neural network for solving (2.1) by the following dynamical system

\[
\begin{align*}
\frac{dw}{dt} &= -\nabla_w L(w, v, \mu) \\
&= - \left( \frac{\partial f}{\partial w_k} + \frac{1}{2} \sum_{j=1}^{n} v_j^2 \frac{\partial g_j}{\partial w_k} + \mu \frac{\partial h}{\partial w_k} \right), \quad k = 1, \ldots, m + s, \\
\frac{dv}{dt} &= \nabla_v L(w, v, \mu) = \text{diag}(v_1, \ldots, v_n) g(w), \\
\frac{d\mu}{dt} &= \nabla_{\mu} L(w, v, \mu) = h(w).
\end{align*}
\]

(2.4) (2.5) (2.6)
with an initial point \((w_0^T, v_0^T, \mu_0^T)^T\) and \(v_j(t_0) \neq 0 (j = 1, ..., n)\).

### 3 Stability and convergence analysis

In this section, we study stability properties for the neural network whose dynamics is described by the differential equations (2.4)-(2.6).

**Theorem 3.1.** \((w^*, v^*, \mu^*)^T\) is the equilibrium point of (2.4)-(2.6) if and only if \(w^*\) is a KKT point of the problem (2.1).

Consider the following Lyapunov function (energy function):

\[
E(w) = \|\Phi(q)\|^2 + \frac{1}{2}\|q - q^*\|^2,
\]

where \(q = (w^T, v^T, \mu)^T\) and

\[
\Phi(q) = \begin{pmatrix}
-\left(\nabla f(w) + \frac{1}{2}\nabla g(w)^T v^2 + \nabla h(w)^T \mu\right) \\
\text{diag}(v_1, ..., v_n) g(w) \\
h(w)
\end{pmatrix}.
\]

Using the above Lyapunov function, we have the following theorem.

**Theorem 3.2** ([3]). If \(\nabla^2 f(w)\) is positive definite and \(\nabla^2 g_j(w) (j = 1, ..., n)\) is positive semi-definite or if \(\nabla^2 f(w)\) is positive semi-definite and \(\nabla^2 g_j(w) (j = 1, ..., n)\) is positive definite, then the proposed neural network in (2.4)-(2.6) is globally stable in the sense of Lyapunov sense and is globally convergent to one KKT point \(q^* = (w^*, v^*, \mu^*)^T\), where \(w^*\) is the optimal solution of the DEA problem (2.1).

### 4 A numerical example

In this section an example is provided to illustrate the effectiveness and efficiency of the proposed neural network. The simulation is carried out in Matlab R2012a and the ordinary differential equation solver engaged is ode45.

Table 1 shows information of six DMUs. Here each of six DMUs produces a single output in the amount \(y\) by using two inputs in the amounts \(x_1\) and \(x_2\). For instance, the CCR original formulation for \(DMU_A\) can be written as

\[
\begin{align*}
\text{minimize} \quad & 1w + 3v2 \\
\text{subject to} \quad & u = 1, \\
& 4v1 + 3v2 - u \geq 0, \\
& 7v1 + 3v2 - u \geq 0, \\
& 8v1 + v2 - u \geq 0, \\
& 4v1 + 2v2 - u \geq 0, \\
& 2v1 + 4v2 - u \geq 0, \\
& 10v1 + v2 - u \geq 0, \\
& v1, v2, u \geq 0.
\end{align*}
\]

We now apply the proposed neural network in (2.4)-(2.6) to solve this problem. Table 2 summarizes the comparison results for Example. From Table 2, w2bb observe the optimal DEA solution obtained from the both methods and shows that the same solution was for the two methods.
Table 2: Input and output data of example.

<table>
<thead>
<tr>
<th>Method</th>
<th>LP-based CCR solution</th>
<th>FNN(^\dagger) solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(v_1) (v_2) (u) (\xi)</td>
<td>(w_1) (w_2) (w_3) (\xi)</td>
</tr>
<tr>
<td>DMU(_A)</td>
<td>0.14 0.14 0.86 0.86</td>
<td>0.14 0.14 0.86 0.86</td>
</tr>
<tr>
<td>DMU(_B)</td>
<td>0.05 0.21 0.63 0.63</td>
<td>0.05 0.21 0.63 0.63</td>
</tr>
<tr>
<td>DMU(_C)</td>
<td>0.08 0.33 1 1</td>
<td>0.08 0.33 1 1</td>
</tr>
<tr>
<td>DMU(_D)</td>
<td>0.17 0.17 1 1</td>
<td>0.17 0.17 1 1</td>
</tr>
<tr>
<td>DMU(_E)</td>
<td>0.21 0.14 1 1</td>
<td>0.21 0.14 1 1</td>
</tr>
<tr>
<td>DMU(_F)</td>
<td>0 1 1 1</td>
<td>0 1 1 1</td>
</tr>
</tbody>
</table>

\(^\dagger\) Feedback Neural Network

5 Conclusions

Based on the KKT optimality conditions, this paper presented a novel neural network to find the optimal relative efficiency among multiple DMUs of a DEA problem. Based on Lyapunov stability, we have studied stability and convergence of the neural network. The principal advantages of proposed neural network are its robustness in hardware implementation and its simple structure. Simulation results on a numerical example have shown that the proposed neural network is reliable and very efficient.

References

The DEA based on an analysis of fraction (DEA-R) we are looking for DMU (Decision Making Units) efficiency, also considering the prices of inputs (costs), efficiency of costs are calculated. Estimates of production function in the DAE-R, have some weak points. The neural networks in this regard, with repetition, training and use of multilayer networks are useful. In this article, using DEA-R models and neural networks, scale efficiency and performance cost are calculated on the Stock Exchange 250.

**Keywords** DEA-R; Neural network.

1 Introduction

The DEA based on an analysis of fraction (DEA-R) was first introduced by Despic et al. Then compare the performance of arithmetic, geometry performance with the performance of the agreement, DEA models studied. Wei et al using DEA-R super performance models looked DEA models in another way. First there is false inefficiency in DEA, secondly the DEA because of positive weights, weights control coms, third in case relative data there, DEA-R models are amenable only) As the proportion of assets to liabilities). In recent years, neural networks are used to estimate the production function. (See Olanrewaju et al.) As with the use of multi-layer network and training network and methods such as the gradient can be used to estimate the scale of efficiency and cost effectiveness. In the second part of this paper introduces the basic concepts of DEA-R and cost efficiency. In the third section, the proposed algorithm is proposed to estimate the production function. Finally conclusions are.

2 Preliminary concepts DEA-R

In this section, by taking n decision-making units (DMUs) with m inputs $X_j = (x_{1j}, x_{2j}, ..., x_{mj})$ and s outputs $Y_j = (y_{1j}, ..., y_{sj})$ as the inputs and outputs are positive. DEA-R model for inputs-oriented with constant returns to scale as follows:

$$\max \gamma_R$$

s.t. $\sum_{j=1}^{n} \lambda_j (x_{ij}/y_{rj}) \leq (x_{io}/y_{ro}) \gamma_R i = 1, ..., m, r = 1, ..., s,$

$\sum_{j=1}^{n} \lambda_j = 1$

$\lambda_j \geq 0 \quad j = 1, ..., n.$

Model (1.1) is a linear programming problem with $(n + 1)$ variables and $(ms + 1)$ constraint used for evaluating $DMU_0$ with $x_0$ inputs and $y_0$ outputs. Now consider the cost vector $C = (c_1, c_2, ..., c_m)$ for inputs, cost efficiency model in DEA-R is proposed as follows:
\[
\begin{align*}
\min & \sum_{i=1}^{m} c_i x_i \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j (x_{ij}/y_r) \leq (x_{i}/y_r) i = 1, \ldots, m, r = 1, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \geq 0 j = 1, \ldots, n.
\end{align*}
\] 

(2.2)

Model (1.2) is a linear programming problem is to keep the output \(y_0\) and the value of \(DMU_0\) considering the proportion of inputs (inputs costs) obtained from model (1.2) also the above model is a linear programming problem with \((m+n)\) variables and \((ms+1)\) constraint.

### 3 Production function estimation using neural networks and DEA

In this section we consider the matrix inputs and outputs as \(A = [X/Y]\) including \(ms\) row and \(n\) column. The purpose of using neural networks to estimate a production function. Using a multi-layer perceptron to predict the performance of decision making units (DMU). Network function is used to estimate the performance measure is. The network, which uses the normal data, for learning, for 250 companies in the Stock Exchange and learn pattern are SCG and LM. Network output is a number between 0 and 1, model (1.1). Also, using model (1.2), the input vector is. Generally learning in neural networks, synaptic weights are modified so that the relationship between input and output neurons with a specific purpose conform. The gradient method is used to train the neural network.

The proposed algorithm is as follows:

Step 1. Model (1.1) and efficiency all the DMU.

Step 2. Training of relevant data, the matrix A and scale efficiency in network.

Step 3. Efficiency units using DEA and Neural Networks. If the result obtained in Step 3, go to Step 4. Otherwise, go to Step 3.

Step 4. The results with respect to the methods of teaching and learning model analysis. If we use the algorithm of model (1.2), the performance cost is calculated.

### Conclusion

In this paper, using models of DEA-R, in addition to scale efficiency, cost efficiency by taking data from 250 companies on the Stock Exchange, and also uses a neural network to estimate the production function. Generally for future work, super efficiency, return to scale, using DEA-R models and neural networks is proposed.

### References


The traditional data envelopment analysis (DEA) models for the observations containing ratio data as input and/or output may result in incorrect efficiency scores. To overcome this shortcoming, a set of modified DEA models has been presented in the literature taking into account the correct convexity of decision making units (DMUs) when a ratio variable is included in the assessment model. The objective of the present paper is to propose a novel set of generalized DEA models for measurement of relative efficiencies of DMUs in the presence of ratio data and non-discretionary factors. The generalized models integrate the previous approaches for dealing with non-discretionary factors.
technology is shown in (2.1), where $D$ is the set of discretionary inputs, $ND$ is the set of ND inputs, and $j_o$ is the unit under assessment.

\[
\begin{align*}
\min & \quad \theta \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} - \theta x_{ij_o} \leq 0, \quad i \in D \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij} - x_{ij_o} \leq 0, \quad i \in ND \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rj_o}, \quad r = 1, 2, \ldots, s \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \geq 0, \quad j = 1, 2, \ldots, n
\end{align*}
\]

(2.1)

This model differs from the traditional VRS model of Banker et al. [1] in that the contraction factor $\theta$ is associated only with discretionary inputs.

3 ND DEA model with ratio data

In this section, the BM input-oriented model (2.1) is modified for situations that at least one of the output data is in the form of ratio.

Assume that for model (2.1), the $k$–th output of outputs is in the form of ratio, where $1 \leq k \leq q$. Suppose that $y_{kj}$ for unit $j$ contain numerator and denominator of $n_{kj}$ and $d_{kj}$ respectively, i.e. $y_{kj} = \frac{n_{kj}}{d_{kj}}, j = 1, 2, \ldots, n$.

In the case of numerator and denominators of the output-ratio variables as presented separated output and input variables, the BM-VRS input-oriented model (2.1) in the presence of this output-ratio should be rewritten as follows:

\[
\begin{align*}
\min & \quad \theta \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} - \theta x_{ij_o} \leq 0, \quad i \in D \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij} - x_{ij_o} \leq 0, \quad i \in ND \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rj_o}, \quad r = 1, 2, \ldots, s, r \neq k \\
& \quad \sum_{j=1}^{n} \lambda_j n_{kj} \geq n_{kj_o}, \quad r = k \\
& \quad \sum_{j=1}^{n} \lambda_j d_{kj} - \theta d_{kj_o} \leq 0, \quad r = k \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \geq 0, \quad j = 1, 2, \ldots, n
\end{align*}
\]

(3.1)

It can be seen that the generalized model (3.1) evaluates the efficiency score of the DUM under evaluation against $n$ DMUs each of them contains $m + 1$ inputs and $s$ outputs. Therefore, the main drawback of
the generalized model (3.1) is that if there are several output-ratios then the number of variables will be increased leading to unstable results. To overcome this drawback, the correct convexity for the ratio variables should be defined as ratio of convex combination of numerator to the convex combination of denominator rather than a simple convex combination of ratio variable. In this case, the convexity assumption (when assessing unit $j_o$) should be taken into the model as follows:

\[
\sum_{j=1}^{n} \lambda_j n_{kj} \geq \sum_{j=1}^{n} \lambda_j d_{kj} = y_{kj_o}.
\]

Hence model (2.1) should be reformulated in the following form:

\[
\begin{align*}
\text{min} & \quad \theta \\
\text{s.t.} & \quad \sum_{j=1}^{n} \lambda_j x_{ij} - \theta x_{i,j_o} \leq 0, \quad i \in D \\
& \quad \sum_{j=1}^{n} \lambda_j x_{ij} - x_{i,j_o} \leq 0, \quad i \in ND \\
& \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r,j_o}, \quad r = 1, 2, \cdots, s, s \neq k \\
& \quad \sum_{j=1}^{n} \lambda_j n_{kj} - y_{kj_o} \sum_{j=1}^{n} \lambda_j d_{kj} \geq 0, \quad r = k \\
& \quad \sum_{j=1}^{n} \lambda_j = 1 \\
& \quad \lambda_j \geq 0, \quad j = 1, 2, \cdots, n
\end{align*}
\] (3.2)

4 Numerical example

In this section, a numerical example is presented using the proposed discretionary-ration DEA models to illustrate their applicability and effectiveness.

Consider a case where ten universities have used one discretionary input ($x_1$ = total expenditure) and one non-discretionary input ($x_2$ = area) to produce two outputs, $y_1$ = % degree awarded and $y_2$ = amounts of research income. Assume that assume that % degree awarded is in the form of output-ratio and can be considered as number of degree awarded ($n$) to number of student ($d$). Data set is listed in Table 1. Using DEA models (2.1), (3.1) and (3.2), we obtained the results shown in Table 2.

As it is expected the higher efficiency scores in model (3.1) as compared to model (2.1) is due to the higher number of variables in the DEA model.

References


Table 1: Data set.

<table>
<thead>
<tr>
<th>DMU</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( n )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_1 )</td>
<td>163</td>
<td>49</td>
<td>0.20</td>
<td>25</td>
<td>1200</td>
<td>6000</td>
</tr>
<tr>
<td>( U_2 )</td>
<td>281</td>
<td>176</td>
<td>0.29</td>
<td>16</td>
<td>2500</td>
<td>8500</td>
</tr>
<tr>
<td>( U_3 )</td>
<td>393</td>
<td>277</td>
<td>0.41</td>
<td>41</td>
<td>6000</td>
<td>14500</td>
</tr>
<tr>
<td>( U_4 )</td>
<td>94</td>
<td>267</td>
<td>0.14</td>
<td>65</td>
<td>290</td>
<td>2050</td>
</tr>
<tr>
<td>( U_5 )</td>
<td>127</td>
<td>356</td>
<td>0.15</td>
<td>69</td>
<td>700</td>
<td>4500</td>
</tr>
<tr>
<td>( U_6 )</td>
<td>118</td>
<td>503</td>
<td>0.15</td>
<td>28</td>
<td>600</td>
<td>4000</td>
</tr>
<tr>
<td>( U_7 )</td>
<td>120</td>
<td>6227</td>
<td>0.13</td>
<td>30</td>
<td>550</td>
<td>4200</td>
</tr>
<tr>
<td>( U_8 )</td>
<td>242</td>
<td>660</td>
<td>0.87</td>
<td>55</td>
<td>10500</td>
<td>12000</td>
</tr>
<tr>
<td>( U_9 )</td>
<td>202</td>
<td>880</td>
<td>0.80</td>
<td>48</td>
<td>8500</td>
<td>10500</td>
</tr>
<tr>
<td>( U_{10} )</td>
<td>96</td>
<td>330</td>
<td>0.16</td>
<td>16</td>
<td>340</td>
<td>2100</td>
</tr>
</tbody>
</table>

Table 2: Efficiency comparison in Models (2.1) and (3.2)

<table>
<thead>
<tr>
<th>DMU</th>
<th>Model (2.1)</th>
<th>Model (3.1)</th>
<th>Model (3.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Input: ( x_1, x_2 )</td>
<td>Input: ( x_1, x_2, d )</td>
<td>Input: ( x_1, x_2 )</td>
</tr>
<tr>
<td></td>
<td>Output: ( y_1, y_2 )</td>
<td>Output: ( n, y_2 )</td>
<td>Output: ( y_1, y_2 )</td>
</tr>
<tr>
<td>( U_1 )</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>( U_2 )</td>
<td>57.57</td>
<td>74.53</td>
<td>47.87</td>
</tr>
<tr>
<td>( U_3 )</td>
<td>100.00</td>
<td>100.00</td>
<td>28.61</td>
</tr>
<tr>
<td>( U_4 )</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>( U_5 )</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>( U_6 )</td>
<td>72.70</td>
<td>74.96</td>
<td>71.44</td>
</tr>
<tr>
<td>( U_7 )</td>
<td>70.00</td>
<td>73.11</td>
<td>70.00</td>
</tr>
<tr>
<td>( U_8 )</td>
<td>100.00</td>
<td>100.00</td>
<td>97.48</td>
</tr>
<tr>
<td>( U_9 )</td>
<td>100.00</td>
<td>100.00</td>
<td>88.51</td>
</tr>
<tr>
<td>( U_{10} )</td>
<td>91.22</td>
<td>99.94</td>
<td>88.18</td>
</tr>
</tbody>
</table>


Data Envelopment Analysis evaluates the efficiency of decision making units via a mathematical programming model, while considering the production process of the units as a black box. However, considering the internal structure of the units, network DEA models have been proposed. In this paper, we present a network DEA model for efficiency analysis in a two-stage production process which is based on collaboration between the two sub-units. The proposed model can also be applied for benchmarking.

**Keywords** DEA; efficiency; improved unit.

1 **Preliminaries**

Data Envelopment Analysis (DEA) is a powerful mathematical programming based technique for evaluating Decision Making Units (DMUs) with multiple inputs and multiple outputs [1]. However, in many real applications the production process in each unit is composed of several sub-processes, while the conventional DEA models ignore the internal structure of the DMUs. In this regard, the original DEA models need to be modified in order to do the evaluation more accurately. Such a DEA model which reflects the internal structure of DMUs is called a network DEA model. Unlike the standard DEA models, network DEA models appear with different structures. For a review of network DEA models the reader is referred to [3, 2]. Among all of network DEA studies, most of them have devoted to the two-stage production processes. The reason is that many general multi-stage processes can be suitably formulated by two-stage ones. In this paper, we propose a two-stage DEA model which is based on improving the performance of units via collaboration between the sub-units. The main feature of the proposed model is that it tries to improve the performance of units by considering the possibility of substitution between intermediate products and/or exogenous inputs and outputs.

Consider a two-stage technology where the production process in each unit is composed of two sub-units connected in series, as follows: The main feature of our structure is that the vector $Z$ of intermediate products has some items in common with the first stage’s exogenous output $Y^1$ and/or the second stage’s exogenous input $X^2$.

To formulate this situation, we split the vectors $Z$, $Y^1$ and $X^2$ respectively as $Z = (Z^X, Z^Y)$, $Y^1 = (Y^1, Y^Z)$ and $X^2 = (X^2, X^Z)$ where $Z^X$ and $X^Z$ are of the same kind, and so are $Z^Y$ and $Y^Z$. Also, for homogeneity we may write $X^1 = (X^1, 0)$ and $Y^2 = (Y^2, 0)$ whenever needed.

The main idea of our model is to improve the performance of each unit by taking into consideration the possibility of collaboration between its sub-units. Observe that since $Z^X$ and $X^Z$ are of the same kind, if we increase $Z^X$ as output of the first sub-unit then we have the possibility of decreasing $X^Z$ as
input of the second sub-unit and so improve the performance of the unit. Similarly, we can increase \( Y^Z \) as output of the first sub-unit by decreasing \( Z^Y \) as input of the second sub-unit, as much as possible. These measures of increasing outputs and / or decreasing inputs can be used to diagnose the efficiency status of the evaluated unit and also to define an efficiency measure.

2 Model Development

Assume that we have \( n \) Decision Making Units (DMUs) as \( DMU_j \) for \( j = 1, \ldots, n \) and associated with \( DMU_j \) we have the vectors \( (X^1_j, 0) \) and \( (X^2_j, X^Z_j) \) as inputs, \( (Y^1_j, Y^Z_j) \) and \( (Y^2_j, 0) \) as outputs, and \((Z^X_j, Z^Y_j)\) as intermediate products.

To evaluate \( DMU_o \), for \( o \in \{1, \ldots, n\} \) we need to construct the production possibility set (PPS) for the above-mentioned technology.

**Definition 2.1.** Considering the above notations, the network PPS is defined as:

\[
P^N = \{ (X^1, (Y^1, Y^Z), (Z^X, Z^Y), (X^2, X^Z), Y^2) \mid (X^1, (Y^1, Y^Z), (Z^X, Z^Y)) \in P^1, (X^2, X^Z), (Z^X, Z^Y), Y^2) \in P^2 \}\]

where \( P^1 \) and \( P^2 \) are the production possibility sets for the first and the second stages, respectively, and are defined as:

\[
P^1 = \{ (X^1, (Y^1, Y^Z), (Z^X, Z^Y)) \mid X^1 \geq \sum_{j=1}^{n} \lambda^1_j X^1_j, Y^1 \leq \sum_{j=1}^{n} \lambda^1_j Y^1_j, Y^Z + Z^Y \leq \sum_{j=1}^{n} \lambda^1_j (Y^Z_j + Z^Y_j), Z^X \leq \sum_{j=1}^{n} \lambda^1_j Z^X_j, Y^1 \geq 0, Y^Z \geq 0, Z^X \geq 0, \lambda^1_j \geq 0, j = 1, \ldots, n \}, \tag{2}
\]

and

\[
P^2 = \{ ((X^2, X^Z), (Z^X, Z^Y), Y^2) \mid X^2 \geq \sum_{j=1}^{n} \lambda^2_j X^Z_j, X^Z + Z^X \geq \sum_{j=1}^{n} \lambda^2_j (X^Z_j + Z^X_j), Z^Y \geq \sum_{j=1}^{n} \lambda^2_j Z^Y_j, Y^2 \geq \sum_{j=1}^{n} \lambda^2_j Y^2_j, \lambda^2_j \geq 0, j = 1, \ldots, n \}. \tag{3}
\]

Now, based on the constructed PPS, the evaluation model can be formulated in the next section.

3 Efficiency Evaluation

To evaluate \( DMU_o \) the following slack-based model is proposed:

\[
\begin{align*}
\text{Max} & \quad 1S^X + 1S^Y \\
\text{s.t.} & \quad (X^1_o, (Y^1_o, Y^Z_o + S^Y), (Z^X + S^X, Z^Y - S^Y), (X^2, X^Z - S^X), Y^2) \in P^N.
\end{align*}
\]
Based on the structure of $P^N$, $P^1$ and $P^2$, the above model can be decomposed into two linear programming problems as:

Max $S^X$ s.t. $X^1_o \geq \sum_{j=1}^{n} \lambda^1_j X^1_j$
$Y^1_o \leq \sum_{j=1}^{n} \lambda^1_j Y^1_j$
$Y^Z_o + Z^1_o \leq \sum_{j=1}^{n} \lambda^1_j (Y^Z_j + Z^X_j)$,
$Z^X_o + S^X_o \leq \sum_{j=1}^{n} \lambda^1_j Z^X_j$,
$\lambda^1_j \geq 0, j = 1, \ldots, n.$

(5)

and

Max $S^Y$ s.t. $X^2_o \geq \sum_{j=1}^{n} \lambda^2_j X^2_j$,
$X^Z_o + Z^2_o \geq \sum_{j=1}^{n} \lambda^2_j (X^Z_j + Z^X_j)$,
$Y^2_o \leq \sum_{j=1}^{n} \lambda^2_j Y^2_j$,
$\lambda^2_j \geq 0, j = 1, \ldots, n.$

(6)

Definition 3.1. Assuming that $S^X*$ and $S^Y*$ are respectively the optimal solutions for problems (5) and (6), the network efficiency measure for $DMU_o$ is defined as:

$$\theta_o = \frac{1}{p} \sum_{i=1}^{p} (1 - \frac{S^x_i}{x_{io} + \hat{x}_{io}})\frac{1}{q} \sum_{r=1}^{q} (1 + \frac{S^y_r}{y_{ro} + \hat{y}_{ro}})$$

(7)

where $p$ and $q$ stand for the dimension of $S^X$ and $S^Y$, respectively.

Note that $DMU_o$ is said to be network efficient iff $\theta_o = 1$. Moreover, the proposed model is capable of determining a benchmark unit associated with each inefficient unit. Actually, the (virtual) improved unit $\hat{DMU}_o$ characterized by

\[
\begin{align*}
(X^1_o, (Y^1_o, Y^Z_o + S^Y*),
(Z^X_o + S^X*, Z^Y - S^Y*),
(X^2, X^Z - S^X*), Y^2_o)
\end{align*}
\]

(8)

can be considered as a benchmark unit for $DMU_o$. The following theorem clears this fact.

Theorem 3.2. The improved unit $\hat{DMU}_o$ is network efficient.

4 Conclusion

The proposed model measures the efficiency score of DMUs by taking into account their internal structure. Moreover it can be used as a benchmark tool for two-stage production processes. Also, the improved DMUs can then be evaluated by a conventional DEA model as a black box in order to examine their CCR efficiency status.

