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A computational method for fractional Fredholm integro-differential equations

M. Mojahedfar* [1] , A. Tari[2],
[1] Department of Mathematics, Shahed University, Tehran, Iran
[2] Department of Mathematics, Shahed University, Tehran, Iran

Abstract

In this paper, a method based on Legendre or any orthogonal polynomials, is developed to find numerical solutions of fractional Fredholm integro-differential equations (FFIDEs).

Keyword Fredholm integro-differential equations, Fractional calculus, orthogonal polynomials.

Mathematics Subject Classification (2010):65R20.

1 Introduction

Fractional calculus is a branch of mathematical analysis which deals with derivatives and integrals of arbitrary order. Fractional integral and differential equations are used to model some practical problems in physics, engineering, economics and biology. In recent years, many numerical methods have been developed for solving fractional integro-differential equations, such as Adomian decomposition method [1], variational iteration method [2], wavelet method [3], Operational Tau method [4] and so on.
In this paper, we study a class of FFIDEs as follows:

$$D^{\alpha}u(x) - \lambda \int_{0}^{1} k(x,t)u^{p}(t)dt = f(x), \tag{1}$$

with initial conditions

$$u^{(i)}(0) = \delta_i, \quad i = 0,1,...,n-1, \quad n-1 < \alpha \leq n, \quad n \in N, \tag{2}$$

where D^{α} is fractional derivative operator of order α in caputo sense, $f(x)$ and $k(x,t)$ are known continious functions and $u(x)$ is a solution to be determined.
Here, we replace the differential and integral parts of Eq.(1) and initial conditions (2) by their operational matrix representation. Then we obtain a system of nonlinear equations and solve it to obtain an approximate solution of the problem.

*Speaker: m.mojahedfar@shahed.ac.ir

2 Preliminaries and notations

In this section, we present some necessary definitions of the fractional calculus [5] which will be used in this paper.

Definition 2.1. *The Riemann-Liouville fractional integral operator J^α of order α is given by*

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t)dt, \quad \alpha > 0,$$
$$J^0 u(x) = u(x).$$

One of important properties of the fractional integral operator is

$$J^\alpha x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} x^{\alpha+\nu}, \tag{3}$$

where $\alpha > 0$ and $\nu > -1$.

Definition 2.2. *The fractional derivative of $u(x)$ in the Caputo sense is defined as*

$$D^\alpha u(x) = J^{n-\alpha} D^n u(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} u^{(n)}(t)dt, \quad n-1 < \alpha \le n, \quad n \in N. \tag{4}$$

3 Description of the method

In this section, we describe the method of this paper for solving fractional Fredholm integro-differential equations by using shifted Legendre polynomials. we first need the following lemma.

Lemma 3.1. *Let $p(x) = \sum_{i=0}^{\infty} p_i x^i = PX_x$ be a polynomial, then we have*

$$D^r p(x) = \frac{d^r}{dx^r} p(x) = P\eta^r X_x, \qquad r = 0, 1, 2, ..., \tag{5}$$

where $P = [p_0, p_1, p_2, ...]$, $X_x = [1, x, x^2, ...]^T$ and $\eta = [\eta_{ij}]_{i,j=0}^{\infty}$, $\eta_{ij} = i\ \delta_{i,j+1}$.

Let $\{\phi_i(x)\}_{i=0}^N$ be a set of Legendre polynomials, then one can write:

$$u(x) = \sum_{j=0}^{N} u_j \phi_j(x) = \underline{u}\Phi X_x \tag{6}$$

where $\underline{u} = [u_0, u_1, ..., u_N]$ is a $(N+1)$ vector of unknown coefficients, $X_x = [1, x, x^2, ..., x^N]^T$ and Φ is lower triangular matrix which converts Legendre base to standard base.
By using (4) and (5), we obtain an operational form of D^α as follows:

$$D^\alpha u(x) = J^{n-\alpha} D^n (\underline{u}\Phi X_x) = J^{n-\alpha}(\underline{u}\Phi\eta^n X_x) = \underline{u}\Phi\eta^n J^{n-\alpha}(X_x), \tag{7}$$

and by (3), we have:

$$J^{n-\alpha}(X_x) = \left[\frac{\Gamma(1)x^{n-\alpha}}{\Gamma(n-\alpha+1)}, \frac{\Gamma(2)x^{n-\alpha+1}}{\Gamma(n-\alpha+2)}, ..., \frac{\Gamma(N+1)x^{n-\alpha+N}}{\Gamma(n-\alpha+N+1)}\right] = B\Pi. \tag{8}$$

where $\Pi = [x^{n-\alpha}, x^{n-\alpha+1}, ..., x^{n-\alpha+N}]^T$ and B is an $(N+1)\times(N+1)$ diagonal matrix with elements

$$B_{i,i} = \frac{\Gamma(i+1)}{\Gamma(n-\alpha+i+1)}, \quad i = 0, 1, ..., N.$$

But $x^{n-\alpha+i}$ can be approximated as:

$$x^{n-\alpha+i} = \sum_{j=0}^{N} a_{i,j}\phi_j(x) = \mathbf{a}_i \Phi X_x, \quad \mathbf{a}_i = [a_{i,0}, ..., a_{i,N}], \quad a_{ij} = \int_0^1 x^{n-\alpha+i}\phi_j(x)dx, \ i, j = 0, ..., N.$$

