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ON SEMI-P-REDUCIBLE FINSLER METRICS

BY

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Abstract. The class of semi-P-reducible manifolds contains the class of Randers manifolds and Landsberg manifolds as special cases. In this paper, we prove that every semi-P-reducible manifold with P-reducible metric reduces to a Landsberg manifold. Then we show that there is not exists P2-like Randers metric.

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1. Introduction

Various interesting special forms of Cartan and Landsberg tensors have been obtained by some Finslerians (see [4], [5], [12], [13]). The Finsler spaces having such special forms have been called C-reducible, P-reducible, general relatively isotropic Landsberg and others (see [10], [11]). In [3], MATSUMOTO introduces the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, MATSUMOTO-HŌJŌ [7] proves that the converse is also true. A Randers metric $F=\alpha+\beta$ is just a Riemannian metric α perturbated by a one form β. Randers metrics have important applications both in mathematics and physics ([14]). Then as a generalization of C-reducible metrics, Matsumoto-Shimada introduce the notion of P-reducible metrics ([8], [9]).

Let us remark some important curvatures in Finsler geometry. For a Finsler metric $F=F(x,y)$, its geodesics are characterized by the system of differential equations $\ddot{c}^i+2G^i(\dot{c})=0$, where the local functions $G^i=G^i(x,y)$ are called the spray coefficients. A Finsler metric F is called a Berwald metric if $G^i=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y\in T_xM$ for any $x\in M$

([18]). The second derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$.

There is a weaker notion of metrics- weakly Landsberg metrics. Set $\mathbf{I}_y := \sum_{i=1}^n \mathbf{C}_y(e_i, e_i, \cdot)$ and $\mathbf{J}_y := \sum_{i=1}^n \mathbf{L}_y(e_i, e_i, \cdot)$, where $\{e_i\}$ is an orthonormal basis for $(T_xM, \mathbf{g}_y)$. $\mathbf{I}_y$ and $\mathbf{J}_y$ is called the mean Cartan and mean Landsberg curvature, respectively. A Finsler metric F is said to be weakly Landsbergian if $\mathbf{J} = 0$.

In [5], Matsumoto-Shibata introduce the notion of semi-C-reducibility by considering the form of Cartan torsion of a non-Riemannian (α, β)-metric on a manifold M with dimension $n \geq 3$. A Finsler metric is called semi-C-reducible if its Cartan tensor is given by $C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}J_j\} + \frac{q}{C^2}I_iI_jI_k$, where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM, h_{ij} is the angular metric and $C^2 = I^iI_i$. The function p is called characteristic scalar of F. If $q = 0$, then F is called C-reducible metric. It is remarkable that, an (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M.

As a generalization of C-reducible metrics, Matsumoto-Shimada introduce the notion of P-reducible metrics (see [6], [16]). A Finsler metric is called P-reducible if its Landsberg tensor is given by following $L_{ijk} = \frac{1}{1+n}\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}$. In [15], Rastogi introduces a new class of Finsler spaces named by semi-P-reducible spaces, which contains the notion of P-reducible metrics, as a special case. A Finsler metric F is called semi-P-reducible if its Landsberg tensor is given by

$$L_{ijk} = \lambda\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} + 3\mu J_iJ_jJ_k, \tag{1}$$

where $\lambda = \lambda(x, y)$ and $\mu = \mu(x, y)$ are scalar functions on TM. We have some special cases as follows: If $\mu = 0$, then F is a P-reducible metric; if $\lambda = 0$, then F is a $P2$-like metric [15] and if $\mu = \lambda = 0$, then F is a Landsberg metric. The geometric meaning of P-reducible Finsler metrics is studied in [16]. Since the class of semi-P-reducible metrics contains the class of C-reducible metrics as a special case, therefore the study of this class of Finsler spaces will enhance our understanding of the geometric meaning of Randers metrics.

In this paper, we prove that every semi-P-reducible manifold with P-reducible metric reduces to a Landsberg manifold. Then we show that there is not exists any P2-like Randers metric.

In this paper, we use the Berwald connection on Finsler manifolds. The h- and the v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on $TM_0 := TM \setminus \{0\}$;

(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;

(iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite, $\mathbf{g}_y(u, v) := \frac{1}{2}\left[F^2(y + su + tv)\right]|_{s,t=0}$, $u, v \in T_xM$.

Let $x{\in}M$ and $F_x{:=}F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM\otimes T_xM\otimes T_xM \to \mathbb{R}$ by $\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}$, $u, v, w \in T_xM$. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i\frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ ([17]). For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where $M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\}$, and $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity is introduced by MATSUMOTO [3]. Matsumoto proves that every Randers metric satisfies $\mathbf{M}_y = 0$. Later on, MATSUMOTO-HŌJŌ [7] proves that the converse is true too.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM\otimes T_xM\otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$ ([2]).