References


A genetic algorithm (GA) is introduced for the no-wait flow shop scheduling problem (NWFSP) by Tseng et al. [5]. In their algorithm, Taguchi method is introduced as a crossover operator. In fact, in this paper, we compare the performance of $L_8(2^7)$ that is common and used their algorithm and $L_9(3^4)$ as a crossover approach with six and three cut points. Comparison of the results shows that the new method for crossover operators are effective and it is efficient and competitive method for solving no-wait flow shop problem.

**Keywords** Scheduling; Flow shop; No-wait flow shop; Taguchi orthogonal arrays.

## 1 Introduction

This paper considers the no-wait flow shop scheduling problem with makespan criterion, which has important applications in modern industries that can be found in Hall and Sriskandarayah’s review paper [1]. In NWFSP, each job has to be processed from the first machine to the last without any interruption and the job sequence is unique on all machines. In addition, each machine can handle no more than one job at a time and each job has to visit each machine exactly once. Therefore, the start of a job on the first machine may be delayed in order to meet the no-wait requirement. Given that the release time of all jobs is zero and set-up time on each machine is included in the processing time, the no-wait flow-shop problem is to schedule jobs that minimize the makespan over all jobs. Due to the fact that the no-wait flow-shop scheduling problem is NP-hard, in the operations research literature, many elegant mathematical models and solution methods have been developed to cope with real-world problems. Recently, meta-heuristics with techniques such as genetic algorithm [5], ant-colony algorithm [4], and particle swarm optimization [2] for the problem under consideration have been developed. The Taguchi method, a robust design approach, uses many ideas from statistical experimental design for evaluating and implementing improvements in products, processes, and equipment. The fundamental principle is to improve the quality of a product by minimizing the effect of the causes of variation without eliminating the causes. On the other hand, among the existing optimization algorithms, the genetic algorithm (GA) has received considerable attention regarding its potential as a novel optimization technique for complex problems and has been successfully applied in various areas. In this paper, a new approach of Taguchi method is introduced as a crossover operator to GA. Indeed, we use a three level orthogonal-array-based crossover operators (OA-crossover) to improve the solutions of GA algorithm. As the experimental results, we compare the performance of our and common OA-crossovers on benchmark problems.

## 2 Description of the algorithm

In GA optimization, solutions are encoded into chromosomes to establish a population being evolved through generations. At each generation, parents are selected and mated from the population to carry out the crossover operator leading to new solutions called children. Then, some of the individuals are mutated or perturbed. Finally, they are pooled together to select new individuals for next generation. This
procedure is repeated until the stopping criterion is achieved. However, in the GA, an Orthogonal-Array-based crossover operator (OA-crossover) to enhance the solution quality is employed. The Taguchi method is inserted between crossover and mutation operations. Then, the systematic reasoning ability of the Taguchi method is incorporated in the crossover operations to select the better solutions to achieve crossover, and consequently enhance the GA. In the GA, The initial population is constructed in such a way that the first individual is constructed by NEH heuristic and the rest is randomly established. Each individual is assigned to the crossover strategies separately and the local search that is applied in algorithm is based on insert method (Ins(x,y) picks out the job at position x and inserts it into position y). In the following the genetic algorithm is described. The details of the OA-crossover will be described in thereinafter. The procedure of the Algorithm 1 explains the components of the proposed GA:

Algorithm 1: Proposed Genetic algorithm

Step 1: Set the population size $NP$.

Step 2: Initialize the population:
- i. $\pi_1 = NEH(\pi_1)$
- ii. $\pi = \{\pi_2, \pi_3 \ldots \pi_{NP}\}$ and evaluate each solution in the population.

Step 3: For $i = 1, 2, \ldots, NP$, repeat the following sub-steps.
- i. For the individual $\pi_i$, select one or two mates $\pi_k$ from the population.
- ii. Produce a new offspring $u_i$ by recombining them with the strategy $S_i$.
- iii. Evaluate the new offspring $u_i$ and apply LocalSearch() to $u_i$.
- iv. If $u_i$ is better than best solution so far, update best so far solution $\pi_B$ and apply LocalSearch() for $u_i$.

Step 4: If the termination criterion is reached, return the best solution found so far $\pi_B$; otherwise go to step 3.

3 OA-crossover

Taguchi method belongs to the class of optimization techniques named global optimizers while the more familiar, traditional techniques are classified as local optimizers. The distinction between local and global search of optimization techniques is that the local techniques produce results that are highly dependent on the starting point or initial guess, while the global methods are highly independent of the initial conditions. Taguchi’s method was developed based on the concept of the orthogonal array (OA), which can effectively reduce the number of tests required in a design process. It provides an efficient way to choose the design parameters in an optimization procedure. The crossover operator is used in genetic algorithms to find better solutions by recombining good genes from different parent chromosomes. One cut point or two cut points were usually used in the crossover operator. But in this paper, we use $L_8(2^7)$ and $L_9(3^4)$ as a crossover approach with six and three cut points separately. The concept of the Taguchi method is to perform the minimum number of experiments needed to minimize the causes of variation and improve product quality. The OA-crossover is described in the Algorithm 2:
Algorithm 16 Proposed Genetic algorithm

Step 1: Let $N$ be the number of pieces into which the user wants to cut solutions $P_1$ and $P_2$ for recombination. Generate the orthogonal array $L(N + 1)(2^N)$.

Step 2: Choose two solutions $P_1$ and $P_2$. Randomly choose $N - 1$ cut points to cut $P_1$ and $P_2$ into $N$ subsequences.

Step 3: Consult the $i$th row of the $L(N + 1)(2^N)$ and generate a sampled new solution $C_i$, for $i = 1, 2, \ldots, N + 1$. The $j$th subsequence of $C_i$ is taken from the $j$th subsequence of $P_1$ if the level of the $j$th factor in row $i$ is 0. Otherwise, the $j$th subsequence of $C_i$ is taken from the $j$th subsequence of $P_2$. Repair $C_i$ whenever it is necessary. (Repair scheme will be described later.)

Step 4: Calculate the evaluation value $E_i$ of each sampled new solution $C_i$, for $i = 1, 2, \ldots, N + 1$. The evaluation value $E_i$ is the fitness value (i.e. the reciprocal of the makespan) of the solution $C_i$.

Step 5: Calculate the main effect $F_{jk}$ of factor $j$ with level $k$, for $j = 1, 2, \ldots, N$ and $k = 0, 1$.

Step 6: Find the best level for each factor. The best level of factor $j$ is $k$ if $F_{jk} = \max \{ F_j0, F_j1 \}$. Use the best levels of all factors to generate another solution $C_1(N + 2)$ (Taguchi method). Repair $C_1(N + 2)$ whenever it is necessary. Calculate the evaluation value $E(N + 2)$.

Step 7: From $C_1, C_2, \ldots, C_1(N + 2)$, choose the one with the best evaluation value to be the new solution.

4 Computational result

The genetic algorithm is implemented in C language. Tests are run on a dual core 3.6 GH, CPU with 4 GB memory. The performance of the proposed algorithms is evaluated with PRD as a parameter that show average of solutions after 5 runs and AvgTime denotes the average CPU time (in second) of 5 runs (R=5). We conducted experiments on 18 problems (rec01, rec03… rec35) provided by reeves [3] that are shown in Table 3. GA(1) and GA(2) refer respectively to GA algorithm with Regarding the results of Table 3, the searching quality of our GA(2) is very superior to that of GA(1) with respect of quality of solutions and CPU time. Due to the fact that $L_8(2^7)$ is common for using as a crossover operator in different algorithms especially in GA, we apply $L_9(3^4)$ and compare their results in GA algorithm. Table 3 shows that GA(2) is really competitive and effective and its solutions are reliable.

5 Conclusion

In this paper, we coded a genetic algorithm [5] for the no-wait flowshop scheduling with respect to the makespan criterion. The algorithm starts with a population of individuals created by a heuristic based on NEH heuristic. The used crossover operator, called OA-crossover, consists $L_8(2^7)$ that is common. But in this paper, we proposed new approach of OA-crossover named $L_9(3^4)$. Then, we copied them into
offspring by keeping the same positions. In order to prevent the algorithm from being stuck into local optima, an improvement procedure by using an insert method is applied. The results show that applying $L_9(3^4)$ as crossover approach remains efficient both in terms of quality of solution and time requirement.

### References


CENTRAL RESOURCE ALLOCATION IN DEA_R MODELS

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In this paper the central resource allocation models based on DEA fraction analysis are suggested. DEA_R resource allocation models are proposed in radial and non-radial modes and results of DEA models are compared. In case of multiple inputs and one output and vice versa the solutions of DEA_R are equal with DEA models. And in case of multiple inputs, multiple outputs in the input oriented DEA_R are efficiently greater than or equal to similar models in DEA.

Keywords DEA, DEA_R.

1 Introduction

Data envelopment analysis for evaluating DMUs was first presented by Farrell for multiple inputs and one output. Then, Charnes, Cooper and Rhodes in an article entitled as CCR measuring the efficiency of decision making units with multiple inputs generalized multiple outputs. DEA models were studied by Banker (1984) in open state on a qualitative scale and the BCC model was proposed. Generally, to evaluate N decision making units a single, N linear programming problems are utilized to find the efficiency and the appropriate model. In the year (2004) Lozano proposed the central resource allocation in DEA. In the proposed models an appropriate model can be achieved through a linear programming problem. In the year (2007), Despic et al. suggested DEA_R model based on the fractional analysis. Then, Wei et al. developed the idea. In this regard, Mozaffari et al. addressed the relationship between DEA, DEA_R. This paper is structured as follows: in section two, the radial model of resource allocation in DEA_R is suggested. In Section three the non-radial DEA models are discussed. Finally, the conclusion of the article is presented in the last section.

2 The radial models of resource allocation based on DEA_R

Suppose n decision making units with the consumption of m inputs $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ are to produce $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^T$ outputs. Given that the DEA models based on the fractional analysis (DEA_R) and central resource allocation models, the new radial model which reduces the total inputs and increases total outputs is presented as follows.

$$\max \psi_R$$

s.t $\sum_{j=1}^{n} \lambda_{jt} \frac{y_{rj}}{x_{ij}} \geq \psi_R \frac{\sum_{j=1}^{n} y_{rj}}{\sum_{j=1}^{n} x_{ij}} \quad i = 1, \ldots, m, \quad r = 1, \ldots, s$ \hspace{1cm} (2.1)

$$\sum_{j=1}^{n} \lambda_{jt} = 1 \quad t = 1, \ldots, n$$

$$\lambda_{jt} \geq 0 \quad t = 1, \ldots, n, j = 1, \ldots, n.$$ \hspace{1cm} (2.2)

The pattern of all decision-making units is obtained through the equation(1.2).

$$\left(\sum_{j=1}^{n} \lambda_{jt} x_{ij}, \sum_{j=1}^{n} \lambda_{jt} y_{rj}\right)$$ \hspace{1cm} (2.3)

It should be noted is the solution for the rest of model (2.1).
3 Non-radial models of resource allocation based on DEA_R

In this section with the aim of decreasing inputs and increasing total outputs and the use of the enhanced Russell model, the proposed input oriented model will be as follows:

\[
\begin{align*}
\text{max} & \quad \frac{1}{m.s} \sum_{i=1}^{m} \sum_{r=1}^{s} \sum_{t=1}^{n} \sum_{j=1}^{n} \lambda_{jt} \frac{y_{ij}}{x_{ij}} \\
\text{s.t} & \quad \sum_{j=1}^{n} \sum_{t=1}^{n} \lambda_{jt} \frac{y_{ij}}{x_{ij}} \geq \frac{1}{\sum_{r=1}^{s} \sum_{t=1}^{n} \sum_{j=1}^{n} \varphi_{ir} \frac{y_{ij}}{x_{ij}}}, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \\
\sum_{j=1}^{n} \lambda_{jt} &= 1, \quad t = 1, \ldots, n \\
\lambda_{jt} &\geq 0, \quad t = 1, \ldots, n, \quad j = 1, \ldots, n, \\
\varphi_{ir} &\geq 1, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s.
\end{align*}
\]

Model (3.1) is a linear programming problem that the number of its qualities is \((n^2 + ms)\) and the number of constraints is \((n + ms)\). Considering \(\lambda_{jt}\) as the solution for the rest of problem (3.1) and inserting it in model (1.1), the pattern for all decision making units is obtained. The difference between model (2.1) and model (3.1) lies in the decrease of the ratio of total inputs to total outputs. Such that in model (2.1) the ratio of \(\theta\) is effective in the increasing and decreasing of outputs while in the model (3.1) it changes in respect to \(\varphi_{ir}\). In the other words, in model (2.1) the decrease and increase are in the same ratio but in model (3.1) they are possible with different ratios. In model (3.1) \(\varphi_{ir}\) is the theoretical priorities for deceasing or increasing the ratio of inputs to outputs.

4 Conclusion

In the data envelopment analysis, first, to evaluate \(n\) decision-making units, \(n\) linear programming problem is solved. Secondly, problems such as the use of non-zero weights and the false inefficiency arises. But DEA_R and resource allocation models resolved the aforementioned problems. Also taking into account the radial and non-radial models, we were to find a pattern for all decision making units. For future studies utilizing Entropy, neural networks and fuzzy logic will of great importance in solving problems of resource allocation and DEA_R.

References


Some products have remarkable short life cycles and experience considerable price drops afterward. Due to this phenomenon in this type of commodities, incentive contracts are usual exercises, especially in the high-tech industries. Increasing innovation in such industries in addition to the uncertain characteristics of market demand may prevent the retailers to order not too much of a product to satisfy the market demand. In this paper, the effect of buyback contract in supply chain coordination is considered using the uncertainty theory introduced by Liu in 2007. This contract is intended to potentially counterbalance the negative impact of double marginalization outcome.

**Keywords** Supply Chain Coordination; Buyback Contract; Uncertainty Theory.

1 Introduction

A supply chain is coordinated when decision makers are acting optimally for the whole supply chain not for their individual profit. Some coordination methods such as buyback contract are used to manage the interaction among the supply chain’s members. The incentive contracts and their effects are ubiquitous in the literature of supply chain coordination when the uncertainty occurs in demand with known probability distribution. However, for the cases when there is not enough historical data, using probability theory might be misleading. Here we propose the uncertainty theory initiated by Liu [1] to investigate the buyback contract effects on supply chain coordination of the newsboy problem.

This paper assumes that product has a long manufacturing lead-time, so that the retailer has a single order opportunity. The manufacturer (supplier) usually provides a incentive buyback contract to the retailer to stock more. For the sake of simplicity, a one-manufacturer and one-retailer supply chain is considered. Shortage or stock-out costs are incurred at the ends of the period. We study the motivation of the manufacturer to use only the buyback contract policy over the declining retail prices in uncertain environment in the sense of Liu [1].

1.1 Uncertain measure, variable and expected value

Let us review the notion of uncertainty. Let $\Gamma$ be the set of all possible outcomes. The $\sigma$-algebra $\mathcal{L}$ defined on $\Gamma$ should contain all events we are concerned about. On the $\sigma$-algebra $\mathcal{L}$, a number $M(\Lambda)$ is assigned to the event $\Lambda$ to indicate the belief degree with which we believe $\Lambda$ will happen. The uncertain measure $M$ must have certain mathematical properties. Liu [1] suggested the following three axioms:

**Axiom 1.** (Normality Axiom) $M(\mathcal{L}) = 1$ for the universal set $\mathcal{L}$.

**Axiom 2.** (Duality Axiom) $M(\Lambda) + M(\Lambda^c) = 1$ for any event $\Lambda$.

**Axiom 3.** (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, it holds

$$M\left(\bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} M(\Lambda_i).$$

The triplet $(\Gamma, \mathcal{L}, M)$ is called an uncertainty space. An uncertain variable is a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, M)$ to the set of real numbers, such that for any Borel set $B$ of real numbers, the set $\{\gamma \in \Gamma | \xi(\gamma) \in B\}$ is an event.
The expected value of an uncertain variable $\xi$ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi > r\}dr - \int_{-\infty}^0 M\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite. For more details on uncertainty theory, the interested reader is referred to [1].

## 2 Model formulation and problem description

The following sequence of events occurs in this game: the supplier offers the retailer a buyback contract; the retailer accepts or rejects the contract; assuming the retailer accepts the contract, the retailer submits an order quantity, $q$, to the supplier; the supplier produces and delivers to the retailer before the selling season; season demand occurs; and finally transfer payments are made between the firms based upon the agreed contract. If the retailer rejects the contract, the game ends and each firm earns a default payoff.

The used notations in addressing the uncertain newsboy problem with a buyback contract are listed in Table 2.1.

### 2.1 Optimal order quantity of the retailer

Let us denote $p - c_r - w$ with $k_r$ which is the net revenue of the retailer for per sold item. One can easily establish the uncertain profit function of the retailer as follows:

$$f_r(\xi, q) = \begin{cases} k_r q - (\xi - q) g_r & q < \xi \\ k_r \xi + (q - \xi) b & q \geq \xi \end{cases}$$

Optimal quantity for the retailer is obtained by solving the following uncertain programming problem:

$$\begin{align*}
\max_{q \geq 0} & \quad E[f_r(\xi, q)] \\
\text{s.t.} & \quad q \geq 0,
\end{align*} \tag{2.1}$$

**Theorem 2.1.** Let an uncertain demand $\xi$ be defined on an uncertainty space. Then the expected uncertain profit value of the retailer is

$$E[f_r(\xi, q)] = \int_0^{k_r q} M_r \left\{ \frac{x - bq}{k_r - b} \leq \xi \leq \frac{(k_r + g_r)q - x}{g_r} \right\} dx.$$ 

Observe that deriving an exact scalar value of the uncertain measure $M_r(\Lambda_r(q))$ and consequently, the objective function of the optimization problem (3.12) is not possible in general unless its uncertainty distribution is available. Here we provide the following lower and upper bounds for the objective function of problem (3.12) as

$$\begin{align*}
\text{lb}_r(q) &= \int_0^{k_r q} \Phi_r\left(\frac{(k_r + g_r)q - x}{g_r}\right) - \Phi_r\left(\frac{x - bq}{k_r - b}\right) dx, \\
\text{ub}_r(q) &= \int_0^{k_r q} \min\{\Phi_r\left(\frac{(k_r + g_r)q - x}{g_r}\right), (1 - \Phi_r\left(\frac{x - bq}{k_r - b}\right))\} dx.
\end{align*}$$

In this way one can conclude the following result.

**Theorem 2.2.** The objective value function of (3.12) is bounded as

$$\begin{align*}
\max_{q \geq 0} \text{lb}_r(q) &\leq \max_{q \geq 0} E[f_r(\xi, q)] \\
\min_{q \geq 0} E[f_r(\xi, q)] &\leq \min_{q \geq 0} \text{ub}_r(q).
\end{align*}$$
<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>The uncertain quantity demanded.</td>
</tr>
<tr>
<td>$f_r(\xi, q)$</td>
<td>The uncertain profit function of the retailer.</td>
</tr>
<tr>
<td>$f_s(\xi, q)$</td>
<td>The uncertain profit function of the supplier.</td>
</tr>
<tr>
<td>$f(\xi, q)$</td>
<td>The uncertain supply chain’s profit function.</td>
</tr>
<tr>
<td>$M_r{\Lambda}$</td>
<td>The belief index measure of the retailer for the event $\Lambda$ of the uncertain demand $\xi$.</td>
</tr>
<tr>
<td>$M_s{\Lambda}$</td>
<td>The belief index measure of the supplier for the event $\Lambda$ of the uncertain demand $\xi$.</td>
</tr>
<tr>
<td>$\Phi_r(.)$</td>
<td>The uncertainty distribution of the retailer for the uncertain demand $\xi$.</td>
</tr>
<tr>
<td>$\Phi_s(.)$</td>
<td>The uncertainty distribution of the supplier for the uncertain demand $\xi$.</td>
</tr>
<tr>
<td>$s$</td>
<td>The net salvage value of unsold goods gained by the supplier.</td>
</tr>
<tr>
<td>$p$</td>
<td>The retail price.</td>
</tr>
<tr>
<td>$c_s$</td>
<td>The supplier’s production cost per unit.</td>
</tr>
<tr>
<td>$c_r$</td>
<td>The retailer’s marginal cost per unit.</td>
</tr>
<tr>
<td>$g_r$</td>
<td>The goodwill penalty cost of each unsatisfied demand for the retailer.</td>
</tr>
<tr>
<td>$g_s$</td>
<td>The goodwill penalty cost of each unsatisfied demand for the supplier.</td>
</tr>
<tr>
<td>$q$</td>
<td>The retailer’s submitted order quantity.</td>
</tr>
<tr>
<td>$w$</td>
<td>The supplier’s receiving from the retailer for per unit purchased with a buyback contract.</td>
</tr>
<tr>
<td>$b$</td>
<td>The retailer’s receiving from the supplier for per unit remaining at the end of the season with a buyback contract.</td>
</tr>
</tbody>
</table>
2.2 Optimal order quantity of the supplier

One can easily establish the uncertain profit function of the supplier as follows:

\[ f_s(\xi, q) = \begin{cases} 
(w - c_s)q - (\xi - q)g_s & q < \xi \\
(w - c_s)\xi + (s - b)(q - \xi) & q \geq \xi.
\end{cases} \]

Optimal quantity for the supplier is obtained by solving the following uncertain programming problem:

\[
\max_{q \geq 0} E[f_s(\xi, q)]
\]

The following theorem resembles analogous result for the expected profit value of the supplier. The proof is omitted.

**Theorem 2.3.** Let \( \xi \) be defined on uncertainty space as an uncertain demand. Then the expected profit value of the supplier is

\[
E[f_s(\xi, q)] = \int_0^{(w-c_s)q} M_s \left\{ \frac{x + (b-s)q}{w - c_s + b - s} \leq \xi \leq \frac{(w - c_s + g_s)q - x}{g_s} \right\} dx.
\]

Let us define

\[
\Lambda_s = \left\{ \frac{x + (b-s)q}{w - c_s + b - s} \leq \xi \leq \frac{(w - c_s + g_s)q - x}{g_s} \right\}.
\]

Then

\[
\Phi_s(\frac{(w - c_s + g_s)q - x}{g_s}) - \Phi_s(\frac{x + (b-s)q}{w - c_s + b - s}) \leq M_s \{\Lambda_s(q)\}
\]

\[
\leq \min(\Phi_s(\frac{(w - c_s + g_s)q - x}{g_s}), (1 - \Phi_s(\frac{x + (b-s)q}{w - c_s + b - s}))).
\]

where \( \Phi_s(x) = M_s\{\xi \leq x\} \) is the uncertainty distribution of the uncertain demand \( \xi \) for the supplier.

Analogously the following theorem presents lower and upper bounds for the objective function of problem (3.12).

**Theorem 2.4.** The objective value function of (2.2) is bounded as \( \max_{q \geq 0} lb_s(q) \leq E[f_s(\xi, q)] \leq \min_{q \geq 0} ub_s(q) \), where

\[
lb_s(q) = \int_0^{(w-c_s)q} \Phi_s(\frac{(w - c_s + g_s)q - x}{g_s}) - \Phi_s(\frac{x + (b-s)q}{w - c_s + b - s}) dx,
\]

\[
ub_s(q) = \int_0^{(w-c_s)q} \min(\Phi_s(\frac{(w - c_s + g_s)q - x}{g_s}), (1 - \Phi_s(\frac{x + (b-s)q}{w - c_s + b - s}))) dx.
\]

2.3 Optimal order quantity of the coordinated supply chain

For notational convenience, let \( c = c_s + c_r \) and \( g = g_s + g_r \). The supply chain’s uncertain profit function is defined as

\[ f(\xi, q) = f_r(\xi, q) + f_s(\xi, q) = \begin{cases} 
(p - c + g)q - g\xi & q < \xi \\
sp + (p - c - s)\xi & q \geq \xi
\end{cases} \]

The following theorem presents closed form of the expected profit value of the supply chain. The proof is similar and ignored.

**Theorem 2.5.** Let \( \xi \) be defined on uncertainty space as an uncertain demand. Then the expected profit value of the supply chain is

\[
E[f(\xi, q)] = \int_0^{(p-c)q} M \left\{ \frac{x - sq}{p - c - s} \leq \xi \leq \frac{(p - c + g)q - x}{g} \right\} dx.
\]
Analogous to Theorem 2.3, one can establish the following bounds for the optimal value of the expected value of uncertain supply chain profit function with buyback contract.

**Theorem 2.6.** The optimal value of the expected value of uncertain supply chain profit function is bounded as \( \max_{q \geq 0} lb(q) \leq E[f(\xi, q)] \leq \min_{q \geq 0} ub(q) \), where

\[
\begin{align*}
lb(q) &= \int_{0}^{k-q} \Phi\left(\frac{(p - c + g)q - x}{g}\right) - \Phi\left(\frac{x - sq}{p - c - s}\right) dx, \\
ub(q) &= \int_{0}^{k-q} \min\left\{\Phi\left(\frac{(p - c + g)q - x}{g}\right), \left(1 - \Phi\left(\frac{x - sq}{p - c - s}\right)\right)\right\} dx.
\end{align*}
\]

and \( \Phi(x) = M_r\{\xi \leq x\} = M_s\{\xi \leq x\} \) provided that the belief index measure of both the retailer and the supplier are identical.