Therefore, we obtain

$$\Pi = [\mathbf{a}_0\Phi X_t, \mathbf{a}_1\Phi X_x, ..., \mathbf{a}_N\Phi X_x]^T = A\Phi X_x, \tag{9}$$

where

$$A = [\mathbf{a}_0, \mathbf{a}_1, ..., \mathbf{a}_N]^T.$$

Substituting (8) into (7) and using (9), we get

$$D^\alpha u(x) = \underline{u}\Phi\eta^n B\Pi = \underline{u}\Phi\eta^n BA\Phi X_x. \tag{10}$$

On the other hand, the functions $f(x)$ and $k(x,t)$ can be written as

$$f(x) = \sum_{j=0}^{N} f_j\phi_j(x) = F\Phi X_x, \quad k(x,t) = \sum_{j=0}^{N}\sum_{i=0}^{N} k_{ij}\phi_i(x)\phi_j(t). \tag{11}$$

Now, we state the following lemma:

Lemma 3.2. *If $u(x) = \underline{u}\Phi X_x$, then $u^p(x) = \underline{u}\Phi U^{p-1}X_x$ where U is an upper triangular matrix with elements*

$$U_{ij} = \sum_{r=0}^{\infty} u_r\Phi_{r,j-i}, \quad j \geq i, \quad i,j = 0,1,2,... \tag{12}$$

Then, by using Lemma 3.2, we can write

$$u^p(x) = C\Phi X_x = \sum_{j=0}^{N} c_j\phi_j(x) \tag{13}$$

where $C = \underline{u}\Phi U^{p-1}\Phi^{-1}$.
By using (11) and (13), we have

$$\begin{aligned}\int_0^1 k(x,t)u^p(t)dt &= \int_0^1 \left(\sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}\phi_i(x)\phi_j(t)\right) \times \left(\sum_{r=0}^{N} c_r\phi_r(t)\right) dt \\ &= \sum_{r=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}c_r\phi_i(x)\int_0^1 \phi_j(t)\phi_r(t)dt = \sum_{r=0}^{N}\sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}c_r\phi_i(x)\delta_{rj} \\ &= \sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}c_j\phi_i(x) = \Lambda\Phi X_x,\end{aligned} \tag{14}$$

where Λ is an $(N+1)$ vector with elements

$$\Lambda_i = \sum_{j=0}^{N} k_{ij}c_j, \quad j = 0,1,...,N.$$

Substituting (10), (11) and (14) into Eq.(1) implies

$$(\underline{u}\Phi\eta^n BA - \lambda\Lambda - F)\ \Phi X_x = 0. \tag{15}$$

Since ΦX_x is a base vector, we obtain a system of nonlinear equations as

$$\underline{u}\Phi\eta^n BA - \lambda\Lambda = F. \tag{16}$$

Now, we convert the conditions (2) to a system of equations. By using (5) and (6), we have

$$u^{(i)}(x)|_{x=0} = \underline{u}\Phi\eta^i X_x|_{x=0}. \tag{17}$$

Substituting (17) into (2) yeilds

$$\underline{u}\Phi\eta^i X_x|_{x=0} = \delta_i, \quad i = 0,1,...,n-1. \tag{18}$$

By solving the system of equations (16) and (18) simultaneously and determining the unknown coefficients u_j, approximate solution $u(x)$ can be calculated from (6).

4 Numerical example

Example 4.1. *Let us consider the following FFIDE*

$$D^{\frac{1}{2}}u(x) - \int_0^1 xt\ u(t)dt = \frac{\frac{8}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} + \frac{x}{12}, \quad x \in [0,1],$$

with the initial condition $u(0) = 0$*. The exact solution of this problem is* $u(x) = x^2 - x$*. Applying this method for* $N = 2$*, we obtain the approximate solution which is the same as the exact solution.*

Example 4.2. *As second example, consider the following nonlinear equation*

$$D^{\frac{3}{4}}u(x) - \int_0^1 xt\ u^2(t)dt = \frac{4x^{\frac{1}{4}}}{\Gamma(\frac{1}{4})} - \frac{x}{4}, \quad x \in [0,1],$$

subject to $u(0) = 0$ *with the exact solution* $u(x) = x$*. Table 1 shows the absolute errors* $(\mathrm{e}(x))$*at some points.*

Table 1: absolute errors of Example 4.2 for $N = 15$.

x	e(x)
0.2	0.4320e-15
0.5	0.2147e-14
0.8	0.4887e-14
1	0.7222e-14

Conclusion

In this work, a computational method based on shifted Legendre polynomials presented for solving FFIDEs by converting it to an algebraic system of equations.

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