Define $\bar{\mathbf{M}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{M}}_y(u, v, w) := \bar{M}_{ijk}(y)u^iv^jw^k$ where $\bar{M}_{ijk} := L_{ijk} - \frac{1}{n+1}\{J_ih_{jk} + J_jh_{ik} + J_kh_{ij}\}$. A Finsler metric F is said to be P-reducible if $\bar{\mathbf{M}}_y = 0$. The notion of P-reducibility was given by Matsumoto-Shimada [8].

3. Main results

In this section, we are going to consider semi-P-reducible Finsler manifold with P-reducible metric. Then we prove the following.

Theorem 3.1. *Let (M, F) be a semi-P-reducible Finsler manifold and $\mu \neq 0$. Suppose that F is a P-reducible metric. Then F reduces to a Landsberg metric.*

Proof. Let F be a P-reducible metric

$$L_{ijk} = \frac{1}{n+1}\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\}. \tag{2}$$

On the other hand, F is a semi-P-reducible metric

$$L_{ijk} = \lambda\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} + 3\mu J_iJ_jJ_k. \tag{3}$$

By (2) and (3), we get

$$\left(\frac{1}{n+1} - \lambda\right)\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} = 3\mu J_iJ_jJ_k. \tag{4}$$

Multiplying (4) with g^{ij} implies that

$$\{(n+1)\lambda + 3\mu J^2 - 1\}J_k = 0. \tag{5}$$

Suppose that $J_k \neq 0$. Then we have

$$\lambda = \frac{1 - 3\mu J^2}{n+1}. \tag{6}$$

Plugging (6) into (3) implies that

$$\begin{aligned} L_{ijk} &= \frac{1}{n+1}\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} \\ &\quad -3\mu J^2\Big\{\frac{1}{n+1}\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\} - \frac{1}{J^2}J_iJ_jJ_k\Big\}. \end{aligned} \tag{7}$$

Since F is a P-reducible metric, thus (7) reduces to the following

$$3\mu J^2\Big\{\frac{1}{n+1}\{J_ih_{jk}+J_jh_{ki}+J_kh_{ij}\}-\frac{1}{J^2}J_iJ_jJ_k\Big\}=0. \tag{8}$$

By (8) and our assumptions, we deduce that the following holds

$$\frac{1}{n+1}\{J_ih_{jk}+J_jh_{ki}+J_kh_{ij}\}=\frac{1}{J^2}J_iJ_jJ_k. \tag{9}$$

This is impossible, since $Rank(h_{jk}J^2)=n-1$ and $Rank(J_jJ_k)=1$. Thus

$$J^2=J_iJ^i=0. \tag{10}$$

Since F is a positive-definite metric, then $J_i=0$. Thus by (2), we conclude that F is a Landsberg metric. □

Example 3.1. Let (M,F) be a 2-dimensional Finsler manifold. We refer to the Berwald's frame (ℓ^i,m^i), where $\ell^i=y^i/F(y)$, m^i is the unit vector with $\ell_im^i=0$ and $\ell_i=g_{ij}\ell^i$. Then the Cartan tensor is given by following $C_{ijk}=Cm_im_jm_k$, where $C:=m_pm^p$ (for more details see [1]). By taking a horizontal derivation of above equation, we get $L_{ijk}=FC_0m_im_jm_k$, where $C_0:=C_{|s}y^s$. By multiplying of above equation with g^{ij}, we can deduce that every Finsler surface is P2-like.

Theorem 3.2. *Let (M,F) be a Finsler manifold of dimension $n\geq 3$. Then there is not exists any P2-like Randers metric.*

Proof. Let F be a P2-like Randers metric on a manifold M of dimension $n\geq 3$. It is easy to see that F is P-reducible. We have

$$\frac{1}{n+1}\{J_ih_{jk}+J_jh_{ki}+J_kh_{ij}\}=\frac{1}{J^2}J_iJ_jJ_k. \tag{11}$$

We have

$$h_{ij}J^i=(g_{ij}-\ell_i\ell_j)J^i=J_j. \tag{12}$$

Contracting (11) with J^i and using (12) yields

$$\frac{1}{n+1}\{h_{jk}J^2+2J_jJ_k\}=J_jJ_k, \tag{13}$$

or equivalently

$$h_{jk}J^2=(n-1)J_jJ_k. \tag{14}$$

By the same argument used in Theorem 3.1, we conclude that there is not exists any P2-like Randers metric on a manifold M of dimension $n\geq 3$. □

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