**References**

This paper presents an ant colony algorithm for solving the well-known job shop scheduling problem which is a typical NP-hard problem. In order to initialize the pheromone trails, a novel mechanism is employed based on an initial sequence. Moreover, the pheromone trail intensities are limited between lower and upper bounds dynamically modified. An artificial ant constructs a complete solution by iteratively applying a pseudo-stochastic rule based on the pheromone trails. A local search is then performed to improve the performance quality of the solution. The computer simulations were made on a set of benchmark problems and the results demonstrated the effectiveness of the proposed algorithm.

Keywords Job shop; scheduling; Ant colony optimization.

1 Introduction

Scheduling is one of the most critical issues in the planning and managing of manufacturing processes. One of the most difficult problems in this area is the job shop scheduling problem (JSSP).

The JSSP which we are concerned can be described as follows [2]: There are \( n \) different jobs to be processed on \( m \) different machines. Each job needs \( m \) operations and each operation needs to be processed without preemption for a fixed processing time on a given machine. There are several constraints on jobs and machines:

- A job can visit a machine once and only once.
- There are no precedence constraints among the operations of different jobs.
- Preemption of operations is not allowed.
- Each machine can process only one job at a time.
- Each job can be processed by only one machine at a time.
- Neither release times nor due dates are specified.

The problem is to find a schedule to minimize the makespan, that is, to minimize the time required to complete all jobs.

Real-life application of the JSSP was discussed in [3]. A fair number of authors tried to increase the performance of the genetic algorithm by incorporating other traditional heuristics in the algorithm. This hybridization can happen with simple techniques, such as local search operator [2] and tabu search technique [5]. Additionally, a genetic algorithm and a scatter search procedure is proposed by [4] to solve the job shop scheduling problem.

In this paper, we apply an ant colony algorithm (ACA) that is based on model given by [1] to solve the JSSP problem. The performance of the proposed approach is evaluated on a set of benchmark problems.
2 ant colony algorithm

The main idea in ant colony optimization (ACO) algorithms is to mimic the pheromone trails used by real ants searching for feed as a medium for communication and feedback. In the ACA, a rather good solution is firstly generated in negligible computation time, and then the pheromone trails are initialized depending on this solution. In other words, unlike most applications of ACO, at the beginning of the ACA an equal initial value is not assigned to all pheromone trails. Each artificial ant starts with an empty sequence and chooses one of the jobs. Then, the ant iteratively appends an unscheduled job to the partial sequence constructed so far until a complete solution is built. At each step, a job is chosen by applying a transition rule based on the pheromone trails. The performance quality of the constructed solution is then improved by means of a local search procedure. Once all ants in the colony have built their solutions, to make the search more directed, the pheromone trails are modified by applying a global updating rule. Moreover, the trail intensities are limited between lower and upper bounds which change dynamically in a new manner.

The general structure of the proposed algorithm is represented as follows:

Algorithm 17 General structure of the ACA

Step 1: Set parameters; generate a seed solution and initialize the pheromone trails.

Step 2: While the termination condition is not met, do the following:
- For each ant in the colony, do:
  - By repeatedly applying the transition rule, construct a solution;
  - Improve the solution quality by the local search;
  - In case of an improved solution, update the best solution generated so far.
  - Modify the pheromone trails according to the global updating rule;
  - Update the minimum and maximum trail bounds, and limit the pheromone trails.

Step 3: Return the best solution found.

Since searching a large neighborhood requires more computational time, so in the local search procedure, a threshold probability $T$ is incorporated for choosing a job to insert into the other positions of a given sequence.

The proposed local search procedure is then represented as follows:

3 Computational results

In order to verify the good performance of the proposed algorithm, we use Lawrence's benchmark problems (LA01-40). The ACA was compared with the HGA algorithm proposed by [2]. The proposed algorithm has been coded in Visual C++ and all test runs have been carried out on a 2.0 GHz Intel Core 2 Duo Processor with 2 GB memory.

Table 1 shows the experimental results. It lists problem name, problem size, the best known solution (BKS) found in the literature and the solution obtained by the ACA and HGA algorithms.

As seen from Table 1, the proposed algorithm is able to find the best known solution for 28 instances.
Algorithm 18 Local search procedure

For each job $j$ ($j = 1, \ldots, N$), do the following:
  
  Generate a random number $R$ uniformly distributed in $[0, 1]$;
  
  If $R \leq T$, do:
    
    For each position $i$ ($i = 1, \ldots, N$), do:
      
      If job $j$ is not in the $i$th position of the current sequence, insert job $j$ in position $i$ without any change in the other sequence, and then calculate the makespan of the newly obtained sequence.
    
    Determine the best sequence among the $N-1$ newly obtained sequences.
    
    If the makespan is improved, replace the current sequence by the best one found.

4 Conclusions

In this paper, an ant colony algorithm is developed for the JSSP. At first, a rather good solution is generated in negligible computation time and then, the trail intensities are initiated based on this solution. It is noteworthy that in initializing, updating as well as limiting the trail intensities, the goal is to guide the search towards the neighborhood around the best solution found. Computational experiments show that the proposed ACA algorithm is effective.

References


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A MIGRATING BIRDS ALGORITHM FOR THE TRAVELLING SALESPERSON PROBLEM WITH HOTEL SELECTION (TSPHS)

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In this paper a new metaheuristic solution is designed to solve the travelling salesperson problem with hotel selection (TSPHS). The applied solution approach is based on the V flight formation of migrating birds. Different numerical analysis confirms that it is efficient enough for solving the problem in a reasonable computational time. To check the efficiency and suitability of the solution algorithm, results of exact solution method solved by commercial software for small instances are compared with results of the applied metaheuristic algorithm. The comparisons confirm that the developed algorithm is capable to find near optimal solutions for the TSPHS. By doing more numerical analysis we concluded the trip duration will decrease by increasing of available hotels. It also decreases if there is permission for over time working by travelers.

Keywords TSPHS; hotel selection; migrating birds algorithm; new metaheuristic.

1 Introduction

The travelling salesperson problem with hotel selection was introduced by Vansteenwegen et al. [1]. The motivation for this problem is that a salesperson can work for a specific time a day and most of the time he/she cannot visit all of customers in that specific day. He/She has to a stay at a hotel and continues his/her job in the next day. There is no limitation for selecting hotel but in the last day he/she should come back to beginning hotel. In contrast to Castro et al. [2] in this paper the hotel selection is determined in a greedy approach. The proposed algorithm selects the most suitable hotel which is near to the last visited customer today and the next one will be visited tomorrow. In this paper, the migrating birds algorithm is used for solving TSPHS, Which were used by Duman[3] before.

2 MATHEMATICAL FORMULATION

There is two differences between formulation of this paper and vansteenwegen et al [1]. First, asymmetric subtour is used in this paper. Second, the constraint (6) in vansteenwegen et al. [1] is changed to:

\[
\sum_{i,j \in A} (C_{ij} + \tau_j)X_{ij}^d - \sum_{i \in A} C_{ih}X_{ih}^d \leq L \quad (2.1)
\]

\(d = 1, \ldots, d\)

The reason of the second change is that, in the real world the time of travelling from last customer node to the hotel is not assumed as working time so it should be purified.

3 A migrating birds algorithm for TSPHS

In this section the proposed solution algorithm for TSPHS is presented in Algorithm 20. MBO algorithm for the TSP in 1st iteration is similar to the presented algorithm by Duman [3]. To check the algorithm performance, the results have been reported in Tables 1 and 2 comparing to an exact solution.

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Algorithm 19 Migrating birds for TSPHS

Step 1: Generate initial solutions \((P)\)

Step 2: while stopping criterion not met do

Step 3: implement MBO for tsp in one iteration

Step 4: for each tour, select its related hotels \((ph)\)

Step 5: while each \(ph\) is infeasible

	mutate \(p\) and make new \(ph\)

Step 6: end while

Step 7: end for

Step 8: replace the best solution in the head

Step 9: change \(Ph\) to \(p\)

Step 10: end while

Step 10: select hotel for the tour on the head

4 Numerical instances

It will be shown in Table 2 that the gap of the MBO solution and exact solution is acceptable. In the

<table>
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</table>

Figure 1, it was realized that the trip duration will be decreased by increasing of available hotels.

In the Figure 2, it was realized that the trip duration will be decreased by increasing the time of working in a day.

The TSPHS is a difficult optimization problem. In this paper, a new meta heuristic solution method for TSPHS is presented. The meta heuristic is a migrating birds optimization. The MBO algorithm solves the TSPHS by distinguish two levels: the performing the MBO algorithm for the tour and hotel selection. In this paper, it was shown that the gap of the MBO solution and exact solution is acceptable. In this paper, the strategy of working in a day is different with the strategy that was mentioned in the Pieter vansteenwegen et al’s article [1]. In this paper the strategy of selecting the hotels is obvious, but in the vansteenwegen et al [1] the strategy of selecting hotel is based on the randomization. In the real case the worker sometimes can work more than 8 hour. The future study of us is researching about the effect of the overtime working on the cost for a travelling salesman who needs to select hotel.
Table 2: The GAP of exact solution (ES) and MBO (time in second).

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5 CONCLUSION AND FUTURE STUDY

The TSPHS is a difficult optimization problem. In this paper, a new meta heuristic solution method for TSPHS is presented. The meta heuristic is a migrating birds optimization. The MBO algorithm solves the TSPHS by distinguish two levels: the performing the MBO algorithm for the tour and hotel selection. In this paper, it was shown that the gap of the MBO solution and exact solution is acceptable. In this paper, the strategy of working in a day is different with the strategy that was mentioned in the Pieter Vansteenwegen et al. [1]. In this paper the strategy of selecting the hotels is obvious, but in the Vansteenwegen et al. [1] the strategy of selecting hotel is based on the randomization. In the real case the worker sometimes can work more than 8 hour. The future study of us is researching about the effect of the overtime.
working on the cost for a travelling salesman who needs to select hotel.

References


THE PICK UP AND DELIVERY PROBLEM WITH CROSS DOCKING CONSIDERING CUSTOMER PREFERENCE TIME WINDOW (PDCDPTW)

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Cross docking is a distribution policy that enables the consolidation of freight from origins to destinations. In a cross dock, freights are unloaded from inbounded trucks and directly reloaded to out-bounded trucks without long storage in it. In this paper we introduce a pickup and delivery problem with cross docking and customer preference time window (PDCDPTW) that considers a preference time window (PTW) inside of a hard time window (HTW). The preference time window is the time window in which customers incline to receive their demands. Deviation from preference time window are allowed but these deviations are accounted for in the objective function. We propose a mixed integer programming model for this problem to minimize transportation cost with maximum customer satisfaction. Real instances of data sets are used to check performance of the proposed model. The results confirm better customer servicing.

Keywords Cross docking; Vehicle routing; Preference time window; Pickup and delivery.

1 Introduction

Cross docking is a new warehousing strategy in logistics that today's is widely used all over the world and attract many researchers attention to study about. It is proved that cross docking in compare with traditional distribution centers deal with cost and storage reduction, shorter delivery lead time, better customer service, fast inventory turn and reduce risk from loss and damage. The most important constraint in cross docking is the limited time products staying inside the cross-dock, e.g., 24 hours in some cases like frozen freights or some kind of drugs, the storage time should be less than an hour.

An important purpose of a cross-dock is to enable the consolidation of shipments according to their destinations to reduce transportation costs. Because of this advantages, today's cross docking is used all over the world. We can mention some successful applications of cross docking which applied with: Yellow Transport (Chicago), American freightways (Atlanta), consolidated Freightways (Portland), Wal-Marts (Arkansas). One of the important problems in cross docking is vehicle routing problem in order to find best tours of pickup and delivery to reduce the transportation cost.

The vehicle routing problem with cross docking was introduced by Lee et al.[1] and solved by tabu search. Liao et al.[2] proposed another tabu search to solve the same problem. wen et al.[3] considered same problem with consolidation, in that problem orders after pickup phase, consolidated at a cross dock then immediately delivered to customers. In consolidation phase, orders are unloaded from inbounded trucks and reloaded to outbounded trucks while if a truck must deliver an order that picked it up itself, consolidation is not necessary. In this paper we consider an extension of pickup and delivery problem with time window (PDCDTW). Assumption of our problem are same as wen et al.[3] and in addition to normal time window we consider preference time window based on customer satisfaction. We use real instances from a dataset with 40 nodes including 20 suppliers and 20 customers with one cross dock and the model was coded in GAMS software to optimality.
2 MATEMATICAL FORMULATION

We consider costumer preference time window so there are differences between formulation of this paper and wen et al.[3]. Related parameters are defined as follows:

\[ [a_p, b_p] : \text{The preference time window of node } i \in N ; \]
\[ M : \text{An arbitrarily large constant}; \]
\[ \alpha_i : \text{The deviation after preference time window}; \]
\[ \beta_i : \text{The deviation before preference time window}; \]

Other parameters and variables are the same with wen et al. [3].

\[
\sum_{(i,j) \in E} \sum_{k \in K} c_{ij} x_{ij} + \sum_{i \in D} \alpha_i + \beta_i \quad (2.1)
\]
\[
\alpha_i \geq \sum_{j : (i,j) \in E} \sum_{k \in K} y^k_{ij} - b p_i \forall i \in D \quad (2.2)
\]
\[
\beta_i \geq a p_i - \sum_{j : (i,j) \in E} \sum_{k \in K} y^k_{ij} \forall i \in D \quad (2.3)
\]
\[
y^k_{ij} \leq s^k_i \quad \forall i,j \in D, k \in K \quad (2.4)
\]
\[
y^k_{ij} \leq M x^k_{ij} \quad \forall i,j \in D, k \in K \quad (2.5)
\]
\[
y^k_{ij} \geq s^k_i - M (1 - x^k_{ij}) \quad \forall i,j \in D, k \in K \quad (2.6)
\]
\[
x^k_{ij} \in \{0, 1\} \quad \forall i \in p, (i,j) \in E \quad (2.7)
\]
\[
y^k_{ij}, s^k_i, \gamma_i, \tau_i \geq 0 \quad \forall i \in N, k \in K \quad (2.8)
\]

The hard and preference time window has been shown in Figure 1

![Figure 1: The hard and preference time window](image)

3 NUMERICAL RESULTS

In this section results of different sensitivity analysis are reported. Figures 2, 3 and 4 show the results obtained for real instances by changing of different parameters. There are two parameters for unloading and reloading freight at cross-dock including fix time for each vehicle and variable time for each pallet in the vehicles.

In the Figure 2, it was realized that the cost will be increased by increasing of unloading and reloading time.

According to the Figure 3, it can be concluded that the number of unloads and reloads will be decreased by increasing of unloading and reloading time.
Figure 2: The effect of increasing time of unload and reload

![Figure 2](image1.png)

Figure 3: The effect of increasing time of unload and reload on the number of reload and unload

![Figure 3](image2.png)

Figure 4: The comparing between VRPCDTW and VRPCDPTW

![Figure 4](image3.png)

Figure 4 compares results of two models VRPCDTW and VRPCDPTW, in different scenarios. In the first one we assumed that length of preference time window is 40 minutes and in the next scenarios we increased its length up to 120 minutes. The results have been reported in Figure 3. It shows that when the preference time window is 120 minutes there is no difference between two models and VRPCDTW and VRPCDPTW will construct same network and routing plans while with tightening of PTW their performance gap will be more and more.
References


Poster Presentation, Day 1
A Multi Objective Linear Programming Approach for Sustainable Plantation

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Multi objective linear programming or goal programming technique was used in order to determine the optimal forest plantation in north of Iran. The objective was maximizing net present value, number of labor and carbon sequestration. Required data including growth, stumpage price, carbon sequestration and number of labor was collected. Regression analysis was used to derive growth models. Expected mean price was estimated using wood price and variable harvesting costs. LINGO software was used for analysis. Results indicated that the optimal plantation for oak, ash and loblolly pine are 149.37, 136.72 and 206.20 ha, respectively.

Keywords Multi objective linear programming; sustainable forest plantation; carbon sequestration.

1 Introduction

The role of planted forests (plantation), in achieving sustainable development and mitigating the effects of climate change is critical. The significance of plantation and recognition of their contributions to a range of development goals are anticipated to increase in coming decades. Plantations can provide an array of social and economic benefits, and can contribute to satisfying the world’s needs for forest products, they should complement the management of, reduce pressures on, and promote the restoration and conservation of natural forests [2]. Bettinger et al. (2009) mentioned the application of goal programming model to the following cases:

1. Determination of present and future production from the land, as well as the demand for the different products.
2. Estimation of physical capacity of the land to produce the various products.
3. Analyze the complementary and competitive relationships among the goals.
4. Determination the feasible set of desirable goals.
5. Express the goals as a single objective function, and design the problem formulation using the appropriate constraints [1].

Goal programming (GP) technique was used in order to determine the optimal harvest volume for the Iranian Caspian forest. Results indicated that the optimum volumes of species were $250.25 m^3 h^{-1}$ for beech, $59 m^3 h^{-1}$ for hornbeam, $73 m^3 h^{-1}$ for oak, $41 m^3 h^{-1}$ for alder, and $32 m^3 h^{-1}$ for other species [2]. Usually in Iran the economics and social aspect of forest plantation is neglected and there are many researches regarding to the ecological aspects of plantation. Therefore, the aim of this study to determine the appropriate species in forest plantation considering economically, socially and ecologically criteria use operation research technique. This research was conducted in Shaforoud forest, north of Iran. The total area of forest management plan is 1524 ha, of this about 922.17 ha is planted [5]. Data such as growth, timber price at road side, harvesting cost, sequestrated carbon and number of labor was collected. Selection species based on ecological conditions: The species were classified within two groups based on the soil type of the area as below: Species in group one are Oak (Quercus castanifolia) and pine (Pinus taeda). Species in group two are ash (Fraxinus excelsior) and maple (Acer velutinum).
2 Main Results

Selection species based on ecological conditions: The species were classified within two groups based on the soil type of the area as below: Species in group one are Oak (Quercus castanifolia) and pine (Pinus taeda). Species in group two are ash (Fraxinus excelsior) and maple (Acer velutinum). The following goal programming model was used in order to determine the appropriate species: The following goal programming model was used in order to determine the appropriate species:

### 2.1 Formulating of GP model

\[
\begin{align*}
\min z &= 0.00425D_{NPV}^- + 312.5D_L^- + 93.20D_{CAR_1}^- + 136.89D_{CAR_2}^- & (2.1) \\
2206.878X_1 + 2623.883X_2 + 4319.964X_3 + 8064.665X_4 + D_{NPV}^- &= 2351376.026 & (2.2) \\
0.065X_1 + 0.065X_2 + 0.065X_3 + 0.065X_4 + D_L^- &= 32 & (2.3) \\
X_1 + X_4 & \leq 355.58 & (2.4) \\
X_2 + X_3 & \leq 151.27 & (2.5) \\
0.244X_1 + 0.3X_4 + D_{CAR_1}^- &= 107.288 & (2.6) \\
0.032X_2 + 0.046X_3 + D_{CAR_2}^- &= 73.05 & (2.7)
\end{align*}
\]

The objective function was to minimize undesired deviation from the goal. Where, $D_{NPV}^-$ is negative deviation of NPV, $D_L^-$ is negative deviation of labor, $D_{CAR_1}^-$ is negative deviation of sequestered carbon of oak and loblolly pine species and $D_{CAR_2}^-$ is negative deviation of sequestered carbon of ash and maple species. Finally we solved the GP model consisting of objective function and constraints using LINGO software. The value of decision variables $X_1, X_2, X_4$ are 149.37, 136.72 and 206.20. It means that the optimal plantation for oak, ash and loblolly pine are 149.37, 136.72 and 206.20 ha, respectively. The maple species is not economically profitable for plantation at the study area based on the parameters value of objective function and the constraints. The results also show that the reduced cost of species $X_4$ is 1.991696 (10 thousands Iranian Rials). It means that if these values added to the correspondent parameter values on the objective function, this species will be profitable for plantation. The amounts of underachieved and overachieved are showed by slack and surplus, respectively. Surpluses for the constraint 4 at model is 14.54231. It means that the amount of 14.54231 ha is higher than the minimum acceptable level (151.27 ha) (Table1). Multi objective linear programming or goal programming technique was used in order to determine the optimal forest plantation in north of Iran. The optimal sustainable forest plantation was to develop based on goal programming and Analytic Hierarchy Process (AHP) methods. This study was carried out in Guilan province, results indicated that the appropriate plantation area for each species are as follow: Acer velutinum (810 ha), Alnus subcordata (348 ha), Pinus taeda (235 ha), Tilia begonifolia (165 ha), Quercus castanifolia (149 ha), Pinus nigra (110 ha) and Fraxinus excelsior (0 ha) [1].

References

Table 1: Results of LP model solved in LINGO software.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Reduced Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{NPV}$</td>
<td>0.000000</td>
<td>0.737852E-02</td>
</tr>
<tr>
<td>$D_L$</td>
<td>0.000000</td>
<td>232.0925</td>
</tr>
<tr>
<td>$D_{CAR_1}$</td>
<td>66.46501</td>
<td>0.000000</td>
</tr>
<tr>
<td>$D_{CAR_2}$</td>
<td>11.18876</td>
<td>0.000000</td>
</tr>
<tr>
<td>$X_1$</td>
<td>149.3759</td>
<td>0.000000</td>
</tr>
<tr>
<td>$X_2$</td>
<td>136.7277</td>
<td>0.000000</td>
</tr>
<tr>
<td>$X_3$</td>
<td>0.000000</td>
<td>1.991696</td>
</tr>
<tr>
<td>$X_4$</td>
<td>206.2041</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Row</th>
<th>Slack or Surplus</th>
<th>Dual Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7726.167</td>
<td>-1.000000</td>
</tr>
<tr>
<td>2</td>
<td>0.000000</td>
<td>0.312852E-02</td>
</tr>
<tr>
<td>3</td>
<td>0.000000</td>
<td>-80.40753</td>
</tr>
<tr>
<td>4</td>
<td>0.000000</td>
<td>21.06301</td>
</tr>
<tr>
<td>5</td>
<td>14.54231</td>
<td>0.000000</td>
</tr>
<tr>
<td>6</td>
<td>0.000000</td>
<td>-93.20000</td>
</tr>
<tr>
<td>7</td>
<td>0.000000</td>
<td>-136.8900</td>
</tr>
</tbody>
</table>


ECONOMIC-STATISTICAL DESIGN OF VSSI-MEWMA-DWL CONTROL CHART WITH MULTIPLE ASSIGNABLE CAUSES USING GENETIC ALGORITHM

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Desirable properties of the multivariate exponentially weighted moving average (MEWMA) control chart such as the ability to detect small shifts in the process parameters have been caused that the MEWMA has been received significant attention from researchers in recent years. This paper proposes an economic-statistical design (ESD) model of the variable sample size and sampling interval (VSSI) MEWMA control chart by using double warning lines (DWL) by considering multiple assignable causes based on Lorenzen and Vance cost function and multivariate Taguchi loss approach. Due to the complexity of the model a genetic algorithm is designed as an optimization technique. A numerical example is provided to illustrate the performance of the model and the solution approach.

Keywords MEWMA control chart; VSSI; DWL; ESD; Genetic algorithm.

1 Introduction

In real environments many types of assignable causes may take place. Hence, developing ESD model of the MEWMA control chart that incorporates multi-assignable causes is important. ESD of VSSI-MEWMA-DWL control chart considers minimization of the expected cost per time unit, $E(A)$, as objective function, a lower limit ($AT S_L$) for in-control average time to signal ($AT S_0$) and an upper limit ($AAT S_U$) for the out-of-control adjusted average time to signal ($AAT S$) as statistical constraints. We extend the Lorenzen and Vance cost function considering multiple assignable causes. The multivariate Taguchi loss approach provided by Kapur and Cho [1] is used to obtain the expected external cost per time unit for the in-control and out-of-control process. In addition the Markov chain approach first proposed by Lee [2] is employed to obtain the desired $AT S_0$ and $AAT S$ of the VSSI MEWMA-DWL control chart.

2 Proposed Model

The MEWMA vector and the plotted chart statistics are defined as follows respectively:

$$z_t = \gamma (x_t - \mu) + (1 - \gamma)z_{t-1},$$

$$T^2_t = z_t' \Sigma_z^{-1} z_t,$$

where $\gamma (0 < \gamma \leq 1)$ is a diagonal weight matrix and $z_0$ represents a zero vector.

The VSSI-DWL MEWMA is a modification of fixed sample rate MEWMA control chart using two warning lines: The warning line $w_h$ is a guideline to switch between long and short sampling intervals $h_1$ and $h_2$ respectively, whereas the warning line $w_n$ is used to switch between small and large sample sizes $n_1$ and $n_2$ respectively. The next sampling point of the MEWMA-DWL control chart depends on the value of $T^2_t$. In the DWL scheme, there are three possible scenarios including: (i) $w_h < w_n$; (ii) $w_h > w_n$ and (iii) $w_h = w_n$. The control chart alarms an out-of-control signal when $T^2_t > H$, where $H$ is an upper control limit. The proposed ESD model that is used to determine the VSSI-MEWMA-DWL chart parameters$(n_1, n_2, h_1, h_2, \gamma, w_n, w_h, H)$ is defined as follows:
Min $E(A)$
subject to
$ATS_0 \geq ATS_L$
$\overline{AAT S} \leq AAT S_U$
$0 \leq w_n, w_h \leq H \leq H_{max}$
$h_{min} \leq h_2 \leq h_1 \leq h_{max}$
$1 \leq n_1 \leq n_2 \leq n_{max}$ (integers)
$0 < \gamma \leq 1$

(2.3)

3 MODIFIED LORENZEN AND VANCE COST FUNCTION

We make the usual assumption that there are multiple assignable causes ($j = 1, 2, ..., s$) where in each time just one of them can take place and change the mean of the process. Hence, the modified Lorenzen and Vance cost function for computing the expected cost per hour is defined as

$$E(A) = \frac{E(c)}{E(T)}$$

(3.1)

$$E(C) = \frac{1}{\lambda} \{c_0 + \sum_{j=1}^{s} \lambda_j c_{1j} [AAT S_j + \bar{n}E + r_{1j}T_{1j} + r_{2j}T_{2j}] + a_4ANF + a_1ANS + a_2ANI + \sum_{j=1}^{s} \lambda_j a_{3j}}$$

$$+ \frac{a_1 + a_2n_1}{\bar{h}_2} \sum_{j=1}^{s} \lambda_j (\bar{n}E + r_{1j}T_{1j} + r_{2j}T_{2j})$$

(3.2)

$$E(T) = \frac{1}{\lambda} \{1 + \sum_{j=1}^{s} \lambda_j (AAT S_j + \bar{n}E + T_{1j} + T_{2j} + (1 - r_{1j})T_{0ANF})\}$$

(3.3)

where $\lambda_j$ is the occurrences rate of $j^{th}$ assignable cause per hour according to Poisson process. Other parameters of the expected cost functions are defined similar to Faraz [3].

