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CROSS AND SUBCROSS UNITARILY INVARIANT NORMS

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Abstract. We characterize building blocks of unitarily invariant norms which are cross or subcross.

1. Introduction and Preliminaries

There has been a great interest in studying unitarily invariant norms on the algebra of $n \times n$ matrices with the complex entries(see, e.g., [1-4] and the references therein). Let M_n denote the algebra of $n \times n$ matrices with the complex entries. The singular values of a matrix A in M_n which are arranged in decreasing order are displayed by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$. Also the diagonal matrix $diag(s_1(A), \cdots, s_n(A))$ and the vector $(s_1(A), \cdots, s_n(A))$ are displayed by $\Sigma(A)$ and $\sigma(A)$ respectively. We write

$$\mathbb{R}^n_{+\downarrow} \equiv \{(x_1, \cdots, x_n) : x_1 \geq x_2 \cdots \ x_n \geq 0\}.$$

For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_{+\downarrow}$, $1 \leq k \leq n$ and $a \geq x_{k+1}$, we denote $(x_1, \cdots, x_k)$ by $x(k)$. Also for a diagonal matrix $x = diag(x_1, \cdots, x_n)$,

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we display

$$diag(\underbrace{a, \cdots, a}_{k\ times}, x_{k+1}, \cdots, x_n)$$

by $x(k, a)$. The matrix $\sum_{i=1}^{k} E_{ii}$, is displayed by I_k, where E_{ii} denotes the matrix which (i,i)th entry is one and zero elsewhere.

A norm on M_n is unitarily invariant if $\|UAV\| = \|A\|$ for all unitary matrices $U, V \in M_n$ and all $A \in M_n$. Typical examples of unitarily invariant norms on M_n are Ky-Fan k-norms defined by

$$N_k(\cdot) = s_1(\cdot) + \cdots + s_k(\cdot),$$

and Schatten (p, n)-norms $(1 \leq p \leq \infty)$ defined by

$$\|\cdot\|_{(p,n)} = (\sum_{i=1}^{n} s_i(\cdot)^p)^{\frac{1}{p}}.$$

For $A \in M_n$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n_{+\downarrow}$ we write

$$\|A\|_\alpha = \alpha_1 s_1(A) + \cdots + \alpha_n s_n(A). \tag{1.1}$$

These norms can be viewed as the building blocks of unitarily invariant norms [5, 6].

A unitarily invariant norm $\|.\|$ on M_n is submultiplicative with respect to ordinary or Hadamard product if and only if $\|E_{11}\| \geq 1$. A norm $\|.\|$, on M_n is a cross norm if

$$\|A \otimes B\| = \|A\|\|B\|,\ A \in M_k,\ B \in M_s,\ 1 \leq ks \leq n, \tag{1.2}$$

where $A \otimes B$ denotes the Kronecker product of A and B. Also it is a subcross norm if the following inequality holds

$$\|A \otimes B\| \leq \|A\|\|B\|, A \in M_k,\ B \in M_s,\ 1 \leq ks \leq n. \tag{1.3}$$

In this paper we characterize building blocks of unitarily invariant norms which are cross or subcross.

2. Main results

As we said unitarily invariant norms of the form (1.1), can be viewed as the building blocks of unitarily invariant norms. Using the following lemmas, we can characterize unitarily invariant norms of this type, which are cross or subcross.

Lemma 2.1. Let $\alpha = (\alpha_1, \cdots, \alpha_{sk}) \in \mathbb{R}^{sk}_{+\downarrow}$, $B \in M_k$, $1 \leq p \leq s$ and $1 \leq q \leq k$. If $\|I_h \otimes I_j\| \leq \|I_h\|\|I_j\|,\ 1 \leq h \leq s,\ 1 \leq j \leq k$,

then the function

$$g^B_{p,q,\alpha} : [S_{q+1}(B), \infty) \longrightarrow \mathbb{R},$$

$$y \longmapsto \|I_p\|_{\alpha(p)}\|\Sigma(B)(q,y)\|_{\alpha(k)} - \|I_p \otimes \Sigma(B)(q,y)\|_{\alpha(pk)},$$

is increasing.

Lemma 2.2. Let $\alpha = (\alpha_1, \cdots, \alpha_{sk}) \in \mathbb{R}^{sk}_{+\downarrow}$, $A \in M_s$, $B \in M_k$, $1 \leq p \leq s$ and $1 \leq q \leq k$. If

$$\|I_h \otimes I_j\| \leq \|I_h\|\|I_j\|,\ 1 \leq h \leq s,\ 1 \leq j \leq k\ ,$$

then the function

$$g^{A,B}_{p,q,\alpha} : [S_{p+1}(A), \infty) \longrightarrow \mathbb{R},$$

$$x \longmapsto \|\Sigma(A)(p,x)\|_{\alpha(s)}\|\Sigma(B)\|_{\alpha(k)} - \|\Sigma(A)(s,x) \otimes \Sigma(B)\|_{\alpha(sk)},$$

is increasing.

Now we are ready to state the two main Theorems of our argument as follow:

Theorem 2.3. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n_{+\downarrow}$, $\|.\|_\alpha$ is a subcross norm if and only if

$$\|I_s \otimes I_k\|_{\alpha(sk)} \leq \|I_s\|_{\alpha(s)}\|I_k\|_{\alpha(k)},\ 1 \leq sk \leq n.$$

Theorem 2.4. For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n_{+\downarrow}$, $\|.\|_\alpha$ is a cross norm if and only if

$$\|I_s \otimes I_k\|_{\alpha(sk)} = \|I_s\|_{\alpha(s)}\|I_k\|_{\alpha(k)},\ 1 \leq sk \leq n.$$

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