4 Solution approach

The proposed ESD model has both continuous and discrete decision variables and can be solved using one of the meta-heuristic methods. In this paper a genetic algorithm is developed to solve the proposed ESD model. Given the process parameters ($p, \lambda_j, T_0, T_{1j}, T_{2j}, \gamma_{1j}, \gamma_{2j}, E$) and the cost parameters ($C_0, C_{1j}, a_1, a_2, a_{3j}, a_5$) we strive to find the near optimum values for the control chart parameters ($n_1, n_2, h_1, h_2, \gamma, w_n, w_h, H$). The genetic algorithm parameters involve initial population size ($N_{pop}$), cross over percentage ($P_c$), and mutation percentage ($P_m$) have been set according to Faraz [3]. The pseudo-code for the genetic algorithm of this paper is given in Algorithm 20.

A numerical example is presented to demonstrate the proposed model. Table 1 displays the optimum values of the parameters of the proposed ESD model. The optimum values of the $ATS_0$, $AAT S$ and $E(A)$ is illustrated in Table 1.
Table 1: The Optimal Parameters of the ESD of VSSI-MEWMA-DWL Control Chart.

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$h_1$</th>
<th>$h_2$</th>
<th>$\gamma$</th>
<th>$w_h$</th>
<th>$w_n$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>12</td>
<td>2.327</td>
<td>0.679</td>
<td>0.429</td>
<td>1.21</td>
<td>1.18</td>
<td>11.972</td>
</tr>
</tbody>
</table>

Table 2: The Optimal values of the $AT\bar{S}_0$, $AAT\bar{S}$, $E(A)$.

<table>
<thead>
<tr>
<th>$AT\bar{S}_0$</th>
<th>$AAT\bar{S}$</th>
<th>$E(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>645</td>
<td>0.8123</td>
<td>5944</td>
</tr>
</tbody>
</table>

5 Conclusion and future research

In this study an ESD model of the VSSI-MEWMA-DWL control chart is proposed to consider multiple assignable causes to improve the efficiency of control charts and provide much faster detection of small and moderate process changes. In contrast to previous models we consider multiple assignable causes and more real-life aspects of manufacturing process than existing models in ESD of MEWMA control chart. Considering the problem as a multi-objective model and incorporating the preventive maintenance into the cost model can be a fruitful area for future research.

References


Algorithm 20 The Pseudo-code of the proposed genetic algorithm.

Begin

Input: \( n_1, n_2, h_1, h_2, \gamma, w_n, w_h, H, AATS_U, ATS_L, H_{\text{max}}, h_{\text{min}} \),
\( h_{\text{max}}, n_{\text{max}}, C_0, C_{ij}, a_1, a_2, a_3, a'_3, p, \lambda_j, T_0, T_{ij}, T_{2j}, \gamma_{ij}, \gamma_{2j}, E, d, P_c, P_m, P_r, N_{\text{pop}} \).

Output: \( n_1^*, n_2^*, h_1^*, h_2^*, \gamma^*, w_n^*, w_h^*, H^*, AATS_0^*, \overline{AATS}^*, E(A)^* \).

// Initialize generation 0

\[ g = 0 \]
\[ k = 1 \]

While \( k \leq N_{\text{pop}} \)

Generate initial chromosome \( P_k \) consisting of 8 genes \( n_1, n_2, h_1, h_2, \gamma, w_n, w_h, H \) randomly.

// Evaluate \( P_k \)

Compute the \( E(A), ATS_0, \overline{AATS} \).

If the \( ATS_0 \) and \( \overline{AATS} \) satisfy the constraints

// Create chromosome \( k + 1 \)

end

end while

do

{ //Create generation \( g+1 \)

// 1. Cross-over

Select \( P_c \times N_{\text{pop}} \) members;

pair them up;

Produce offspring: in each pair three similar genes of chromosomes have been replaced with each other randomly;

Insert the offspring into population of \( g + 1 \);

// 2. Mutation

Select \( P_m \times N_{\text{pop}} \) members;

In each member, two genes are mutated by size \( d \);

//3. Copy:

Select \( P_r \times N_{\text{pop}} \) members of current population randomly;

Insert into population of \( g+1 \);

// Evaluate population

Compute the fitness function: \( E(A), ATS_0, \overline{AATS} \);

// Increment:

\[ g = g + 1 \]

Compute the fitness function: \( E(A), ATS_0, \overline{AATS} \).

}
SOLVING MADM PROBLEMS IN INTUITIONISTIC FUZZY ENVIRONMENTS BASED ON COMBINING CHOQUET INTEGRAL AND TOPSIS METHODS

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In some Multi Attribute Decision Making (MADM) problems, TOPSIS and Choquet Integral (CI) methods, are used to rank or choose the best alternative, individually. In this paper, to gain the advantages of these methods, we combine them and propose a new hybrid method for aggregation of intuitionistic fuzzy-valued information in MADM problems. First, some definitions and preliminaries are defined. Then, we’ll define a new method for aggregation of Intuitionistic Fuzzy Numbers (IFNs) based on TOPSIS and CI which is called TOPSIS-CI. It is caused to interdependent or interactive characteristics among the decision maker’s preference criteria are considered. Finally, this method is illustrated by numerical example.

Keywords Choquet Integral, Distance Functions, Intuitionistic Fuzzy Numbers, Fuzzy measure.

1 Introduction

There are many real problems with a finite set of solutions and we have to choose the best one or to rank them based on a finite set of criteria. Such problems are called MADM problems and there exist many methods to solve them [2]. Choquet integral [2] is an aggregation method which is able to manage interactive criteria. Another more useful method in MADM problems is In 1983, Atanassov introduced the Theory of Intuitionistic Fuzzy Sets (IFSs) [1], to model the vague and uncertain situations and it is widely used. In this paper, we combined TOPSIS and CI methods, which is called TOPSIS-CI method, to solve MADM problems with intuitionistic fuzzy information.

2 Definitions and Preliminaries

In this section some needed concepts and definitions are reviewed.

Let \( X \neq \emptyset \) be a set and \( \Lambda \) be the family of subsets of \( X \) with empty set \( (\phi) \). An extended real valued set function \( \mu \) is a mapping as \( \mu : \Lambda \rightarrow [\pm \infty] \). It is called \( \sigma \)-additive on \( \Lambda \), if and only if \( \mu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i) \), whenever \( \{E_i\}_{i=1}^{n} \) is a pairwise disjoint sequence of sets in \( \Lambda \) and \( \bigcup_{i=1}^{n} E_i \in \Lambda \) [5]. A nonnegative set function \( \mu : \Lambda \rightarrow [0, \infty] \) is a measure if it is \( \sigma \)-additive and there exists \( E \in \Lambda \) such that \( \mu(E) < \infty \). Moreover, it satisfies in \( \lambda \)-rule, \( \lambda \in (\frac{1}{\sup \mu}, \infty) \cup \{0\} \) be constant value that \( \sup \mu = sup\{\mu(E) | E \in \Lambda\} \), if and only if for all pairwisely disjoint finite sequence \( \{E_i\}_{i=1}^{n} \) of elements of \( \Lambda \) [5]:

\[
\mu(\bigcup_{i=1}^{n} E_i) = \begin{cases} 
\frac{1}{\lambda}(\Pi_{i=1}^{n}(1 + \lambda \mu(E_i)) - 1), & \lambda \neq 0; \\
\sum_{i=1}^{n} \mu(E_i), & \lambda = 0.
\end{cases}
\]

Now, let \( \mu \) be a measure, it is called monotone if \( \mu(\phi) = 0 \) and for all \( E, F \in \Lambda \) which, \( E \subseteq F \) then \( \mu(E) < \mu(F) \). A \( \lambda \)-measure is a measure which is satisfied in \( \lambda \)-rule and there exists \( E \in \Lambda \) such that \( \mu(E) < \infty \). The following Theorem tells us how to determine the values of \( \lambda \).

Theorem 2.1. Let \( X = \{x_1, x_2, ..., x_n\} \), \( n \geq 2 \), \( g_\lambda \) on \( P(X) \) be a \( \lambda \)-measure with \( g_\lambda(x_i) = a_i \geq 0 \), \( i = 1, 2, ..., n \) and \( g_\lambda(X) = b > a_i \); then \( \lambda \) obtained uniquely.
Finally, a monotone measure \( \mu : \Lambda \rightarrow [0, 1] \) is called fuzzy measure which is applied in fuzzy integral.

**Definition 2.2.** [2] Let \( C = \{c_1, c_2, ..., c_n\} \) be the criteria set, \( f(c_i) \) be evaluation values with respect to \( i \)th criterion and \( \mu(c_i) = w_i \) be the weight of \( i \)th criterion. Then the Choquet integral of \( f \) with respect to fuzzy measure \( \mu \) obtained as

\[
(C) \int f d\mu = \sum_{i=1}^{n} f(c_i^+) \cdot [\mu((c_i^+, c_{i+1}^+), ..., c_n^+)) - \mu((c_{i+1}^+), ..., c_n^+))],
\]

where \((c_1^+, c_2^+, ..., c_n^+)\) is a permutation of \((c_1, c_2, ..., c_n)\) such that \( f(c_1^+) \leq f(c_2^+) \leq ... \leq f(c_n^+) \) and \( \mu((c_i^+, c_{i+1}^+), ..., c_n^+) \) with \( \mu(c_{n+1}) = 0 \) are joint measure of \( c_i^+, c_{i+1}^+, ..., c_n^+ \) which obtained based on the concept of \( \lambda \)-fuzzy measure.

TOPSIS which is a more useful method in MADM problems, is based on the following steps [2]:

- **Step 1** Normalize the decision matrix to \( N = [v_{ij}]_{m \times n} \).
- **Step 2** Calculate \( NW = [v_{ij} \times w_i]_{m \times n} = [v_{ij}] \).
- **Step 3** Determine the PIS and NIS as two subjective alternatives:
  \[
  A^+ = \{d_1^+, d_2^+, ..., d_m^+\} = \{(max, v_{ij} | j \in B) \& (min, v_{ij} | j \in C)\},
  A^- = \{d_1^-, d_2^-, ..., d_m^-\} = \{(min, v_{ij} | j \in B) \& (max, v_{ij} | j \in C)\}.
\]
- **Step 4** Calculate the Euclidean distance of each alternative from subjective alternatives i.e for \( i = 1, 2, ..., m \):
  \[
  S_i^+ = \sqrt{\sum_{j=1}^{m} (v_{ij} - d_j^+)^2}, \quad S_i^- = \sqrt{\sum_{j=1}^{m} (v_{ij} - d_j^-)^2}.
  \]
- **Step 5** Calculate \( c_i = \frac{S_i^-}{S_i^+ + S_i^-} \), \( i = 1, 2, ..., m \) and rank them.
- **Step 6** Reorder the alternatives according to \( c_i \)'s ranking.

W. J. Qiang and Z. Zhong introduced Trapezoidal Intuitionistic Fuzzy Numbers (TrIFNs) to model uncertainty:

**Definition 2.3.** [4] A TrIFN such as \( \bar{A} \) represented by \( \bar{A} = (\{b_1, a_1, b_2, a_2, a_3, b_3, a_4, b_4\}, w_a, w_u) \), where \( b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4 \), is a special case of IFN with membership function \( \mu_{\bar{A}} \) and nonmembership function \( \nu_{\bar{A}} \) which are defined as:

\[
\mu_{\bar{A}}(x) = \begin{cases} 
0, & x < a_1 \text{ or } x > a_4 \\
\frac{x-a_1}{a_2-a_1} w_a, & a_1 \leq x < a_2 \\
w_a, & a_2 \leq x < a_3 \\
\frac{a_4-x}{a_4-a_3} w_a, & a_3 \leq x < a_4 
\end{cases}
\]

\[
\nu_{\bar{A}}(x) = \begin{cases} 
1, & x < b_1 \text{ or } x > b_4 \\
\frac{b_2-x+u_a(x-b_1)}{b_2-b_1} u_a, & b_1 \leq x < b_2 \\
u_a, & b_2 \leq x < b_3 \\
\frac{x-b_3+u_a(b_4-x)}{b_4-b_3} u_a, & b_3 \leq x < b_4.
\end{cases}
\]

The Hamming distance and Euclidean distance between TrIFNs are defined as follow:
Definition 2.4. [3] Let \( \tilde{A} = [a_1, a_2, a_3, a_4; \mu_1, \nu_1] \) and \( \tilde{B} = [b_1, b_2, b_3, b_4; \mu_2, \nu_2] \) be TriFNs, then Hamming distance and Euclidean distance between them, which are displayed by \( d_h(\tilde{A}, \tilde{B}) \) and \( d_e(\tilde{A}, \tilde{B}) \) respectively, are defined as follow:

\[
d_h(\tilde{A}, \tilde{B}) = \frac{1}{4}[(\mu_1 - \nu_1)a_1 - (\mu_2 - \nu_2)b_1] + [(\mu_1 - \nu_1)a_2 - (\mu_2 - \nu_2)b_2] + [(\mu_1 - \nu_1)a_3 - (\mu_2 - \nu_2)b_3] + [(\mu_1 - \nu_1)a_4 - (\mu_2 - \nu_2)b_4],
\]

\[
d_e(\tilde{A}, \tilde{B}) = \frac{1}{4}[(\mu_1 - \nu_1)a_1 - (\mu_2 - \nu_2)b_1]^2 + [(\mu_1 - \nu_1)a_2 - (\mu_2 - \nu_2)b_2]^2 + [(\mu_1 - \nu_1)a_3 - (\mu_2 - \nu_2)b_3]^2 + [(\mu_1 - \nu_1)a_4 - (\mu_2 - \nu_2)b_4]^2.
\]

3 TOPSIS-CI Method

In this section we combine TOPSIS and CI methods as the most widely used methods and introduce a new method named TOPSIS-CI. Let \( A = [\tilde{a}_{ij}]_{m \times n} \) be decision matrix, where \( \tilde{a}_{ij} = [a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4] \) are trapezoidal fuzzy numbers, \( B \) be the set of benefit criteria and \( C \) be the set of cost criteria, then Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS) are defined as following, respectively:

\[
A^+ = \{ \tilde{a}_{ij}^+, \tilde{a}_{ij}^2, \ldots, \tilde{a}_{ij}^n \} = \{ (\max_i \tilde{a}_{ij}^1, \max_i \tilde{a}_{ij}^2, \max_i \tilde{a}_{ij}^3, \max_i \tilde{a}_{ij}^4), \max_i \nu_{ij}, \min_i \mu_{ij} \mid j \in B \} \text{ and } \{ (\min_i \tilde{a}_{ij}^1, \min_i \tilde{a}_{ij}^2, \min_i \tilde{a}_{ij}^3, \min_i \tilde{a}_{ij}^4), \min_i \nu_{ij}, \max_i \mu_{ij} \mid j \in C \}.
\]

\[
A^- = \{ \tilde{a}_{ij}^-, \tilde{a}_{ij}^2, \ldots, \tilde{a}_{ij}^n \} = \{ (\min_i \tilde{a}_{ij}^1, \min_i \tilde{a}_{ij}^2, \min_i \tilde{a}_{ij}^3, \min_i \tilde{a}_{ij}^4), \min_i \nu_{ij}, \max_i \mu_{ij} \mid j \in B \} \text{ and } \{ (\max_i \tilde{a}_{ij}^1, \max_i \tilde{a}_{ij}^2, \max_i \tilde{a}_{ij}^3, \max_i \tilde{a}_{ij}^4), \max_i \nu_{ij}, \min_i \mu_{ij} \mid j \in C \}.
\]

Then, we define distance matrices \( D^+ = [d^+_{ij}]_{m \times n} \) and \( D^- = [d^-_{ij}]_{m \times n} \) from PIS and NIS, based on defined distances between IFNs, where \( d^+_{ij} \) is the distance between \( \tilde{a}_{ij}^+ \) and \( \tilde{a}_{ij}^- \), \( d^-_{ij} \) is the distance between \( \tilde{a}_{ij}^- \) and \( \tilde{a}_{ij}^- \) for \( j = 1, 2, \ldots, n \). Finally, each row of matrices \( D^+ \) and \( D^- \) will be aggregated using Choquet integral. The aggregation distance of \( i \)th alternative, for \( i = 1, 2, \ldots, n \) from PIS and NIS alternatives, which are denoted by \( c_i^+, c_i^- \) respectively, are calculated as:

\[
(C) c_i^+ = \int d^+_{ij}d\mu = \sum[d^+_{i(j)} - d^+_{i(j-1)}]\mu\{(j), (j + 1), \ldots, (n)\},
\]

\[
(C) c_i^- = \int d^-_{ij}d\mu = \sum[d^-_{i(j)} - d^-_{i(j-1)}]\mu\{(j), (j + 1), \ldots, (n)\}.
\]

Then, the alternatives will be ranked based on ranking order

\[
s_i = \frac{c_i^-}{c_i^- + c_i^+}, \quad i = 1, 2, \ldots, m.
\]

4 Numerical Example

In this section, we solve an example, adapted from [2], which the assessment are expressed by TriFNs.

Example 4.1. Consider four banks \( A, B, C \) and \( D \) to be ranked based on four criteria \( c_1, c_2, c_3 \) and \( c_4 \) which, the importance of each criteria is expressed by \( w = (0.2, 0.35, 0.15, 0.35) \). The decision matrix is
denoted as

\[
A = \begin{pmatrix}
(6.7, 7.0, 7.5, 9.0); .60, .30, (8.7, 9.0, 9.10); .80, .15 \\
(3.7, 4.0, 5.0, 5.5); .80, .15, (3.7, 4.0, 5.0, 5.5); .40, .57 \\
(1.7, 2.3, 3.0, 4.0); .35, .55, (6.7, 7.0, 7.5, 9.0); .85, .05 \\
(6.7, 7.0, 7.5, 9.0); .85, .05, (6.7, 7.0, 7.5, 9.0); .55, .35 \\
(1.7, 2.3, 3.0, 4.0); .50, .40, (6.7, 7.0, 7.5, 9.0); .60, .30 \\
(6.7, 7.0, 7.5, 9.0); .70, .25, (8.7, 9.0, 9.10); .80, .15 \\
(8.7, 9.0, 9.10); .60, .15, (3.7, 4.0, 5.0, 5.5); .90, .057 \\
(8.7, 9.0, 9.10); .70, .15, (3.7, 4.0, 5.0, 5.5); .40, .35 \\
\end{pmatrix}
\]

Subjectives alternatives PIS and NIS are obtained as

\[
PIS = \left( \frac{(6.7, 7.0, 7.5, 9.0); .85, .05}{} \right) \\
(8.7, 9.0, 9.10); .35, .55, (3.7, 4.5, 5.5); .4.57 \\
(1.7, 2.3, 3.4); .5.4 \\
(3.7, 4.5, 5.5); .4.35 \right).
\]

Based on the Humming distance, we calculate positive and negative distance matrices from PIS and NIS, which are displayed by PD and ND as follow:

\[
PD = \begin{pmatrix}
3.78 & 1.38 & 4.78 & 5.47 \\
3.10 & 8.11 & 1.65 & 1.77 \\
6.59 & 1.30 & 0.92 & 3.90 \\
0.00 & 5.83 & 0.00 & 7.51 \\
\end{pmatrix}, \quad
ND = \begin{pmatrix}
2.82 & 6.74 & 0.00 & 2.04 \\
3.51 & 0.00 & 3.12 & 5.74 \\
0.00 & 6.81 & 3.85 & 3.61 \\
6.59 & 2.28 & 4.77 & 0.00 \\
\end{pmatrix}.
\]

After aggregation of each row of these matrices based on CI method and then construct the relative distance (RD) from NIS based on TOPSIS method, we have \(RD = \{0.4873, 0.4243, 0.5633, 0.3778\}\), and then \(D < B < A < C\).

References

BEE ALGORITHM FOR SOLVING AN INVERSE PARABOLIC SYSTEM

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In this paper a numerical approach combining the least squares method and bee algorithm is proposed for the determination of the source term in an inverse parabolic system (IPS). A numerical experiment confirm the utility of this algorithm as the results are in good agreement with the exact data. Results show that a reasonable estimation can be obtained by the bee colony algorithm within a Intel(R) core(TM) i3 cpu with 2.20 GHz.

Keywords Bee algorithm, The least squares method, Inverse parabolic system.

1 Introduction

Most of scientific problems and phenomena especially in mechanical engineering occur parabolic problems.

In parabolic problems, we are usually facing a problem where, problem conditions, initial conditions and boundary conditions, are identified and in the main equation only the equation main function is unknown. In fact, there is just one unknown factor at the problem. These problems are called direct problems. On the contrary there is another category of problems wherein, addition to unknown main factor at the equation, there are other characteristics at equation at its conditions. This type of problems, are called inverse problems[1].

In this paper, an inverse parabolic system(IPS) is considered and solved by bee colony algorithm. Inverse parabolic systems arises, for example, in the study of chemical reactions, and in a wide variety of mathematical biology and physical situations.

we consider the following IPS , for $0 < x < 1, 0 < t < t_M$.

\[
\begin{aligned}
U_t(x, t) - U_{xx}(x, t) + a(x, t)U(x, t) + b(x, t)V(x, t) &= f(x, t), \\
V_t(x, t) - V_{xx}(x, t) + a(x, t)V(x, t) + b(x, t)U(x, t) &= g(x, t), \\
U(x, 0) &= w_1(x), \\
V(x, 0) &= w_2(x), \\
0 &\leq x \leq 1, \\
U(0, t) &= p_1(t), \\
V(0, t) &= p_2(t), \\
0 &\leq t \leq t_M, \\
U(1, t) &= q_1(t), \\
V(1, t) &= q_2(t), \\
0 &\leq t \leq t_M,
\end{aligned}
\] (1.1a)

and the overspecified condition

\[ s(t) = U(a, t), \quad 0 \leq t \leq t_M, \] (1.1c)

where $g(x, t), a(x, t)$ and $b(x, t)$ are continuous known real-valued functions, $w_1(x), w_2(x), p_1(t), p_2(t), q_1(t)$ and $q_2(t)$ are infinitely differentiable known real-valued functions and $t_M$ represents the final existence time for the time evolution of the problem, while function $f(x, t)$ is unknown which remains to be determined from some interior temperature measurements. In this paper we will numerically investigate our IPS. Problem (1.1) can be solved in least-square sense and a cost function for bee algorithm, can be defined as a sum of squared differences between measured temperatures and calculated values of $U(x, t)$ by considering guesses estimated values of $f(x, t)$ by bee algorithm.

\[ f(\text{Guesses estimated values of } f(x, t)) = \sum_{j=1}^{m} (U_j - s_j)^2, \] (1.2)

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where $U_j, j = 1, 2, 3, ..., m$, are calculated by solving the direct heat problem. To do this, we consider prior guess for $f(x, t)$. Also $s_j = s(t_j), j = 1, 2, 3, ..., m$, are measured temperatures. To find optimal solution $f(x, t)$, equation (1.2) must be minimum.

2 Bee algorithm

In computer science and operations research, the Bees Algorithm is a population-based search algorithm which was developed in 2005. The algorithm starts by scout bees being placed randomly in the search space. Then the fitnesses of the sites visited by the scout bees are evaluated and Bees that have the highest fitnesses are chosen as selected bees and sites visited by them are chosen for neighbourhood search. Then, the algorithm conducts searches in the neighbourhood of the selected sites, assigning more bees to search near to the best sites. Searches in the neighbourhood of the best sites are made more detailed by recruiting more bees to follow them than the other selected bees. Together with scouting, this differential recruitment is a key operation of the Bees Algorithm. The remaining bees in the population are assigned randomly around the search space scouting for new potential solutions. These steps are repeated until a stopping criterion is met. At the end of each iteration, the colony will have two parts, those that were the fittest representatives from a patch and those that have been sent out randomly. the algorithm performs a kind of neighbourhood search combined with random search and can be used for both combinatorial and functional optimization [2]. Algorithm 21 shows the pseudo code of basic bees algorithm.

Algorithm 21 Pseudo code of the basic bees algorithm

Step 1: (Initialization) Initialize population with random solutions.

Step 2: Evaluate fitness of the population.

Step 3: While (stopping criterion not met)
//Forming new population.

Step 4: Select sites for neighbourhood search.

Step 5: Recruit bees for selected sites (more bees for best e sites) and evaluate fitnesses.

Step 6: Select the fittest bee from each patch.

Step 7: Assign remaining bees to search randomly and evaluate their fitnesses.

Step 8: End While.

3 Main Results

The aim of this section is to see the applicability of the present algorithm described in Section 2 for solving our IPS. Asexpected IPS (1.1a) is ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem.
Now, we give the following examples in $0 < x < 1, 0 < t < 1$.

\[
\begin{align*}
U_t(x, t) - U_{xx}(x, t) + 2U(x, t) + 3V(x, t) &= f(x, t), \\
V_t(x, t) - V_{xx}(x, t) + 2V(x, t) + 3U(x, t) &= 0
\end{align*}
\]  

(3.1a)

\[
\begin{align*}
U(x, 0) &= e^{-2x} + e^{2x} - \cos(\sqrt{2}x) - \sin(\sqrt{2}x), \\
V(x, 0) &= 1/3e^{-x} + e^{-2x} + e^{2x} - \cos(\sqrt{2}x) + \sin(\sqrt{2}x), \\
U(0, t) &= e^{-t}, \\
V(0, t) &= (10/3)e^{-t}, \\
U(1, t) &= (1123/176)e^{-t}, \\
V(1, t) &= (3807/449)e^{-t},
\end{align*}
\]  

(3.1b)

by the overspecified condition

\[
s(t_j) = U(0.5, t_j) + \sigma R, \quad t_j = 0.05 \times j, \quad j = 0, 1, 2, \ldots, 20,
\]  

(3.1c)

Here the exact values of $f(x, t)$, $U(x, t)$ and $V(x, t)$ are $e^{-t}e^{-x}, e^{-t}(e^{-2x} + e^{2x} - \cos(\sqrt{2}x) - \sin(\sqrt{2}x))$ and $e^{-t}(1/3e^{-x} + e^{-2x} + e^{2x} - \cos(\sqrt{2}x) + \sin(\sqrt{2}x))$, respectively.

**Remark 3.1.** In an IPS there are two sources of error in the estimation. The first source is the unavoidable bias deviation (or deterministic error). The second source of error is the variance due to the amplification of measurement errors (stochastic error). The global effect of deterministic and stochastic errors is considered in the mean squared error or total error, [3].

\[
S = \left[ \frac{1}{(N-1)(M-1)} \sum_{i=1}^{N} \sum_{j=1}^{M} (\hat{f}_{i,j} - f_{i,j})^2 \right]^{\frac{1}{2}},
\]  

(3.2)

where $(N-1)(M-1)$ is the total number of estimated values, $\hat{f}_{i,j}$ is calculated values from interpolated equation and $f_{i,j}$ is exact values of $f(x, t)$.

In our example, a population of 40 individuals is used. Each individual estimates unknown $f(x, y)$ and fitness function is 1.2. Table 1 shows the results for 1000, 5000 and 10000 iterations of bee algorithm. Also figure 1 and 2 show exact and numeric $f(x, t)$ for 10000 iterations.

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Best fitness</th>
<th>Time(s)</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>0.00811</td>
<td>61.47</td>
<td>0.0526</td>
</tr>
<tr>
<td>5000</td>
<td>0.00349</td>
<td>561.90</td>
<td>0.0469</td>
</tr>
<tr>
<td>10000</td>
<td>0.00077</td>
<td>5323.52</td>
<td>0.0048</td>
</tr>
</tbody>
</table>

Table 1: The results of bee algorithm for a population of 40 for determining unknown $f(x, t)$.
Figure 2: Numeric $f(x, t)$.

References


One of the important key of the supply chain performance is its service level which is directly can be affected by variable demands, lead times, traveling times and production times. For this reason, we need to stock the inventory in order to meet costumers demand at the particular time. The focus of this paper is on inventory positioning based on the given configuration of the supply chain, especially for multi-echelon networks, to minimize the holding costs. Inventory positioning policy allows the supply chain to be responsive to uncertainty and variability. Therefore, the main challenge in the inventory positioning field of study is about the question of where and how much inventories should we stock in the supply chain stages to face the uncertainty. The structure of this work is that we first address the problem and the existing approaches which are known as stochastic-service and guaranteed-service models, and then introduce the assumptions and notations of the proposed optimization model as well as the constraints in order to find the best positions of inventory to be stocked at the minimum holding costs.

Keywords supply chain network; inventory positioning; stochastic-service model; guaranteed-service model; base-stock inventory.

1 Introduction

Supply chain network consists of nodes correspond to stages which are the candidate places for inventory positioning and could represent anything from a single step in manufacturing to collection of steps of assembly and test operations. It also contains arcs that are denote the precedence relationship between stages. We have two approaches, guaranteed-service and stochastic-service models, for optimizing inventory positioning that are differ in the way of replenishment policy. In the stochastic-service model, delivery time to one stage can change based on the availability of materials or machinery at its upper stages while in the guaranteed-service model, we have deterministic delivery time between stages. So despite the guaranteed-service model, the replenishment time for the stage in the stochastic-service model is stochastic [1]. In the stochastic service time, each stage has to take into account the possibility that its upstream suppliers meet the demand from safety stock inventories. Therefore, it may make a situation that the supplier will not be able to meet all the demand form its stock and hence, impose a delay to the stage for receiving the materials. This stochastic delay, makes the replenishment time to be stochastic. For more information around stochastic service time see [2]. But in the guaranteed-service model each stage has a commitment to provide the materials in a deterministic lead time through out-sourcing or stocking sufficient inventory. In both approaches, each stage reacts in a same way under all the demand conditions. As a result, the safety stock policy is only designed to meet a portion of the demand.

At the recent works, it is assumed that each stage does not stock any initial inventory and the travel time between the stages are zero. We address the initial inventory as a decision variable in our work in order to find the better inventory policy as well as considering travel times.

In the following sections the details of the proposed model are provided in more details.
2 Assumptions

The assumption of the model we proposed is as follows.

1. Decentralized control: there is no central decision maker controlling the performances of all the stages. Therefore, each stage manages its own inventory with regard to the control policies of its adjacent upstream and downstream stages.

2. Network topology: we assume acyclic network which has no directed cycle. Thus, receiving material from downstream stages such as returning the defected material is not allowable.

3. Stationary demand: demand is stationary with mean of $\mu$ and standard deviation of $\sigma$ for each period of time.

4. Deterministic lead time: each stage has constant process time which includes waiting time, production time and travel time that provide deterministic lead time for its downstream stages. Thus, we follow the guaranteed-service time policy.

5. Variety of final goods: we consider one type of good is prepared at the demand node. Demand node is the stage that has direct connection with customers.

6. Production capacity: each stage has capacitated machinery that leads to limited production capacity. The importance of considering this assumption is because of its effect on the amount of base-stock inventory.

7. Inventory control policy: since we have decentralized control, each stage in the network has its own control policy. Periodic-review base-stock is the common policy implemented at each stage of single echelon network. For multi echelon network, shared stocking policy is more useful due to the effect of risk pooling. These mentioned policies are not the only valid ones implemented in the networks. We can determine a new policy based on the specific network we are dealing with. For more information, readers can see [3] and [4]. But in our proposed model, we also consider the shared stocking policy for controlling the inventory.

3 Notations and Decision Variables

We defined the following notations and parameters for formulating the problem.

- $N$: Set of stages in the network; $N = \{1, 2, \ldots, n\}$ with indices $i, j$.
- $D$: Set of demand nodes in the network.
- $A$: Set of arcs in the network;
- $p_i$: Processing time of stage $i$.
- $t_{ji}$: Travel time to stage from stage $i$.
- $h_i$: Holding cost at stage $i$.
- $g_i$: Guaranteed service time to end customers at demand node $i$.
- $\sigma_i$: The standard deviation of demand at stage $i$.
- $c_i$: Production capacity of stage $i$.

Decision variables are as follows.
$S_i$ The outbound service time at stage $i$. It means that stage should send all the material to its downstream stages at time.

$SI_i$ The inbound service time at stage $i$. It means that stage $i$ should receive all the materials from its upstream stages at time. So it gets ready to start its production at this time.

$k_i$ Safety factor of stage $i$.

4 Mathematical Formulation

Our proposed model for formulating the inventory positioning problem at the guaranteed-service time is as follows.

$$
\text{min} \quad \sum_{i \in N} \sum_{j \in (i,j)} h_i k_i \sigma_i \sqrt{SI_i + p_i - S_i - t_{ij}} + \sum_{i \in N} \sum_{j \in (j,i)} h_i k_i \sigma_i \sqrt{S_j + t_{ji} - SI_i} \\
\text{s.t.} \quad SI_i + p_i \geq S_i + t_{ij} \quad \forall i \in N \quad \forall (i, j) \in A \tag{4.1}
$$

$$
S_j + t_{ji} - SI_i \geq 0 \quad \forall (j, i) \in A \tag{4.2}
$$

$$
S_i \leq g_i \quad \forall i \in D \tag{4.3}
$$

$$
SI_i + p_i - S_i - t_{ij} \leq c_i \quad \forall i \in N \tag{4.4}
$$

$$
k_i \geq \frac{z(1-\alpha)}{2} \quad \forall i \in N \tag{4.5}
$$

$$
S_i, SI_i \geq 0 \quad \forall i \in N \quad \forall (j, i) \in A \tag{4.6}
$$

The objective function of the model is the minimization of total holding costs that include initial inventory and based-stock inventory costs shown in (1.1). Since stage starts its task after receiving all the materials, $SI_i$, the replenishment time is at $SI_i + p_i - S_i - t_{ij}$ time. This expression should always be a nonnegative one to avoid extra holding cost. Hence, if $S_i + t_{ij} \geq SI_i + p_i$ happens, we have to postpone the $SI_i$ time. This fact is expressed in constraint (1.2). There is a situation that we can stock an initial inventory in order to start the production before receiving the ordered materials. Due to the holding cost of the initial inventory, the amount of it should be determined through the model. In constraint (1.3), $S_j + t_{ji}$ denotes the receiving time of materials stage $i$ ordered to stage $j$. Therefore, stage $i$ needs to store materials for $S_j + t_{ji} - SI_i$ period of time to meet the lead time it promises to its downstream stages. Demand nodes are the ones with direct connection to the end costumers. Constraint (1.4) shows that outbound service time of these nodes should not exceed the guaranteed service time. Production capacity is limited via constraint (2.1). Minimum safety factor is also controlled with constraint (2.3) and finally, decision variables bounds are determined in constraint (2.4).

Acknowledgement

In performing our work, we had to take the help and guidelines of a respected person, who deserve our greatest gratitude. The completion of this work makes a strong impression on us to continue this filed seriously. Hence, we would like to express our sincere gratitude to Dr. Farzad Dehghanian, assistant professor of industrial engineering, Ferdowsi University of Mashhad for giving us the munificent consultations for this work.
References


SOLVING THE PROBLEM OF GATE ASSIGNMENT TO THE FLIGHTS USING A MODIFIED SHUFFLED FROG LEAPING ALGORITHM

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Gate is one of the most important sources in an airport. Gate assignments to the flights have influence on passengers’ satisfaction, air traffic, increasing agencies performance; flights delay time and airport expenses. Something that complicates assignment problem is those unexpected changes in which the best gate reassignment should be done as soon as possible. Most of studies on gate assignment are based on mathematical models and since assignment of \( M \) gates to \( N \) flights has \( M^N \) possible cases, it is included in an NP-hard problem. It means in order to solve problem, meta-heuristics should be applied.

**Keywords** Gate assignment; Meta-heuristics; Frog Leaping algorithm.

1 Introduction

With the ever-increasing development of technology, the need for effective global communication, and the high demand for aviation facilities, airports are considered to be critical factors relating to social welfare and the economic state. The complexity of flight schedules in metropolitan airports has resulted in the gate assignment problem becoming of particular importance.

Gate assignment operations should be carried out in the shortest time due to the probability of flight disruption. As the gate assignment and reassignment problem is non-deterministic polynomial-time hard, it is impossible to solve the problem with classical solution methods, which is why a MSLF algorithm has been applied.

2 Literature review


3 PROBLEM STATEMENT

Here we introduce the mathematical model.
3.1 Parameters

\( f_i \) = Number of passengers;
\( f_{ik} \) = Number of flight \( i \) passengers to flight \( k \);
\( w_j \) = Gate \( j \) passengers' walking distance;
\( w_{jl} \) = Transferring passengers; walking distance from gate \( j \) to gate \( l \), \( l:1,2,... \);
\( A_i \) = Scheduled arrival time for flight \( i \);
\( G_i \) = Sufficient stand time for flight \( i \) in gate;
\( T_i \) = Latest available time for gate \( j \) at the end of the timetable;
\( B_j \) = Earliest available time for gate \( j \) at the beginning of the timetable;
\( m \) = Available gates in the scheduled time with index \( j \);
\( n \) = Flights Number in the scheduled time with index \( i, k \);
\( H_j \) = Total working time for each gate;
\( \bar{H} \) = Mean working time of the gates throughout the planning period.

3.2 Variables

\( A_{ij} \) = Arrival time of flight \( i \) to gate \( j \);
\( X_{ij}, X_{kl} \) = Decision variable zero; which is one if flight \( i \) or \( k \) is assigned to gate \( j \) or \( l \) is equal to one, otherwise it is zero.
\( Y_{ijkl} \) = Decision variable zero; which is one if flight \( i \) is assigned to gate \( j \) and flight \( k \) is assigned to gate \( l \) is equal to one, otherwise it is zero.

3.3 Model

\[
\begin{align*}
\text{Min} & \sum_{i=1}^{n} \sum_{j=1}^{m} (f_i - \sum_{k=1}^{n} f_{ik})w_j x_{ij} + \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{m} \sum_{l=1}^{m} f_{ik} w_{jl} y_{ijkl} \\
\text{Min} & \sum_{i=1}^{n} \sum_{j=1}^{m} (A_{ij} - A_i x_{ij}) \\
\text{Min} & \sum_{j=1}^{m} |H_j - \bar{H}| \\
\text{s.t.} & \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij} = 1 \\
& \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m} Y_{ijkl} = 1 \\
& X_{ij} + X_{kl} - 2 \cdot Y_{ijkl} \geq 0 \quad i \neq k \quad j \neq l \\
& A_{ij} \leq T_j x_{ij} \\
& B_j x_{ij} \leq A_{ij} \\
& A_{ij} \geq A_k x_{ij} \\
& A_{ij} + G_i x_{ij} \leq A_{kj} + (1 - x_{kj}) M \quad \forall i < k \\
\end{align*}
\]
\[ H = \frac{\sum_{i=1}^{n} G_i}{m} = \frac{\sum_{j=1}^{m} H_j}{m} \]  

\[ X_{ij} = (0, 1) \]  

\[ Y_{ijkl} = (0, 1) \]  

\[ A_{ij} \geq 0 \]

The first objective calculates summation of walking distance for non-transferring and transferring passengers. The second objective is minimizing delay times occurring due to the gate occupation. The third objective was to minimize each gate’s work time deviation from the mean of all gates’ work times. According to the equation (1), every flight is assigned to one gate. In the equation (5) and (6), \( Y_{ijkl} \) is equal to unit constant if both flight \( i \) and \( k \) are assigned to gate \( j \) and \( l \). Equations (7) and (8) show that the arrival time for each flight must be in planning duration. In equation (9), flights are not allowed to enter before a defined time in the programming schedule. Equation (10) emphasises the non-overlapping flights in gates. Equation (11) shows that gate \( j \) occupancy is equal to total occupancy time in that duration. Equation (12) calculates average gates occupancy time in the whole time table.

4  SOLVING THE MULTIPURPOSE OF MATHEMATICS USING L-P METRIC METHOD

L-P metric method is one of the methods to find the Parreto answers (non-dominant) for the multipurpose problem.

\[ L_p = \left\{ \sum_{j=1}^{k} \gamma_j \left[ \frac{f_j(x_j^*) - f_j(x)}{f_j(x_j^*) - f_j(\tilde{x}_j^*)} \right]^p \right\}^{\frac{1}{p}} \]  

\( f_j(x_j^*) \) is the best objective amount and \( f_j(\tilde{x}_j^*) \) is the worst objective amount. The solutions of L-P metric problems are the solutions of Parreto of multipurpose problem and the solutions for the amounts of \( p \) are calculated as 1, 2 or infinite.

5  MSFL DESCRIPTION

SFLA is a population based optimization algorithm inspired from the memetic evolution of a group of frogs when searching for food. Since the algorithm has an insufficient learning mechanism and cause premature convergence and lead the algorithm to be trapped in local optimum easily, a new method is presented for local search in the memeplexes. The steps of the MSFLA is given as below:

1- Initialize (Population size (N), Number of memeplexes (m), Number of evolution step within each memeplex);
2- Generate population (P) randomly;
3- Evaluate the fitness of (P);
4- Sort (P) in descending order;
5- Partition (P) into m memeplex;
6- LOCAL search;
7- Shuffle the memeplexes;
8- If convergence criteria is satisfied, then return the best solution, else go to step (4);
9- End.
6. CONCLUSION

6.1 METHA-HEURISTIC ALGORITHM OF MSLF

The complexity of airport operations and increase flights to major airports and traffic of them caused gate assignment problem is of particular importance. These operations should be carried out in the shortest time. As the gate assignment and reassignment problem is NP-hard, it is impossible to solve the problem with classical solution methods. In order to investigate the model objectives namely minimizing the walking distance, minimizing the delay times and minimizing each gate’s work time deviation from the mean of all gates’ work times, a case study research has been conducted in one of the international airports in Iran. The result of comparison of the problem answer in small scale in Lingo and MSLF algorithm was very desirable. Also the problem was investigated with the real scale and data and the results indicated costs decrease in a logical period of time.

7 REFERENCES


In this paper Nelder-Mead(NM) simplex search method combines with least squares method for the determination of temperature in an inverse heat conduction problem (IHP). The performance of NM is established with an examples of IHP. Numerical results are obtained by implementation NM on 2.20GHz clock speed CPU. 

**Keywords** Nelder-Mead simplex search method, The least squares method, inverse heat conduction problem.

### 1 Introduction

Most phenomena in the real world can be described through heat conduction equations which have attracted lost of attention from among scientists. For example, consider that a room begins to war up because the sun is shining into it. Apparently, the physical property of interest here is the temperature which is a function of the time and space. The temperature variations can be represented by a heat conduction problem (HCP) as the governing equation. In heat conduction problems, there is a problem in the main equation of which the problem conditions, initial conditions, and boundary conditions can be identified, but the main function of which is unknown. In other words, there is one unknown factor in the problem equation. This type of problem is called direct problem. However, there is another type of problems wherein, in addition to the unknown main factor, there are other characteristics in the equation and its conditions. This form of problem is called inverse problems.

On the contrary there is another category of problems wherein, addition to unknown main factor at the equation, there are other characteristics at equation at its conditions. This type of problems, are called inverse problems[1].

In this paper, inverse heat conduction problems of the following form are dealt with:

\[
U_t(x, t) = U_{xx}(x, t), \quad 0 < x < 1, \quad 0 < t < t_M \tag{1.1a}
\]

\[
T(x, 0) = f(x), \quad 0 \leq x \leq 1, \tag{1.1b}
\]

\[
T(0, t) = p(t), \quad 0 \leq t \leq t_M, \tag{1.1c}
\]

\[
T(1, t) = q(t), \quad 0 \leq t \leq t_M, \tag{1.1d}
\]

and the overspecified condition

\[
U(a, t) = s(t), \quad 0 \leq t \leq t_M, \tag{1.1e}
\]

where \( f(x) \) is a continuous known function, \( p(t) \) and \( q(t) \) are infinitely differentiable known functions, and \( t_M \) represents the final existence time for the time evolution of the problem, whereas function \( q(t) \) is unknown and remains to be determined from some interior temperature measurements. In this paper, Nelder-Mead simplex search method is used to solve IHCPs. The purpose of the study was to find an unknown boundary condition in IHCPs by using over-specified condition. Problem (1.1) can be solved in least-square sense and a cost function can be defined as a sum of squared differences between measured temperatures and calculated values of \( U(x, t) \) by considering guesses as the estimated values of \( q(t) \).
where \( U(a,t_j), j = 1, 2, 3, \ldots, m, \) are calculated by solving the direct HCP. To this end, prior guess for \( q(t) \) can be considered. In addition, \( s_j = s(t_j), j = 1, 2, 3, \ldots, m, \) are measured temperatures at \( x = a. \) Equation (1.2) must be minimum so that the optimal solution of \( q(t) \) can be found.

2 Nelder-Mead simplex search method for solving IHCP

This simplex search method was first proposed by Spendley, Hext, and Himsworth, and was later refined by Nelder and Mead [2]. Their method is one of the most efficient pattern search methods currently available. This method is a derivative-free line search method particularly designed for traditional unconstrained minimization scenarios, such as the problems of nonlinear least squares, nonlinear simultaneous equations, and other types of function minimizations. In this method, for \( N \) vertices of an initial simplex, the cost function for each vertex is first evaluated. Then, the previous vertex is replaced by a better newly reflected point, which can be approximately located in the negative gradient direction. In the minimization problem with three initial simplex vertices, the method can be mentioned as follow[3]:

- \( x_h: \) Vertex with highest cost function value.
- \( x_s: \) Vertex with the second highest cost function value.
- \( x_l: \) Vertex with lowest cost function value.
- \( x_c: \) The centroid of vertices except \( x_h. \)

1. Reflection. Reflect \( x_h \) (figure1) and find \( x_0 \) such that

\[
x_0 = 2x_c - x_h
\]

Figure 1: Reflection \( x_h \) toward \( x_0. \)

2. If \( f(x_l) < f(x_0) < f(x_s), \) replace \( x_h \) by \( x_0 \) and return to step 1.

3. Expansion. If \( f(x_0) < f(x_l) \) then expansion operation makes \( x_{00} \) (figure2). Based on the function value, replace \( x_h \) by either \( x_0 \) or \( x_{00}. \) More specifically

(a) If \( f(x_{00}) < f(x_l), \) replace \( x_h \) by \( x_{00}. \)
(b) If \( f(x_{00}) > f(x_l), \) replace \( x_h \) by \( x_0. \)

Then, return to step 1.
4. Contraction. If \( f(x_0) > f(x_s) \), then, the contraction operation of \( x_{00} \) is performed (done) by considering the following two cases:

(a) If \( f(x_0) < f(x_h) \) (figure 3) find \( x_{00} \) such that

\[
x_{00} = \frac{1}{2} x_0 - \frac{1}{2} x_c
\]

(b) If \( f(x_0) \geq f(x_h) \) (figure 4) find \( x_{00} \) such that

\[
x_{00} = \frac{1}{2} x_h - \frac{1}{2} x_c
\]

There are two more cases to consider in using 4.a or 4.b

(c) If \( f(x_{00}) < f(x_h) \) and \( f(x_{00}) < f(x_0) \) then replace \( x_h \) by \( x_{00} \) and return to step 1.

(d) If \( f(x_{00}) \geq f(x_h) \) or \( f(x_{00}) > f(x_0) \) then reduce size of simplex by halving distances from \( x_l \) and return to step 1.

The process terminates when either the number of iterations has exceeded a preset amount, or the simplex size is smaller than a given value.

In this paper, an attempt was made to consider vectors that estimate unknown \( q(t) \) as the vertices of the simplex to be put in equation (1.2) as the cost function. The initial vertices were randomly generated. In order for the unknown \( q(t) \) to be found, the final vertex was interpolated at the end of NM simplex search method.
3 Main Results

An example may demonstrate, numerically, some of the results for the unknown boundary condition in the reverse problem (1.1) by NM simplex search method. In this example, let us consider the following one-dimensional IHCP,

\begin{align}
  U_t(x, t) + U(x, t)U_x(x, t) &= U_{xx}(x, t), \quad 0 < x < 1, \quad 0 < t < t_M \\
  U(x, 0) &= \frac{1}{2} - \frac{1}{2} \tanh\left(\frac{x}{4}\right), \quad 0 \leq x \leq 1, \\
  U(0, t) &= \frac{1}{2} - \frac{1}{2} \tanh\left(-\frac{t}{8}\right), \quad 0 \leq t \leq t_M, \\
  U(1, t) &= q(t), \quad 0 \leq t \leq t_M, \\
  s(t_j) &= U(0.5, t_j), \quad t_j = 0.05 \times j, \quad j = 0, 1, 2, \ldots, 20.
\end{align}

(3.1a) (3.1b) (3.1c) (3.1d) (3.1e)

and the overspecified condition

where the unknown function is the continuous function \( q(t) \) as, \( q(t) = \left(\frac{1}{2} - \frac{1}{2} \tanh\left(\frac{1}{4}(1 - \frac{t}{2})\right)\right) \). Figure 1 presents exact and numeric \( q(t) \) for 100 iterations by implementing Nelder-Mead simplex search method for the above example.

![Figure 5: Exact and numeric \( q(t) \) for 100 iterations by implementing Nelder-Mead simplex search method at example.](image)

References


A Numerical Study on Optimization of PMEDM Process Parameters for Ti-Co Alloy Using Genetic Algorithm

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Nowadays, electrical discharge machining (EDM) has become one of the most extensively used non-traditional material removal process. In the present work, a study has been made to model and optimize the process parameters of powder mixed electrical discharge machining (PMEDM). Metal removal rate (MRR) and electrode wear rate (EWR) have been considered, as the process characteristics, to plan and analyze the experiments. Grain size of the aluminum powder (S), concentration of the powder (C), discharge current (I) and pulse on time (T) are chosen as control variables to study the process performance. The experimental results are used to develop the regression models based on second order polynomial equations for the different process characteristics. Then, a genetic algorithm (GA) has been employed to determine optimal process parameters for any desired output values MRR and EWR.

Keywords

ABSTRACT

Nowadays, electrical discharge machining (EDM) has become one of the most extensively used non-traditional material removal process. In the present work, a study has been made to model and optimize the process parameters of powder mixed electrical discharge machining (PMEDM). Metal removal rate (MRR) and electrode wear rate (EWR) have been considered, as the process characteristics, to plan and analyze the experiments. Grain size of the aluminum powder (S), concentration of the powder (C), discharge current (I) and pulse on time (T) are chosen as control variables to study the process performance. The experimental results are used to develop the regression models based on second order polynomial equations for the different process characteristics. Then, a genetic algorithm (GA) has been employed to determine optimal process parameters for any desired output values MRR and EWR.

1 INTRODUCTION

Ti-Co Alloy is one of the most difficult-to-cut hot worked alloys. Formation of complex shapes (of this material) along with reasonable speed and surface finish is very difficult by traditional machining. Among the non-traditional methods of machining processes, electrical discharge machining (EDM) has drawn a great deal of attention because of its broad industrial applications including different dies and tools [1].

In the EDM, machining control variables include the work piece polarity, pulse on time, pulse off time, open discharge voltage, discharge current, dielectric fluid, grain size and concentration powder particles in the dielectric. Among these the most significant parameters are the followings [5]:

- Grain size of aluminum powder particles (S_{Al}, \mu m)
- Concentration of aluminum powder particles (C_{Al}, \text{g/l})
- The discharge current (I_p, A)
- The pulse on time (T_p, \mu S)

Cobalt bonded tungsten carbide is a widely used, high strength material produced by compacting techniques of powder metallurgy and high-temperature sintering. In the conventional EDM machining of this material, the machined surface would usually have a significant amount of cracks and spalling which decreases the hardness, wear and corrosion resistance of this alloy.
To enhance the machined surface properties and prevent the surface defects, a technique called powder mixed electrical discharge machining (PMEDM), is now being used. In this method, fine powder of a specific material (usually Aluminum) is mixed into the dielectric fluid of EDM to increase the process quality.

2 PROBLEM STATEMENT

Selection of appropriate machining parameters for any particular material in EDM is difficult, and relies heavily on operators’ experience [2]. Usually, the desired machining parameters are determined based on experience or handbook values. However, this does not ensure that the selected machining parameters result in optimal or near optimal machining performance for any given material and EDM environment. To resolve this problem, in the present study, a GA based optimization procedure has been utilized to determine the optimal machining parameters in the PMEDM of Tungsten-Cobalt alloy. The proposed approach can easily be extended to any other materials and machining environments. To achieve this, first based on experimental data a set of mathematical models have been develop to relate important process-control parameters (powder grain size (S), powder concentration (C), discharge current (I) and pulse on time (T)) to the machining response outputs (material removal rate (MRR) and electrode wear rate (EWR)). The adequacy of the proposed models is verified with ANOVA. Then, the developed models are embedded into a GA algorithm to determine the best EDM process parameters for any target values of machining outputs. Computational results show that GA method can be used effectively for solving such complicated and highly non linear equations in prediction and optimization of process parameters.

For illustrative purposes, the data obtained by Kung et. al [3] is used here. In our research, different regression models have been examined on the experimental data; among which the curvilinear model proves to be the best fit. The second order curvilinear mathematical models, representing the relationship between process parameters and EDM machining characteristics, can be stated as follows:

\[
MRR = -0.00751 + 0.02925 \cdot S + 0.00107 \cdot C + 0.02100 \cdot I + 0.0001 \cdot T + 0.00094 \cdot S^2 + 0.00030 \cdot C^2 + 0.00038 \cdot S \cdot I + 0.000004 \cdot S \cdot T + 0.000004 \cdot I \cdot T
\]

\[
EWR = 98.51523 - 61.85566 \cdot S - 0.40935 \cdot C + 11.27579 \cdot S^2 - 0.00024 \cdot C^2 + 0.3935 \cdot S \cdot C + 0.7200 \cdot S \cdot I + 0.0418 \cdot S \cdot T - 0.1800 \cdot C \cdot I - 0.00035 \cdot C \cdot T
\]

The separable influence of individual machining parameters and the interaction between these parameters are also investigated by using analysis of variance (ANOVA). This study highlights the development of mathematical models for investigating the influences of machining parameters on performance characteristics and the proposed mathematical models that proven to fit and predict values of performance characteristics close to those readings recorded experimentally with a 95% confidence interval.

The above models may now be used to predict the best process parameters in such a way that desired output values are obtained from the machining operations. For this purpose, we first define the prediction (error) function as follow:

\[
\text{Minimize } J = \alpha_1 \cdot \left( MRR - MRR_d \right)^2 + \alpha_2 \cdot \left( EWR - EWR_d \right)^2
\]

Where:

MRR and EWR are metal removal rate and electrode wear rate given by (1) and (2) respectively. In the same manner, MRRd and EWRd are the target output values for the electrical discharge machining operation. The objective is to set the process parameters at the levels that these machining characteristics are achieved.

The coefficients \( \alpha_1 \) and \( \alpha_2 \) represent weighing importance of different output parameters in the error function. In the prediction process, the purpose is to minimize this objective function. By doing so, the process parameters are calculated in such way that the PMEDM parameters approach their desired values. To minimize the above error function, a Genetic Algorithm is employed to find the best machining variables with respect to process specifications.
Genetic algorithm can be applied to solve a variety of combinatorial optimization problems including problems in which the objective function is discontinuous, non differentiable, or highly nonlinear [4]. A complete description of this algorithm and some of its applications can be found in [4].

3 RESULTS AND DISCUSSIONS

In this section a numerical example is presented to illustrate the performance of proposed procedure and solution technique. The target values for PMEDM are given in Table 2. Without loss of generality, in this example the values of all components of EDM (S, C, I, T) are considered to have the same importance and therefore, constants $\alpha_1$ and $\alpha_2$ are set to unity. The error functions of given in (3), along with PMEDM modeling (1) to (2) are embedded into GA algorithm. The tuning parameters for the algorithm are presented in Table 1. The objective is to minimize the error function which is used as the fitness criterion in evaluating of solutions in each generation.

**Table 1.** Genetic Algorithm parameters settings

<table>
<thead>
<tr>
<th>Mutation rate</th>
<th>Crossover mechanism</th>
<th>Crossover rate</th>
<th>Population size</th>
<th>No. of Generations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>Scatter</td>
<td>80%</td>
<td>30</td>
<td>800</td>
</tr>
</tbody>
</table>

The comparison of the calculated and desired values is shown in Table II. The largest error is around 5.5% while most parameters deviate from their desired values by less than 1%. These results illustrate that the proposed procedure can be efficiently used to determine optimal process parameters for any desired output values of PMEDM machining operations.

**Table2.** Comparison between target and calculated values

<table>
<thead>
<tr>
<th>No</th>
<th>Target MRR, EWR</th>
<th>Prediction MRR, EWR</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2103 23.12</td>
<td>0.2122 23.1407</td>
<td>0.8954 0.0895</td>
</tr>
<tr>
<td>2</td>
<td>0.1564 25.14</td>
<td>0.1611 25.1343</td>
<td>2.9174 0.0219</td>
</tr>
<tr>
<td>3</td>
<td>0.2044 21.44</td>
<td>0.2121 21.4463</td>
<td>3.6304 0.0294</td>
</tr>
<tr>
<td>4</td>
<td>0.2454 21.02</td>
<td>0.2484 20.9887</td>
<td>1.2077 0.1491</td>
</tr>
<tr>
<td>5</td>
<td>0.2278 22.23</td>
<td>0.2411 22.2068</td>
<td>5.5164 0.1045</td>
</tr>
<tr>
<td>6</td>
<td>0.1798 16.77</td>
<td>0.1822 16.7450</td>
<td>1.3172 0.1493</td>
</tr>
<tr>
<td>7</td>
<td>0.1989 21.62</td>
<td>0.2005 21.5303</td>
<td>0.7980 0.4157</td>
</tr>
<tr>
<td>8</td>
<td>0.2028 23.54</td>
<td>0.2025 23.5609</td>
<td>0.1481 0.0887</td>
</tr>
<tr>
<td>9</td>
<td>0.2213 23.52</td>
<td>0.2288 23.4803</td>
<td>3.0581 0.1682</td>
</tr>
</tbody>
</table>

4 CONCLUSIONS

The paper has presented the use of GA method for the optimization of the electrical discharge machining process with performance characteristics. The second order polynomial models developed for metal removal rate and electrode wear rate have been used for optimization. If the requirement is a lower electrode wear rate or higher material removal rate, a suitable combination of variables can be selected. Optimization will help to increase production rate considerably by reducing machining time.
5 REFERENCES


Poster Presentation, Day 2
Robust DEA Under Discrete Uncertain Data: An Application for Iranian Hospital Emergency Departments

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This paper presents a Data Envelopment Analysis (DEA) model with uncertain data for performance evaluation of the emergency department in Hospitals. The application of mathematical programming models in the important case study such as considering efficiency of emergency departments main contribution to this study. The paper used model basis of DEA with 4 scenarios for calculating the efficiency of 6 hospitals in Tehran. The result from the model indicates that 2nd DMU (hospital number 2) has better performance compared with other hospitals’ ED.

Keywords Robust DEA (RDEA); Discrete uncertain data; Healthcare; Emergency department (ED).

1 INTRODUCTION

Due to the uncertainty of the real world, the necessities of taking into account the uncertainty to increase the accuracy of the mathematical models is needed. There are many approaches of considering the imprecise data. For example, Sadjadi and Omrani [1] and Shokouhi et al. [2] proposed DEA models with uncertain data, which are based on Ben-Tal and Nemirovski [3] and Bertsimas et al. [4] approaches. Another method considered to discrete data is on scenario based, introduced by Mulvey et al. [5]. Hafezalkotob et al. [6] developed a model take into data envelopment analysis (DEA) to evaluate the Iranian electricity companies such Sadjadi el at. [1] case study. Hafezalkotob et al. [6] proposed a method based on scenario to calculate DMU’s efficiency under discrete data. They studied RDEA model to a scenario-based description of the uncertain data. By using robust optimization approach of Mulvey et al. [5], developed DEA formulation to consider discrete uncertainty in input and output parameters as a set of possible scenarios. They explored the effect of the discrete uncertain data on the degree of operational efficiency achieved by the Iranian electricity distribution companies. Conducted a literature review, we found a few studies have done in this area. This paper used proposed model of Hafezalkotob et al. [6] in another case study. Our primary concern in this study is evaluating the efficiency of emergency departments (EDs) under uncertainty about input/output data. The emergency department of a hospital is a complex unit where the fight between life and death is always a hair’s breadth away. Efficiency of emergency departments is one of the principal indicators of ED’s performance. There have been different ways to achieve efficiency, but among all methods, DEA methodology has been an effective approach for efficiency in healthcare [7]. Jong Soon Park et al. [8] measured and benchmarked the operating efficiency of a regionally-based hospital.

(The paper used proposed model of Hafezalkotob et al. [6] basis of DEA with 4 scenarios for calculating the efficiency of 6 hospitals in Tehran. Each hospital used as a DMU.)

2 THE MODEL

The paper used proposed model of Hafezalkotob et al. [6] basis of DEA with 4 scenarios for calculating the efficiency of 6 hospitals in Tehran. Each hospital used as a DMU. Regarding model (2.1) to (2.4), $x_{ij}$ and $y_{rj}$ are noted as input and output of $DMU_j$ ($j = 1, \ldots, n$), respectively (input set is $i = 1, \ldots, m$ and output set is $r = 1, \ldots, t$). A set of scenario of probable input and output data is indicated by
(2.1)
\[
\max \sum_{s \in \Omega} \sum_{r=1}^{t} p_s u_r y_{ros} - \gamma \sum_{s \in \Omega} p_s \delta_{so} - \lambda \sum_{s \in \Omega} p_s (Q_s^+ + Q_s^-)
\]
s.t.
\[
\sum_{i=1}^{m} v_i x_{io} + \delta_{so} = 1, \quad \forall s \in \Omega, \quad \text{(2.2)}
\]
\[
\sum_{r=1}^{t} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \delta_{sj} = 0, \quad \forall s \in \Omega, \quad j = 1, \ldots, n \quad \text{(2.3)}
\]
\[
\sum_{r=1}^{t} p_s u_r y_{ros} - p_s \left( \sum_{s' \in \Omega} \sum_{r=1}^{t} p_{s'} u_{r} y_{ros} \right) = Q_s^+ - Q_s^-, \quad \forall s \in \Omega, \quad v_i, u_r, \delta_{sj}, Q_s^+, Q_s^- \geq 0, \forall i, r, s \quad \text{(2.4)}
\]

With incidence probability \( p_s \), for each scenario \( s \in \Omega \). Related to [6] proposed method the first term of objective function is the expected efficiency of the DMUs. \( \delta_{sj} \) is error variable under scenario \( S \) for \( DMU_j \) weighted by parameter \( \lambda \). The second term is the variance of the efficiency weighted by the goal programming parameter \( \gamma \), which adjusts how much \( DMU_j \) can go function (2.1) penalizes a norm of the infeasibilities. The coefficients \( \gamma \) and \( \lambda \) are user-defined parameters which identify the importance of variance and infeasibility terms, respectively. The transformation variables \( Q_s^+ \) and \( Q_s^- \) are noted for the quadratic term of the variance in the objective function to convert NLP to LP problem. Term \( \sum_{s \in \Omega} \sum_{r=1}^{t} p_s u_r y_{ros} \) in constraint (2.4) is the expected value of efficiencies which indicates the amount of DMU’s efficiency under probable scenarios. Since one of the variables \( Q_s^+ \) and \( Q_s^- \) takes the positive value, this constraint measures the expected deviation of efficiency from its expected value such as variance.

3 MAIN RESULTS

Four different scenarios were used to calculate ED’s efficiency. Table (1) indicates ED’s efficiency changing in parameters Gama and Lambda. As Fig.(1) shows, 2nd DMU has better performance compared with other hospitals’ ED.

4 FIGURE AND TABLE

4.1 TABLE

| Table 1: Objective function regarding to different parameters \( \lambda \) & \( \gamma \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( H \) | \( H \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \lambda \) |
| \( \gamma = 0.8 \) | \( \gamma = 0.8 \) | \( \gamma = 0.8 \) | \( \gamma = 0.8 \) | \( \gamma = 0.5 \) | \( \gamma = 0.6 \) | \( \gamma = 0.4 \) | \( \gamma = 0.2 \) |
| 1 | 0.440923 | 0.468181 | 0.493899 | 0.518486 | 0.561188 | 0.578447 | 0.597666 |
| 2 | 0.627334 | 0.655835 | 0.677993 | 0.704125 | 0.694283 | 0.716174 | 0.726658 |
| 3 | 0.842424 | 0.440343 | 0.497841 | 0.555505 | 0.506367 | 0.516913 | 0.523414 |
| 4 | 0.117052 | 0.212705 | 0.285135 | 0.369573 | 0.299216 | 0.313190 | 0.326973 |
| 5 | 0.584447 | 0.447303 | 0.510431 | 0.573527 | 0.526458 | 0.573527 | 0.596045 |
| 6 | 0.871242 | 0.157941 | 0.228769 | 0.295927 | 0.239397 | 0.256025 | 0.260652 |

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4.2 FIGURE

References


Prioritization of research and development (R&D) projects is of paramount importance due to their outstanding role in the durability of businesses. It is necessary for a thriving business to define its R&D activities in concordance with its strategic tendencies and objectives. Hence, it is necessary for R&D projects to be evaluated and prioritized according to the business strategies. Balanced scorecard (BSC) is a potent tool to define the evaluation criteria based on strategic preferences. Thereafter, the evaluation process could proceed with data envelopment analysis (DEA) method to evaluate the candidate R&D projects according to the BSC-defined criteria. Based on this, the present research is designed to evaluate the R&D projects in a banking system utilizing BSC–DEA approach. In addition to the approach ability to accord with strategic preferences, it is empowered to cope with three more technical complexities, i.e. the DEA model is capable of: embedding ordinal and interval data as nondeterministic subjective viewpoints as well as cardinal data into the model; weighing different significance levels of BSC perspectives; possessing enough discriminatory power in the presence of abundant criteria. The four features mentioned in the preceding statement are the main contributions of this research.

Keywords Balanced scorecard; Research and development; Efficiency; Data envelopment analysis; Grey theory.

1 Introduction

The balanced scorecard (BSC) is a managerial tool to analyze the strategic information in modern businesses according to the strategic objectives and policies. The most outstanding mission of BSC is the definition of evaluation criteria for a strategic decision making process. Based on this feature, numerous studies are conducted to perform an evaluation process applying BSC in conjunction with data envelopment analysis (DEA) approach. DEA models are employed in these studies as quantitative nonparametric techniques to measure the efficiency of decision making units (DMUs) according to an assortment of input and output criteria. In this field, a noteworthy research is conducted in [2] to select the best set of information technology projects by considering cardinal and ordinal data. Recently, a network-structured BSC–DEA approach is developed in [5] to evaluate the supply chain performance utilizing network DEA and DEMATEL methods.

It is established beyond doubt that research and development (R&D) projects are important in satisfying the strategic necessities of corporations. Hence, in numerous studies in the literature, these projects are evaluated according to the strategic corporate objectives and the objectives are embedded in the evaluation criteria utilizing BSC approach. In this direction, a multi-criteria approach for evaluating R&D projects in different stages of their life cycle is developed in [3]. This study integrated the BSC structure into DEA model through a set of balance constraints. Another outstanding study proposing a framework for the analysis of relationships between the four perspectives of BSC in R&D projects was conducted by [4]. To this end, it developed different models of efficiency utilizing DEA.

One feature of R&D projects which is never considered in the former BSC–DEA studies is the nondeterministic and vague nature of projects. Most of time, R&D projects are prone to qualitative measures or nondeterministic subjective viewpoints. This fact has become a thrusting reason for the present research to develop a BSC–DEA model based on grey theory which is able to cope with interval, ordinal,
and cardinal data altogether.

The assortment of BSC perspectives are likely to gain different levels of significance. This issue is more notable in the present research where it intends to consider extra perspectives (Uncertainty and Innovative) other than customary ones (Financial, Customer, Internal Processes, and Learning and Growth). Additionally, the abundance of evaluation criteria compared with the small number of candidate DMUs causes a considerable reduction in the discriminatory power of DEA model. Hence, the main contribution of this research is to cover the following subjects simultaneously:

1. Considering the business strategies and policies,
2. Coping with interval, cardinal, and ordinal data,
3. Weighing the significance level of BSC perspectives,
4. Improving the discriminatory power of DEA model.

2 Methodology and Case Study

2.1 Balanced scorecard

In this section, it is intended to define the criteria which are important in the evaluation of DMUs (R&D projects in banking sector). BSC process is applied here so that the criteria are determined in agreement with the strategic objectives of the banking system. To this end, six general perspectives and a number of input and output criteria are extracted by delving into the literature. Then, the final criteria as well as the perspective weights are determined by a panel of R&D experts in a Delphi decision process as follows:

**Inputs**: Investment cost\(^\dagger\) (0.4–0.7); Human resource\(^\dagger\) (0.3–0.6).

**Outputs**: 

*Financial perspective* (0.10–0.34): Discounted cash flow\(^\dagger\); Increased financial profitability\(^\dagger\). 

*Customer perspective* (0.08–0.34): Compliance with customer needs\(^\dagger\); Performance improvement metrics\(^\dagger\); Perception of customer. 

*Internal business perspective* (0.14–0.38): Compliance with stakeholders needs\(^\dagger\); Congruence; Importance; Synergy with other operations; Degree of influence of external regulation on R&D; Match with budget. 

*Learning and growth perspective* (0.10–0.23): Propriety position; Durability; Training and experience\(^\dagger\). 

*Uncertainty perspective* (0.18–0.44): Probability of technical and commercial success\(^\dagger\); Program complexity; Technology skill base; Availability of people and facilities; Market need. 

*Innovation perspective* (0.10–0.33): Patents; Returns from technology developed internally\(^\dagger\). 

\(^\dagger\) Measured in interval data 
\(^\ddagger\) Measured in ordinal data

The numbers in the parentheses represent for the lower and upper bounds of weights attached to the perspectives. Next, the aforementioned criteria are applied in a decision making process to evaluate the following DMUs:

\(A_1\): Finding new markets for present services
A2 Modifying present services to provide new services
A3 Defining new services related to present services
A4 Defining new services unrelated to present services (without the aid of joint venture)
A5 Defining new services unrelated to present services (with the aid of joint venture)

2.2 DEA model

Based on the problem and its specifications, the DEA model should be capable of dealing with interval and ordinal numbers and perspective weights as well as customary DEA functions. Hence, the grey BSC–DEA model is defined based on [2] and [3] as follows:

\[
\text{max } w_0 = \sum_r Y_{r0} \tag{2.1}
\]

s.t.
\[
\sum_i X_{i0} = 1 \tag{2.2}
\]
\[
\sum_r Y_{rj} - \sum_i X_{ij} \leq 0, \quad \forall j \tag{2.3}
\]
\[
- \sum_{r \in O} Y_{r0} + L_{O_k} \sum_r Y_{r0} \leq 0, \quad \forall k = 1, \ldots, K_O \tag{2.4}
\]
\[
\sum_{r \in O} Y_{r0} - U_{O_k} \sum_r Y_{r0} \leq 0, \quad \forall k = 1, \ldots, K_O \tag{2.5}
\]
\[
- \sum_{i \in I} X_{i0} + L_{I_k} \sum_i X_{i0} \leq 0, \quad \forall k = 1, \ldots, K_I \tag{2.6}
\]
\[
\sum_{i \in I} X_{i0} - U_{I_k} \sum_i X_{i0} \leq 0, \quad \forall k = 1, \ldots, K_I \tag{2.7}
\]
\[
X_{ij} \in \rho_i^- \tag{2.8}
\]
\[
Y_{rj} \in \rho_r^+ \tag{2.9}
\]
\[
X_{ij} \geq \varepsilon, \quad \forall i, j \tag{2.10}
\]
\[
Y_{rj} \geq \varepsilon, \quad \forall r, j \tag{2.11}
\]

where \( X \) (\( Y \)) refers to the weighted inputs (outputs) decision matrix; \( K_I \) (\( K_O \)) is the number of input (output) perspectives; \( I_k \) (\( O_k \)) is the set of criteria included in the input (output) perspective \( k \); \( L_I \) and \( U_I \) (\( L_O \) and \( U_O \)) are respectively the lower and upper bounds of importance attached to the input (output) perspectives. Also, relation (2.8) (relation (2.9)) represents the input (output) constraints embedding nondeterministic interval or ordinal data into the model, as well as the deterministic and cardinal data. Moreover, \( \varepsilon \) in constraints (2.10) and (2.11) is determined by being maximized instead of the objective function (2.1) while taking constraints (2.2) to (2.11) into consideration.

**Theorem 2.1.** The grey BSC–DEA model (2.1) to (2.11) is feasible if the model is feasible without relations (2.8) and (2.9). Theorem 2.1 is simply proved considering that the model is always feasible without relations (2.8) and (2.9) according to [3].
3 Results

By applying the grey BSC–DEA model (2.1) to (2.11) to the case study, the candidate DMUs are evaluated as displayed in Table 1. The model is developed using A&P [1] approach so that the efficient DMUs could be discriminated.

Table 1: Efficiency measurement for R&D projects.

<table>
<thead>
<tr>
<th>DMU</th>
<th>$\varepsilon$</th>
<th>Efficiency</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.010</td>
<td>2.251</td>
<td>1</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.009</td>
<td>1.883</td>
<td>2</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.007</td>
<td>0.834</td>
<td>3</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.004</td>
<td>0.453</td>
<td>5</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.011</td>
<td>0.819</td>
<td>4</td>
</tr>
</tbody>
</table>

As it was anticipated, 40% of DMUs are evaluated as efficient due to the abundance of criteria comparing with the small number of candidate DMUs. Thus, A&P approach is utilized to provide a thorough discrimination of candidate DMUs. Therefore, the project $A_1$ is evaluated as first, implying that finding new markets for present services seems far more legitimate.

References


Efficiency Measure by Generalized Fuzzy Data Envelopment Analysis Model

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Data envelopment analysis (DEA) is a non-parametric technique to measure the efficiencies of a set of decision making units (DMUs) with common crisp inputs and outputs. In real-world problems, however, inputs and outputs typically have some levels of fuzziness. To analyze a DMU with fuzzy input/output data, previous studies provided the fuzzy DEA (FDEA) model and proposed an associated evaluating approach. Nonetheless, numerous deficiencies must be improved in mentioned models. Therefore, the present paper proposes a generalized FDEA model for evaluating efficiency which can evaluate SDMU and the traditional FDEA model.

Keywords Data envelopment analysis, Efficiency, Fuzzy linear programming problem, Sample DMU.

1 Introduction

DEA is a non-parametric method for evaluating the efficiency of DMUs like bank branches, schools, transport sectors etc. on the basis of multiple inputs and outputs. Charnes, Cooper and Rhodes (CCR) [1] developed the DEA approach in 1978. After the paper of CCR, there was an exponential growth in number of publications on DEA. In more general cases, the data for evaluation are often collected from investigations employing a polling approach, where in natural language, such as good, medium, and bad, are used to represent a type of general situation of the examined entities rather than a specific case. Thus, several studies proposed the fuzzy DEA (FDEA) model for input and output data [2, 3, and 4]. However, there are still many places that need to be improved, such as the selected special point, the types of fuzzy number, the -cut or -level approach and the type of the FDEA model when evaluating FDEA model. The above-mentioned facts indicate that studies on FDEA models still focus on the special DEA model and fuzzy number and often apply only a single fuzzy number and the -cut approach to one FDEA model. Muren, proposes a generalized FDEA (GFDEA) model. The GFDEA model not only generalized the FDEA model but also contained the two key model-DEA models, which are the FDEA model with fuzzy sample target DMU and the FDEA model with sample target DMU [5]. In this paper, DEA model is extended to be a generalized fuzzy CCR model for evaluating efficiency and ranking of DMUs with fuzzy data.

2 DEA model with fuzzy data

This paper is proposing a model which is the extension of CCR model to a fuzzy framework. Let a set of $n$ DMUs has $m$ fuzzy inputs $X_{ij}, \quad i = 1, \ldots, m$ and $s$ fuzzy outputs $Y_{rj}, \quad r = 1, \ldots, s$ i.e., inputs and outputs are approximately known and not precisely measured. Thus, fuzzy CCR model is given by fuzzy linear programming problem (LPP) as follows:

$$ E_k = \max \frac{\sum_{r=1}^{s} u_{rk} \hat{Y}_{rk}}{\sum_{i=1}^{m} v_{ik} \hat{X}_{ik}} $$

s.t. $\frac{\sum_{r=1}^{s} u_{rk} \hat{Y}_{rk}}{\sum_{i=1}^{m} v_{ik} \hat{X}_{ik}} \leq 1 \quad \forall j = 1, 2, \ldots, n$

$u_{rk} \geq \varepsilon \quad \forall r = 1, \ldots, s, \quad v_{ik} \geq \varepsilon \quad \forall i = 1, \ldots, m$ (2.1)
The efficiency score evaluated from the model should be fuzzy because this model contains fuzzy parameters. The inputs and outputs can be represented by different level of confidence interval by \( \alpha \)-cuts \[3\].

\[
(E_k)^L = \max \frac{\sum_{r=1}^s u_{rk}(Y_{rk})^L}{\sum_{i=1}^m v_{ik}(X_{ik})^L} \\
s.t. \frac{\sum_{i=1}^m v_{ik}(X_{ik})^U}{\sum_{r=1}^s u_{rk}(Y_{rk})^L} \leq 1 \quad \forall j = 1, 2, \cdots, n \\
\frac{\sum_{i=1}^m v_{ik}(X_{ik})^L}{\sum_{r=1}^s u_{rk}(Y_{rk})^U} \leq 1 \quad \forall j = 1, 2, \cdots, n, \ j \neq k \\
u_{rk} \geq \varepsilon \quad \forall r = 1, \cdots, s, \ v_{ik} \geq \varepsilon \quad \forall i = 1, \cdots, m \tag{2.2}
\]

Similarly we can obtain \((E_k)^U\). The theory of fractional linear programming \[6\] make them possible to replace model (1.2) with an equivalent LPP.

\[
(E_k)^U = \max \sum_{r=1}^s u_{rk}(Y_{rk})^L \\
s.t. \sum_{i=1}^m v_{ik}(X_{ik})^U = 1 \\
\sum_{r=1}^s u_{rk}(Y_{rk})^L - \sum_{i=1}^m v_{ik}(X_{ik})^L \leq 0 \\
\sum_{r=1}^s u_{rk}(Y_{rk})^U - \sum_{i=1}^m v_{ik}(X_{ik})^L \leq 0 \quad \forall j = 1, \cdots, n, \ j \neq k \\
u_{rk} \geq \varepsilon \quad \forall r = 1, \cdots, s, \ v_{ik} \geq \varepsilon \quad \forall i = 1, \cdots, m \tag{2.3}
\]

3 DEA MODEL WITH GENERALIZED FUZZY DATA

**Definition 3.1.** Suppose DMU is one decision making unit in a decision making problem, all the data that in which have the same input and output data with DMU is called sample decision making unit (SDMU) based on this decision making problem \[7\].

After improving FDMU to FSDMU, the generalized FDEA model can be easily obtained.

\[
E_k = \max \frac{\sum_{r=1}^s u_{rk}Y_{rk}}{\sum_{i=1}^m v_{ik}X_{ik}} \\
s.t. \frac{\sum_{i=1}^m v_{ik}X_{ik}}{\sum_{r=1}^s u_{rk}Y_{rk}} \leq 1 \quad \forall j = 1, 2, \cdots, n \\
u_{rk} \geq \varepsilon \quad \forall r = 1, \cdots, s, \ v_{ik} \geq \varepsilon \quad \forall i = 1, \cdots, m \tag{3.1}
\]

By theory of fractional linear, we have

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\[ (E_k) = \max \sum_{r=1}^{s} u_{rk}(\tilde{Y}_{Srk}) \]

s.t. \[ \sum_{i=1}^{m} v_{ik}(\tilde{X}_{Si}) = 1 \]
\[ \sum_{r=1}^{s} u_{rk}(\tilde{Y}_{rk}) - \sum_{i=1}^{m} v_{ik}(\tilde{X}_{ik}) \leq 0 \quad \forall j = 1, \cdots, n \]
\[ u_{rk} \geq \varepsilon \quad \forall r = 1, \cdots, s, \quad v_{ik} \geq \varepsilon \quad \forall i = 1, \cdots, m \] (3.2)

This model transformed to a crisp model by using fuzzy parameters. The distinctions between an SDMU and DMU are presented as follows. The reference sets in FDEA model are the efficient FDMUs, while in the generalized FDEA model, they can be the efficient FDMUs, normal FDMUs, inefficient FDMUs, special FDMUs, and non FDMUs. These five types of DMUs are called fuzzy sample DMUs (FSDMUs). The difference between Eqs 1.1 and 1.4 is the target fuzzy DMU (FDMU). \((X, Y)\) in Eq. 1.1 is an FDMU, whereas \((X_s, Y_s)\) in Eq. 1.2 it is an FSDMU.

4 Conclusion

DEA has wide application to evaluate the relative efficiency in a set of DMUs by using multiple to common crisp inputs and outputs. The existing DEA models are usually limited to common crisp inputs and outputs. In some cases, input and output data of DMUs can’t be precisely measured, for example, quality of service, quality of input resource, degree of satisfaction etc. so, the uncertain theory has played an important role in DEA. In these cases, the data with crisp number will not satisfy the real needs and this restriction will diminish the practical flexibility of DEA in application. Thus, the data can be represented as linguistic variable characterized by fuzzy sets. This paper attempts to extend the traditional DEA model to a fuzzy framework, thus proposing a generalized fuzzy CCR model. The generalized FDEA model is the generalization of FDEA model. It can not only evaluate the inner DMU, but also arbitrarily evaluate the given sample DMU. The extension and use of the different fuzzy numbers in one FDEA model make the FDEA model more general.

References

Finding Non-Zero Multiplier Solutions in Data Envelopment Analysis

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While evaluating decision making units via multiplier DEA models, there exists the possibility of finding multiple optima specially for the DMUs in the relative interior of a face of production possibility set which is not full dimension. With respect to the importance of marginal rates and obtained weights of multiplier models in analyzing sensitivity and robustness of under assessment DMUs, we are looking for non-zero solutions in multiplier models. In other words, the more positive the solution components of multiplier models are, the more robust the under assessment DMUs will be. This signifies that by small variations in input and output data of efficient DMUs it still remains efficient.

Keywords Data envelopment analysis; non-zero weights; multiplier models.

1 Introduction

Considering the basal importance of multiplier weights in data envelopment analysis (DEA) induces us to study this subject to find a suitable methodology. This area faces with some well known complications including alternate optima which is the cause of extreme efficient DMUs in multiplier models [2]. In the optimality of the conventional DEA models, possibly, there exist zero in multiplier weights and this implies that the corresponding inputs and outputs are not fully utilized in DEA evaluation. Generally, zero multiplier weight for an input or output conveys a message that these are not regarded as important in appraisal process. It may seem that changing the non-negative condition of variables to a strict one can be beneficial for getting rid of such a complication. But in this way, the feasible region of the model is not still closed, and therefore, the boundary methods, specially, simplex algorithm can not be used. To overcome such a problem we will propose a new method for finding nonzero multiplier solutions while simplex method is not useless.

2 generating non-zero multiplier solutions

Consider an input vector \( x = (x_1, ..., x_m) \in \mathbb{R}^s \) used to produce an output vector \( y = (y_1, ..., y_m) \in \mathbb{R}^s \) in a technology involving \( n \) decision making units. The input orientation of the CCR envelopment and multiplier model are shown bellow:

\[
\begin{align*}
\text{Min } & \theta \\
\text{s.t. } & \sum_{k=1}^{n} \lambda_k x_j \leq \theta x_o \\
& \sum_{j=1}^{n} \lambda_j y_j \geq y_o \\
& \lambda_j \geq 0. \quad (j = 1, ..., n) \\
\end{align*}
\] (2.1)

\[
\begin{align*}
\text{Max } & u^T y_o \\
\text{s.t. } & u^T y_j - v^T x_j \leq 0 \quad (j = 1, ..., n) \\
& v^T x_o = 1, \\
& v \geq 0, \ u \geq 0.
\end{align*}
\] (2.2)
where $u \in \mathbb{R}^s$ and $v \in \mathbb{R}^m$.

Consider the following primal and dual problems in which $(P)$ and $(D)$ indicate primal and dual problems, respectively.

$\text{(P)}$ \hspace{1cm} \begin{align*}
\text{Max } Z &= cx \\
\text{s.t.} \quad Ax &\leq b, \\
& x \geq 0.
\end{align*}

$\text{(D)}$ \hspace{1cm} \begin{align*}
\text{Min } Y &= yb \\
\text{s.t.} \quad yA &\geq c, \\
& y \geq 0.
\end{align*}

By introducing slack variables we have:

$\text{(P)}$ \hspace{1cm} \begin{align*}
\text{Max } Z &= cx \\
\text{s.t.} \quad Ax + w &= b, \\
& x \geq 0, w \geq 0.
\end{align*}

$\text{(D)}$ \hspace{1cm} \begin{align*}
\text{Min } Y &= yb \\
\text{s.t.} \quad yA - \xi &= c, \\
& y \geq 0, \xi \geq 0.
\end{align*}

The following theorem identifies a relationship between primal and dual optimal solutions which is known as strong complementary slackness theorem.

**Theorem 2.1.** If problems $(P)$ and $(D)$ are feasible then optimal solutions such as $(\bar{x}, \bar{w})$ and $(\bar{y}, \bar{\xi})$ will be respectively found for $(P)$ and $(D)$ which satisfy:

$$\bar{y}_i + \bar{w}_i > 0, \quad i = 1, \ldots, m,$$
$$\bar{x}_j + \bar{\xi}_j > 0, \quad j = 1, \ldots, n.$$ 

**Definition 2.2.** If a solution satisfies the strong complementary slackness theorem then it will be called SCSC solution.

**Definition 2.3.** Let $(\theta^*_o, \lambda^*_o, s^*_o)$ be an unique optimal solution of (2.1) then $DMU_o$ is said to be extreme efficient iff $\theta^*_o = 1, \lambda^*_o = 1, \lambda^*_j = 0, (j = 1, \ldots, n, j \neq o).$

**Lemma 2.4.** Suppose $DMU_o$ is efficient then the CCR multiplier model has an optimal solution in which $(u^*, v^*) > 0$.

With respect to the above lemma, the following system which is the optimal solution region of the multiplier model (2.2), corresponding to $DMU_o$ which is efficient, certainly has a solution that $(u^*, v^*) > 0$:

$$\begin{align*}
v^T x_o &= 1, \\
v^T y_j - u^T x_j + s_j &= 0, \quad j \neq o, \\
u^T y_o &= 1, \\
u \geq 0, v \geq 0, s \geq 0,
\end{align*}
$$

where $DMU_o$ is an efficient $DMU$.  

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Theorem 2.5. Suppose that DMU, be an extreme efficient DMU then the CCR multiplier model has an optimal solution such that \((u^*, v^*) > 0, s_j^* > 0\) for all \(j \neq o\).

Proof. By considering the definition of extreme efficient DMUs it is clear that the envelopment model has an unique optimal solution that \(\lambda_i^* = 1, \lambda_i^* = 0 (j = 1, ..., n, j \neq o)\) as well as \(s_i^* = s_i^* = 0 (r = 1, ..., s, i = 1, ..., m)\). By using strong complementary slackness theorem the multiplier model has an optimal solution such that \((u^*, v^*) > 0, s_j^* > 0\) for all \(j \neq 0\). \(\Box\)

Theorem 2.6. Let \(X\) be the set of feasible solutions of non-homogeneous system of constraints

\[
Ap + Bq = d \quad p \geq 0, q \geq 0,
\]

where \(A\) and \(B\) are matrices of order \(m \times n\) and \(m \times k\), respectively. Further, let \((p_j^*, w_j^*, j = 1, ..., n + 1, q_j^*, j = 1, ..., k)\) be an optimal solution to the following LP problem:

\[
\begin{align*}
Max & \sum_{j=1}^{n+1} w_j \\
\text{s.t.} & \quad \sum_{j=1}^{n} a_j (p_j + w_j) + \sum_{j=1}^{k} b_j q_j - d (p_{n+1} + w_{n+1}) = 0, \\
& \quad 0 \leq w_j \leq 1, p_j \geq 0, \quad j = 1, ..., n + 1, \\
& \quad q_j \geq 0, \quad j = 1, ..., k.
\end{align*}
\]

then, \((\frac{p_j^* + w_j^*}{p_{n+1} + w_{n+1}}, j = 1, ..., n; \frac{q_j^*}{p_{n+1} + w_{n+1}}, j = 1, ..., k)\) is an element of \(X\) such that the number of positive components of \((\frac{p_j^* + w_j^*}{p_{n+1} + w_{n+1}}, j = 1, ..., n)\) is maximized.

The above theorem has been extracted from [1].

It is clear that all solutions \((u, v)\) of system (2.7) are the optimal solutions of (2.2). Let

\[
A = \begin{bmatrix}
0 & x_o^t \\
y_1^t & -x_1^t \\
. & . \\
y_o^{t-1} & -x_o^{t-1} \\
y_{o+1} & -x_{o+1}^t \\
. & . \\
y_n^t & -x_n^t \\
y_o^t & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0_{1 \times (n-1)} \\
I_{(n-1) \times (n-1)} \\
0_{1 \times (n-1)}
\end{bmatrix},
\]

\[
d = \begin{bmatrix}
1 \\
0_{(n-1) \times 1} \\
1
\end{bmatrix},
\]

\[
p = \begin{bmatrix}
u \\
v
\end{bmatrix}, q = s
\]

Using the above assignments for \(A, B, d, p\) and \(q\), system (2.8) is system (2.7). Let \((p^*, q^*, w^*, p_{n+1}^*, w_{n+1}^*)\) is an optimal solution of (2.9) corresponding to (2.10-13). Because DMU, is efficient so by Theorems 2.5 and 2.6 we conclude \((u^*, v^*) = (\frac{p^* + w^*}{p_{n+1}^* + w_{n+1}^*})\) is a strictly positive vector, i.e., \((u^*, v^*)\) is a non zero optimal multiplier solution of (2.2).
3 conclusion

This study proposed a new model for obtaining non-zero multiplier solutions of the multiplier DEA models to deal with the difficulties of multiple optima and zero multipliers for generating more robust solutions. As an important further research in the area of SCSC solutions we are investigating in special group of SCSC solutions which are named analytic centers [3]. Analytic center is a center most solution in the sense that it maximizes the product of positive components among all SCSC solutions. When we use analytic center solutions of the multiplier models, we can find the most variations of the inputs and outputs of an extreme efficient DMU to be unchanged, that is, this extreme efficient DMU remain extreme efficient after that variation. In other words, we can analyze the stability of the extreme efficient DMUs using the analytic center solutions.

References


EQUITABLE EFFICIENCY IN MULTIOBJECTIVE PROGRAMMING

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A problem that sometimes occurs in multiobjective optimization is the existence of a large set of efficient solutions. Hence the decision making based on selecting a unique preferred solution is difficult. Considering models with equitable efficiency relieves some of the burden from the decision maker by shrinking the solution set. This paper focuses on solving multiobjective optimization problems by introducing the concept of equitable efficiency.

Keywords Pareto; Nondominated; Equitability; Multiobjective programming.

1 Introduction

Multiobjective programming has been studied for many years and multiobjective methods have found applications in diverse areas of human life. It is well-known that any multiobjective optimization problem starts usually with an assumption that the criteria are incomparable, i.e., different criteria may have different units and physical interpretations. Many applications, however, arise from situations which present equitable criteria. Equitability is based on the assumption that the criteria are not only comparable (measured on a common scale) but also anonymous (impartial). The latter makes the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria, and therefore models equitable allocation of resources.

The equitable preference was first known as the generalized Lorenz dominance [3, 4]. Kostreva and Ogryczak [1] are the first ones who introduced the concept of equitability into multiobjective programming. They have shown equitable efficiency to be a refinement of Pareto efficiency by adding, to the reflexivity, strict monotonicity and transitivity of the Pareto preference order, the requirements of impartiality and satisfaction of the principle of transfers. Then Kostreva et al. [2] presented the theory of equitable efficiency in greater generality. They have developed scalarization approaches to generating equitably efficient solutions of linear and nonlinear multiobjective programs.

2 Main Results

Throughout this article the following notation is used. Let $R^m$ be the Euclidean vector space and $y', y'' \in R^m$, $y' \leq y''$ denotes $y'_i \leq y''_i$ for all $i = 1, \ldots, m$. $y' < y''$ denotes $y'_i < y''_i$ for all $i = 1, \ldots, m$. $y' \leq y''$ denotes $y'_i \leq y''_i$ but $y' \neq y''$. The set $\{y \in R^m : y' \geq 0\}$ is denoted by $R^m_{+}$.

Consider a decision problem defined as an optimization problem with $m$ objective functions. For simplification we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows:

$$\begin{align*}
\text{min} \quad & \{f_1(x), f_2(x), \ldots, f_m(x)\}, \\
\text{subject to} \quad & x \in X
\end{align*}$$

(2.1)

where $x$ denotes a vector of decision variables selected from the feasible set $X$ and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is a vector function that maps the feasible set $X$ into the objective (criterion) space $R^m_{+}$. We refer to the elements of the objective space as outcome vectors. An outcome vector $y$ is attainable if it expresses outcomes of a feasible solution, i.e., $y = f(x)$ for some $x \in X$. The set of all attainable outcome vectors will be denoted by $Y = f(X)$.

In single objective minimization problems, we compare the objective values at different feasible decisions.
to select the best decision. Decisions are ranked according to the objective values at those decisions and the decision resulting in the least smallest objective value is the most preferred decision. Similarly, to make the multiobjective optimization model operational, one needs to assume some solution concept specifying what it means to minimize multiobjective functions. The solution concepts are defined by the properties of the corresponding preference model. We assume that solution concepts depend only on the evaluation of the outcome vectors while not taking into account any other solution properties not represented within the outcome vectors. Thus, we can limit our considerations to the preference model in the objective space \( Y \).

In the following, some basic concepts and definitions of preference relations are reviewed from [1].

**Definition 2.1.** Let \( y', y'' \in \mathbb{R}^m_+ \) and let \( \prec \) be a relation of weak preference defined on \( \mathbb{R}^m_+ \times \mathbb{R}^m_+ \). The corresponding relations of strict preference \( \prec \) and indifference \( \simeq \) are defined as follows:

\[
y' \prec y'' \iff \left( y' \preceq y'' \text{ and not } y'' \preceq y' \right),
\]

\[
y' \simeq y'' \iff \left( y' \preceq y'' \text{ and } y'' \preceq y' \right).
\]

**Definition 2.2.** Preference relations satisfying the following axioms are called rational preference relations:

1. Reflexivity: for all \( y \in \mathbb{R}^m_+ \), \( y \preceq y \).
2. Transitivity: for all \( y', y'', y''' \in \mathbb{R}^m_+ \), \( y' \preceq y'' \) and \( y'' \preceq y''' \) \( \Rightarrow \) \( y' \preceq y''' \).
3. Strict monotonicity: for all \( y \in \mathbb{R}^m_+ \), \( y - \epsilon e_i \preceq y \) for \( \epsilon > 0 \) where \( e_i \) denotes the \( i \)-th unit vector in \( \mathbb{R}^m_+ \).

The rational preference relations allow us to formalize the Pareto-optimal solution concept with the following definitions.

**Definition 2.3.** The outcome vector \( y' \in Y \) rationally dominates \( y'' \in Y \) iff \( y' \prec y'' \) for all rational preference relations \( \preceq \).

An outcome vector \( y \) is rationally non-dominated if and only if there does not exist another outcome vector \( y' \) such that \( y' \) rationally dominates \( y \). Analogously, a feasible solution \( x \in X \) is an efficient or Pareto-optimal solution of the multiobjective problem (2.1) if and only if \( y = f(x) \) is rationally non-dominated.

It has been shown in [1], the outcome vector \( y' \in Y \) rationally dominates \( y'' \in Y \) if and only if \( y' \preceq y'' \). As a consequence, we can state that a feasible solution \( x \in X \) in a Pareto-optimal solution of the multiobjective problem (2.1), if and only if, there does not exist \( x' \in X \) such that \( f_i(x') \leq f_i(x) \) for \( i = 1, 2, \ldots, m \), where at least one strict inequality holds.

Let \( \preceq \) be a preference relation defined on \( \mathbb{R}^m_+ \).

**Definition 2.4.** \( \preceq \) is said to be impartial if

\[
(y_1, y_2, \ldots, y_m) \preceq (y_{\tau(1)}, y_{\tau(2)}, \ldots, y_{\tau(m)})
\]

for all \( y \in \mathbb{R}^m_+ \), where \( \tau \) stands for an arbitrary permutation of components of \( y \).

**Definition 2.5.** \( \preceq \) is said to satisfy the principle of transfers, \( y_i > y_j \Rightarrow y - \epsilon e_i + \epsilon e_j \preceq y \), for all \( y \in \mathbb{R}^m_+ \), where \( 0 \leq \epsilon \leq y_i - y_j \).

**Definition 2.6.** A preference relation \( \preceq \) defined on \( \mathbb{R}^m_+ \) is called an equitable rational preference relation if it is reflexive, transitive, strictly monotonic, impartial and satisfies the principle of transfers.

The equitable rational preference relations allow us to define the concept of equitably efficient solution.

**Definition 2.7.** Let \( y', y'' \in Y \). We say that \( y' \) equitably dominates \( y'' \), iff \( y' \prec y'' \) for all equitable rational preference relations \( \preceq \), and that denoted by \( y' \preceq_e y'' \).

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An outcome vector \( y \) is equitably nondominated if and only if there does not exist another outcome vector \( y' \) such that \( y' \) equitably dominates \( y \). Analogously, a feasible solution \( x \) is called an equitably efficient solution of the multiobjective problem (2.1) if and only if \( y = f(x) \) is equitably nondominated.

**Definition 2.8.** Let \( y \in \mathbb{R}^m_+ \).
1. Let \( \Theta : \mathbb{R}^m_+ \to \mathbb{R}^m_+ \) be the ordering map defined as \( \Theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_m(y)) \), where \( \theta_i(y) \geq \theta_j(y) \) for \( i = 1, 2, \ldots, m \), and \( \Theta \) is a permutation of the set \( \{1, 2, \ldots, m\} \).
2. Let \( \Theta : \mathbb{R}^m_+ \to \mathbb{R}^m_+ \) be the cumulative ordering map defined as \( \Theta(y) = (\bar{\theta}_1(y), \bar{\theta}_2(y), \ldots, \bar{\theta}_m(y)) \), where \( \bar{\theta}_i(y) = \sum_{j=1}^i \theta_j(y) \) for \( i = 1, 2, \ldots, m \).

To make it practical, equitable efficiency can be defined in terms of vector inequalities. In order to do that, we define a certain preference relation.

**Definition 2.9.** Suppose that \( y', y'' \in Y \) are two outcome vectors. The relation \( \leq_{ie} \) is defined as follows.
\[
y' \leq_{ie} y'' \iff \Theta(y') \leq \Theta(y'').
\]

Below, we will discuss the relationship between two preference \( \succeq_{ie} \) and \( \leq_{ie} \). In order to do that, we need to the following proposition.

**Proposition 2.10.** ([1], Proposition 2.2) If \( \Theta(y') \leq \Theta(y'') \), then \( \Theta(y') \leq \Theta(y'') \) or there exists a finite sequence of vectors \( y^0 = y'', y^1, \ldots, y' \) such that \( y^k - y^{k-1} = \epsilon_k e_i \), \( 0 < \epsilon_k < y^{k-1} - y^{k-1} \) for \( k = 1, 2, \ldots, i \) and \( \Theta(y') \leq \Theta(y'') \).

**Theorem 2.11.** For any two vectors \( y', y'' \in Y \).
\[
y' \succeq_{ie} y'' \iff y' \leq_{ie} y''.
\]

**Proof.** The relation \( \leq_{ie} \) is an equitable rational preference relation. Thus
\[
y' \succeq_{ie} y'' \implies y' \leq_{ie} y''.
\]
Conversely suppose that \( y' \leq_{ie} y'' \), we deduce \( \Theta(y') \leq \Theta(y'') \). Due to Proposition 2.10, we have \( \Theta(y') \leq \Theta(y'') \) for any equitable rational preference relation. Thus \( y' \succeq_{ie} y'' \).

By applying Theorem 2.11 and Definition 2.9, we have the following statement.

**Corollary 2.12.** Suppose that \( y', y'' \in Y \). We have
\[
y' \succeq_{ie} y'' \iff \Theta(y') \leq \Theta(y'').
\]

Note that Corollary 2.12 permits one to express equitable efficiency for problem (2.1) in terms of the standard efficiency for the multiobjective problem with objectives \( \Theta(f(x)) \):
\[
\min \{ \Theta(f(x)) : x \in X \}.
\]

**Theorem 2.13.** Let \( x \in X \) be a feasible solution. \( x \) is efficient solution of the multiobjective problem (2.5) if and only if it is equitably efficient solution of the multiobjective problem (2.1).

**Proof.** By applying Corollary 2.12, we obtain the desired result.
References


Purpose of this paper is to provide a framework for evaluating the overall performance of decision-making units by means of a data envelopment analysis (DEA) model with balanced scorecard approach. This paper presents the development of a conceptual framework, which aims to evaluate performance of Saipa Malleable Company. The proposed conceptual framework has two main contributions. Firstly, it determines causal relationship and mutual influence between four perspectives of the balance scorecard (BSC) with the DEMATEL technique. Secondly, it combines two well-known managerial tools, balance scorecard (BSC) as a comprehensive framework for determining Saipa Malleable Company criteria, and data envelopment analysis (DEA) for measuring efficiency.

Keywords DEA; BSC; DEMATEL.

1 Introduction

In the past carried out performance, evaluation relies to financial criteria but this viewpoint do not consider to effective qualitative criteria on future. Performance management in meaning of performance evaluation based on its strategic program and using of information in improving strategic program, without regard to the other extent expect the field of financial would be impossible. In between the two of the most prominent scientists management Kaplan and Norton, offered a new management system under the title of balanced scorecard (BSC) with realization four main factors. So, in 1992, a modern management system indicated under name of balanced scorecard (BSC) by Kaplan and Norton. However, with using the BSC approach, we cannot rank the decision-making units. In this situation, we use Data Envelopment Analysis (DEA) model to ranking the decision-making units. DEA is a nonparametric powerful tool in analyzing efficiency with multiple inputs and outputs, which can consider both qualitative and quantitative measures. Charnes et al (1978) proposed the DEA model to produce the efficiency frontier based on the concept of Pareto optimum. The main aim of this paper is a new combination of the data envelopment analysis (DEA) model and balance scorecard (BSC). In the first stage, we determine the efficiency measures. For identifying the measures, we use the BSC approach. In the next stage, we determine the relationships between four perspectives of BSC approach with DEMATEL method. With using the DEMATEL method, we can identify the measures of efficiency deeply. Finally, we create a modified DEA model to performance efficiency and ranking the decision-making units.

2 Literature review

2.1 Data Envelopment Analysis

DEA was first proposed by Charnes, Cooper and Rhodes (CCR) in 1978. The evolutionary form of CCR model was suggested in 1984 by Banker et al. In subsequent years, several models were developed by a large number of researchers. Orientation, disposability, diversification, and return to scale are different aspects that can be seen in these models.
2.2 Balanced Scorecard

The BSC is a conceptual framework for translating an organization’s strategic objectives into a set of performance measures distributed among four perspectives: financial, customer, internal business processes, and learning and growth.

2.3 DEMATEL

DEMATEL is a method for making the causality structure and the mutual influential strength between one element and another element and the base of problems could be useful to set the strategies against the problems.

3 Proposal Model

Wu (2011) introduced a DEA model for detecting the most efficient unit. In this paper, we use this model as a base model.

\[
M^* = \min_M \quad s.t.
\]
\[
M - d_j \geq 0 \quad j = 1, 2, \cdots, n
\]
\[
\sum_{i=1}^{m} X_{ij} \leq 1 \quad j = 1, 2, \cdots, n
\]
\[
\sum_{r=1}^{s} Y_{rj} - \sum_{i=1}^{m} X_{ij} + d_j - \beta_j = 0 \quad j = 1, 2, \cdots, n
\]
\[
0 \leq \beta_j \leq 1 \quad d_j \in \{0, 1\} \quad j = 1, 2, \cdots, n
\]
\[
X_{ij} \in \rho^- \quad Y_{rj} \in \rho^+
\]
\[
X_{ij} \geq \varepsilon^* \quad \forall i, j
\]
\[
Y_{rj} \geq \varepsilon^* \quad \forall r, j
\]

\(d_j\) is a binary variable that is represented the deviation variable of DMU. DMU is most efficient if and only if \(d_j = 0\). The constraint \(\sum_{j=1}^{n} d_j = 0\) forces among all the DMUs for determining only single most efficient unit. Amin and Toloo (2007) epsilon model is proposed to determine the non-Archimedean epsilon:

\[
\varepsilon^* = \min_{\varepsilon} \quad s.t.
\]
\[
\sum_{i=1}^{m} X_{ij} \leq 1 \quad j = 1, 2, \cdots, n
\]
\[
\sum_{r=1}^{s} Y_{rj} - \sum_{i=1}^{m} X_{ij} \leq 0 \quad j = 1, 2, \cdots, n
\]
\[
X_{ij} \in \rho^- \quad Y_{rj} \in \rho^+
\]
\[
X_{ij} \geq \varepsilon \quad \forall i, j
\]
\[
Y_{rj} \geq \varepsilon \quad \forall r, j
\]

The proposed method is described as follows:

Step 0: Let \(T = \phi\) and \(e = \) number of DMUs to be ranked.
Step 1: Solve following model:
\[
M^* = \min_{M} M
s.t.
\begin{align*}
M - d_j & \geq 0 \quad j = 1, 2, \ldots, n \\
\sum_{i=1}^{m} X_{ij} & \leq 1 \quad j = 1, 2, \ldots, n \\
\sum_{r=1}^{k} Y_{rj} - \sum_{i=1}^{m} X_{ij} + d_j - \beta_j & = 0 \quad j = 1, 2, \ldots, n \\
\sum_{j=1}^{n} d_j & = n - 1 \\
d_j & = 1 \quad \forall j \in T \\
0 & \leq \beta_j \leq 1 \quad d_j \in \{0, 1\} \quad j = 1, 2, \ldots, n \\
X_{ij} & \in \rho_i \\
Y_{rj} & \in \rho_r^+
\end{align*}
\]

Suppose in optimal solution \(d^*_j = 0\).

Step 2: Let \(T = T \subset \{t\}\).

Step 3: If \(|T| = e\), then stop; otherwise go to Step 1.

Indeed in Step 1 of proposed algorithm, a DMU is identified as most CCR-efficient unit. After entering this DMU to \(T\) in Step 2, in Step 3 if all DMUs are ranked, the algorithm finishes, else it goes to next iteration. Above step are iterated \(e\) times.

In second step of suggested approach, this DEA method could be used to prioritize units on the basis of decision table.

In next section, proposed method is used for Saipa Malleable Company. At first is gotten relation between four BSC perspectives with DEMATEL as for the questionnaire that has been filled by experts of Saipa Malleable Company. Then is investigated efficiency of three units, Machinery, Cast Iron, and Aluminum of Saipa Malleable Company with a combination of DEA-BSC methods.

4 Conclusions

In this paper is used combination of BSC and DEA. At first cause and effect relations between four perspectives of BSC are analyzed by using DEMATEL method. Then Performance evaluation carried out on the three existing unit (Machinery, Aluminum and Cast Iron) in Saipa Malleable Company and units are ranked as for efficiency, the combination of these two models can provide comprehensive model to evaluate efficiency. Via the result of DEMATEL causality analysis, is observed that "Internal process perspective (P)" contains feedback relation. "Learning and growth (L)" is found the most important influential factor of Saipa Malleable company as well as it also affects other perspectives. "Financial perspective (F)" could be affected by the other three perspectives.

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A Bi-Level Programming Formulation and a Hybrid $K$-Means Algorithm for Identifying the Optimal Customer Satisfaction Scheme Under Attack

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Developing a responsive service system play a crucial role in developing a long-term reputation in the marketplace. Nowadays, some terrorists are planning acts of sabotage to destabilize the economy and some facilities are more vulnerable to imminent attack. In this paper, for choosing the best scheme of establishing new facilities and identifying the most cost-effective way of demand satisfaction, a bi-level model is proposed. This model can be regarded as a static Stackelberg game between a malicious interdictor as the leader and a system defender as the follower. In the upper level problem (ULP), the most destructive interdiction strategy in the presence of limitation on the maximum available interdiction budget is detected. In the lower level problem (LLP), the system defender, firstly, locates a specified number of facilities in the most sensible candidate sites. Subsequently, she tries to identify the least cost allocation pattern of customers to the facilities. In this paper, a comprehensive enumeration is used to identify all feasible interdiction strategies. The LLP can be considered as a capacitated clustering problem; therefore, the K-means clustering algorithm is applied to solve the LLP for each interdiction strategy.

Keywords interdiction problem; bi-level programming; location and allocation problem; $K$-means clustering algorithm.

1 Introduction

The probability of disruption occurrences in service systems is considerable; however, the managerial effort is often only focused on decreasing operational costs in normal conditions [1]. Intentional attack of terrorists is one of the main sources of disruption in facilities. In the literature, these problems are categorized as interdiction problems. The interdiction problem has a two-player game nature between an external enemy and a system defender. One of these players is considered as a leader who makes decisions independently and the other is considered as a follower whose decision is subordinate to the leader’s one. Consequently, this problem is predominantly formulated as a bi-level mathematical formulation whose first formulation dates back to 1934 when it has been formulated by H.v. Stackelberg in market economy [2]. Pursuant to [3], the interdiction problems can be categorized according to two features:

- The depth of damage in each system component (partial or full interdiction),
- The type of components which are vulnerable to intentional attacks (facilities of the system, namely nodal interdiction or transportation pathways, namely network interdiction).

The problem which is studied in this paper belongs to the full nodal interdiction category. Therefore, when a facility involved in the interdiction scenario of the attacker, it will completely become unable to serve customers. After the attack, the system defender establishes a predetermined number of facilities at the most beneficial candidate sites and thereupon she tries to achieve the most cost-effective assignment patterns of customers to the facilities, including available former established facilities and the newly established facilities.

To formulate this problem, we benefit from bi-level programming. According to [4], "bi-level programming can be viewed as a static version of the non-cooperative, two person game introduced by H.v
Stackelberg and being static implies that each player has only one move. In the basic model of this static game, control of the decision variables is partitioned amongst the players who seek to optimize their individual payoff functions and perfect information is assumed so that both players know the objective and feasible choices available to the other.

Bi-level programming is associated with inherent complexity and common solvers are unable to obtain the optimal solution directly. To identify the worst-case scenario of the attacker a comprehensive enumeration is used. For each feasible and non-dominated interdiction scenario, the LLP must be solved. The LLP is a binary integer programming (BIP); hence even the LLP is singly NP-hard. In this paper, to solve the LLP, we benefit from K-means clustering algorithm. According to [5], the K-means algorithm is the most widely squared error-based clustering algorithm and its reasonable time and space complexity and its order independent characteristic (i.e., for a given initial cluster centers, the algorithm generates the same partitions regardless of the pattern order) are the indispensable reasons for its popularity. In the rest of this paper the details of the proposed problem are provided in more detail.

2 Notations and decision variables

To give a formal description of the developed model, some notations and variables are introduced as follows: Index sets:

- \( I \) set of customer nodes, \( I = 1, 2, \cdots, m \)
- \( L \) set of former established facilities, \( L = 1, 2, \cdots, q \)
- \( J \) set of candidate facility sites (locations), \( J = 1, 2, \cdots, n \)

Parameters:

- \( p \) the number of new facilities that must be established
- \( \tau \) cost of interdicting a former established facility
- \( b_{tot} \) interdiction budget of the attacker
- \( w_i \) demand of customer \( i \)
- \( d_{ij} \) Euclidean distance between customer \( i \) and facility at site \( j \)

Decision variables:

- \( z_j \) binary variable, equal to 1 if a new facility is established at candidate site \( j \)
- \( x_{ij} \) binary variable, equal to 1 if customer \( i \) is allocated to a new facility which is established at candidate site \( j \)
- \( y_{il} \) binary variable, equal to 1 if customer \( i \) is allocated to former established facility \( l \)
- \( u_l \) binary variable, equal to 1 if former established facility \( l \) loses its service availability due to interdiction

3 Bi-level mathematical formulation

\[
\begin{align*}
\max_u & \quad Z_{att}(U) \\
\text{s.t.} & \quad \sum_{l=1}^{p} \tau u_l \geq b_{tot} \quad (3.2) \\
& \quad u_l \in \{0, 1\}, \forall l \in L \quad (3.3)
\end{align*}
\]
Where $Z_{att}(U)$ is the optimal objective value of the following binary integer programming problem:

$$Z_{att}(U) = \min_{x,y,z} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}d_{ij}w_i + \sum_{i=1}^{m} \sum_{l=1}^{q} y_{il}d_{il}w_i$$

$$s.t. \sum_{i=1}^{m} x_{ij} \leq z_j m; \forall j \in J$$

$$\sum_{j=1}^{n} z_j = p$$

$$\sum_{i=1}^{m} y_{il} \leq m - mu_l; \forall l \in L$$

$$\sum_{j=1}^{n} x_{ij} + \sum_{l=1}^{q} y_{il} = 1; \forall i \in I$$

$$x_{ij}, y_{il}, z_j \in \{0, 1\}$$

In the above formulation, 1.1-1.3 represent the ULP, and 1.4-2.4 represent the LLP of the bi-level model. Expression 1.1 shows the interdictor’s objective function consisting of the shipment cost of goods from newly established and former established facilities to the customers. Constraint 1.2 is the budget constraint of the interdictor. Binary constraints on the interdictor’s decision variables shown in 1.3 conclude the ULP. The defender’s objective in 1.4 is similar to 1.1 except for the direction of optimization. Constraints 2.1 state the customers can be allocated to a candidate site on condition that at this candidate site a new facility has been established. Constraint 2.3 imposes establishing exactly $p$ new facilities at candidate sites on the system defender. Constraints 2.4 state a former established facility is unable to serve the customers if it has been interdicted. Constraints 2.5 enforce each customer allocated to exactly one non-interdicted facility. Finally, constraints 2.6 assure the binary characteristic of the decision variables.

### 4 Solution Procedure

To solve this bi-level model a hybrid of a comprehensive enumeration method and $K$-means clustering algorithm is recommended. In the interest of simplification, an abbreviated form of the suggested procedure is reported as follows:

#### 4.1 Comprehensive enumeration phase

With regard to the interdiction budget, identify all feasible interdiction strategies. If the residual budget of an interdiction strategy is equal or more than $\tau$, consider the strategy as a dominated one. For each feasible and non-dominated strategy runs the $K$-means clustering algorithm.

#### 4.2 $K$-means clustering algorithm

An interdiction strategy which is built in the comprehensive enumeration is the input of this algorithm. $k$ centers in this problem are segmented into two categories:

- Fixed centers: the location of the non-interdicted former established facilities,
- Variable centers: the location of the new $p$ established facilities.
To identify the variable centers, in the first iteration of each interdiction strategy, select \( p \) initial clusters randomly from the given \( n \) candidate sites. Then, assign each customer to the nearest center, including fixed centers and variable centers. Afterwards, compute the new centers of the clusters only for variable centers. If no change occurs in the location of variable centers, the algorithm terminates. Otherwise, again assign the customer demand to the nearest centers and continue the procedure until the stopping criterion is met.

Finally, since this problem has a discrete solution space, the variable centers, which are not located at a candidate sites are estimated with the location of the nearest candidate sites. Subsequently, the objective value of the problem with respect to the analyzed interdiction strategy is calculated.

4.3 Identifying the near optimal solution

Since the interdictor is the leader of this bi-level optimization, the strategy which causes the maximum objective value in \( K \)-means clustering algorithm is chosen. The output of the \( K \)-means clustering algorithm for the chosen strategy suggests the most sensible scheme for establishing new facilities and the most cost-effective demand satisfaction pattern in the wake of the most destructive interdiction scenario.

References


In this paper we recall multi-objective programming and semidefinite programming problems, then we introduce multi-objective semidefinite programming problem and prove some theorems about it. 

**Keywords** Multi-objective; Semidefinite programming; Efficacy.

**1 Introduction**

In this section we recall multi-objective programming and semidefinite programming problem.

**1.1 Multi-objective programming problem**

Multi-objective optimization is an area of multiple criteria decision making. A multi-objective problem is an optimization problem that has multiple objective functions, and can be formulated as follows:

\[
\begin{align*}
\min & \quad f_1(x), f_2(x), \ldots, f_k(x) \\
\text{s.t.} & \quad g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m
\end{align*}
\] (1.1)

Obviously if \(k=1\), problem 1.1 is an ordinary optimization problem. We recall some definitions and some theorems about multi-objective problem.

**Definition 1.1.** A feasible point \(x\) is called efficient if there exist no feasible point \(y\) such that \(f_i(x) \geq f_i(y)\) for all \(i = 1, 2, \ldots, k\), for some \(1 \leq j \leq k\), \(f_j(x) > f_j(y)\).

**Definition 1.2.** A feasible point \(x\) is called weak efficient if there exist no feasible point \(y\) such that \(f_i(x) > f_i(y)\) for all \(i = 1, 2, \ldots, k\).

If objective functions and constraints are linear, problem 1.1 is called multi-objective linear programming (MOLP). There is some approaches to solve an MOLP problem, one of them is the weighted sum method.

**Theorem 1.3.** A feasible point \(x^*\) is efficient if and only if there exist \(\lambda_i > 0\), \(i = 1, 2, \ldots, k\) such that \(x\) is optimal for the following problem

\[
\begin{align*}
\min & \quad \sum_{i=1}^{k} \lambda_i f_i(x) \\
\text{s.t.} & \quad g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m
\end{align*}
\] (1.2)

**1.2 Semidefinite programming**

Semidefinite programming is a main branch of convex optimization which introduced in 1990’s. The general form of this kind of problem is as follows:

\[
\begin{align*}
\min & \quad C \cdot X \\
\text{s.t.} & \quad A_i \cdot X = b_i, \quad i = 1, 2, \ldots, m \\
& \quad X \succeq 0.
\end{align*}
\] (1.3)
where $C$, $A_i$, and $X$ are $n$-by-$n$ symmetric matrices and $b_i \in \mathbb{R}$. The notation $\succeq$ shows the matrix $X$ must be semidefinite matrix, and the notation $\bullet$ is inner product in matrix space. Many practical problems in combinatorial optimization can be modeled or approximated as semidefinite programming problems such as max cut problem. The dual of semidefinite programming can be defined as follows:

$$
\max \sum_{i=1}^{m} b_i y_i \\
\text{s.t.} \sum_{i=1}^{m} y_i A_i \preceq C.
$$

(1.4)

Analogously to linear programming, we have weak duality and strong duality theorems.

**Theorem 1.4.** The value of objective function problem 1.4 for any feasible is a lower-bounds for the value of objective function problem 1.3 for any feasible solution. Also The value of objective function problem 1.3 for any feasible is a upper-bounds for the value of objective function problem 1.4 for any feasible solution.

For proof see [1].

**Theorem 1.5.** Under well known condition if $X^*$ is an optimal point for primal and $y^* = (y_1^*, y_2^*, ..., y_m^*)$ is an optimal point for dual we have

$$
C \bullet X^* = \sum_{i=1}^{m} b_i y_i^*.
$$

And also if $X^*$ and $y^*$ are feasible point and equality $C \bullet X^* = \sum_{i=1}^{m} b_i y_i^*$ is hold then $X^*$ and $y^*$ are optimal.

For proof see [1].

### 2 Multi-objective semidefinite programming

In this section we introduce multi-objective semidefinite programming and present some theorems to find efficient point.

A multi-objective semidefinite programming problem is a multi-objective problem as follows:

$$
\min \quad C_1 \bullet X, C_2 \bullet X, ..., C_k \bullet X \\
\text{s.t.} \quad A_i \bullet X = b_i, \quad i = 1, 2, ..., m \\
X \succeq 0.
$$

(2.1)

**Theorem 2.1.** Suppose that $\lambda_i > 0, i = 1, 2, ..., k$ and consider following problem

$$
\min \quad \sum_{i=1}^{k} \lambda_i C_i \bullet X \\
\text{s.t.} \quad A_i \bullet X = b_i, \quad i = 1, 2, ..., m \\
X \succeq 0.
$$

(2.2)

Let $X^*$ be an optimal solution of problem 2.3 then $X^*$ is an efficient point for problem 2.1.

**Proof.** Suppose that $X^*$ is not efficient, then ther exist a feasible point $Y$ such that $C_i \bullet Y \leq C_i \bullet X^*$ and for some $j$, $C_j \bullet Y < C_j \bullet X^*$, so $\lambda_j C_j \bullet Y \leq \lambda_j C_j \bullet X^*$ and also $\lambda_j C_j \bullet Y < \lambda_j C_j \bullet X^*$ therfore $\sum_{i=1}^{k} \lambda_i C_i \bullet Y < \sum_{i=1}^{k} \lambda_i C_i \bullet X^*$, it is contradic to optimality of $X^*$.

□
**Theorem 2.2.** Under well known condition if $X^*$ be an efficient point then there exist $\lambda_i > 0, i = 1, 2, \ldots, k$ such that $X^*$ is an optimal point for the following problem

$$
\min \sum_{i=1}^{k} \lambda_i C_i \cdot X
$$

s.t. \hspace{1em} A_i \cdot X = b_i, \hspace{1em} i = 1, 2, \ldots, m

$$
X \succeq 0.
$$

**Proof.** Suppose that $X^*$ is an efficient point, consider the following problem:

$$
\min \sum_{j=1}^{k} C_j \cdot X
$$

s.t. \hspace{1em} A_i \cdot X = b_i, \hspace{1em} i = 1, 2, \ldots, m

$$
C_j \cdot X \leq C_j \cdot X^*, \hspace{1em} j = 1, 2, \ldots, k
$$

$$
X \succeq 0.
$$

Suppose that $X^*$ is not optimal for problem 2.4 and let $X^{**}$ be an optimal point then we have $\sum_{j=1}^{k} C_j \cdot X^{**} < \sum_{j=1}^{k} C_j \cdot X^*$ and also we have $C_j \cdot X^{**} \leq C_j \cdot X^*$. So $C_l \cdot X^{**} < C_l \cdot X^*$ for some $1 \leq l \leq k$. So $X^*$ is not an efficient point, the contradiction implies $X^*$ be optimal.

Now consider the dual of problem 2.4:

$$
\min \sum_{j=1}^{k} C_j \cdot X^* y_j + \sum_{i=1}^{m} w_i b_i
$$

s.t. \hspace{1em} \sum_{i=1}^{m} w_i A_i + \sum_{j=1}^{k} C_j y_j \leq \sum_{j=1}^{k} C_j
$$

$$
y_j \leq 0.
$$

From the strong duality theorem, we have

$$
\sum_{j=1}^{k} C_j \cdot X^* y_j^* + \sum_{i=1}^{m} w_i^* b_i = \sum_{j=1}^{k} C_j \cdot X^*
$$

so

$$
\sum_{i=1}^{m} w_i^* b_i = \sum_{j=1}^{k} C_j \cdot X^* - \sum_{j=1}^{k} C_j \cdot X^* y_j^* = \sum_{j=1}^{k} C_j \cdot X^*(1 - y_j^*).
$$

Note that $1 - y_j^* > 0$. Now set $\lambda_j = 1 - y_j^* > 0$ and consider the following problem

$$
\min \sum_{j=1}^{k} (1 - y_j^*) C_j \cdot X
$$

s.t. \hspace{1em} A_i \cdot X = b_i, \hspace{1em} i = 1, 2, \ldots, m

$$
X \succeq 0.
$$
and the dual of problem 2.6 is as follows:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{m} w_i b_i \\
\text{s.t.} & \quad \sum_{i=1}^{m} w_i A_i \preceq \sum_{j=1}^{k} (1 + y_j^*) C_j \\
& \quad C_j \cdot X \preceq C_j \cdot X', \quad j = 1, 2, \ldots, k \\
& \quad h_j \geq 0, \quad X \succeq 0.
\end{align*}
\]

(2.7)

(2.8)

Abiously \( X^* \) and \( w^* \) are feasible and on the other hand the value of objective functions problem 2.7 and problem 2.6 in \( X^* \) and \( w^* \) are equal. So from the duality theorem we have \( X^* \) and \( w^* \) are optimal.

If we have a feasible point we can find an efficient point by solving the following problem:

\[
\begin{align*}
\max & \quad \sum_{j=1}^{k} h_j \\
\text{s.t.} & \quad A_i \cdot X = b_i, \quad i = 1, 2, \ldots, m \\
& \quad C_j \cdot X + h_j = C_j \cdot X', \quad j = 1, 2, \ldots, k \\
& \quad h_j \geq 0, \quad X \succeq 0.
\end{align*}
\]

(2.9)

where \( X' \) is a feasible point and any optimal solution is efficient. We prove this statement.

**Theorem 2.3.** If \( X' \) is a feasible point for problem 2.1 then any optimal solution of problem 2.9 is an efficient point.

**Proof.** Suppose that \( X^* \) is an optimal solution of problem 2.9 and \( X^* \) is not efficient. So there exist some feasible point such as \( X'' \) which \( C_j \cdot X'' \leq C_j \cdot X^* \) and for some \( 1 \leq l \leq k, C_j \cdot X'' < C_j \cdot X^* \) hence we can reduce the objective value of problem 2.9.

References


