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A modified homotopy perturbation method coupled with the Fourier transform for nonlinear and singular Lane–Emden equations

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ABSTRACT

In the present work the modified homotopy perturbation method (HPM), incorporating He's polynomial into the HPM, combined with the Fourier transform, is used to solve the nonlinear and singular Lane–Emden equations. The closed form solutions, where the exact solutions exist, and the series form solutions, where the exact solutions do not exist, are obtained. Thereafter, the Pade approximant is used to provide the trend of monotonic convergence. Moreover, the concept of equilibrium, which arises from the nature of Lane–Emden differential equations when the space coordinate approaches infinity $x \to \infty$, is shown by the monotonic approach of the results toward a constant value.

1. Introduction

Many differential equations that govern physical real-life problems are singular and nonlinear. Owing to these obstacles, the relevant differential equations may not be solved by standard semi-analytical methods, such as the homotopy perturbation method (HPM) [1–21], variational iteration method (VIM) and Adomian decomposition method (ADM) [22–30]. In addition, there are other approaches available for solving nonlinear equations, such as the modified mapping method, the bifurcation method and G'/G-expansion method [31–40]. The travelling wave characteristics are investigated by using the nonlinear Klein–Gordon and Schrödinger's equations employing the trigonometric function series and the G'/G-expansion method [31–40]. These differential equations need to be treated in a special way mathematically. And the Lane–Emden equations belong to this group. The Lane–Emden equations have many applications in the fields of radioactively cooling, self-gravitating gas clouds and in the mean-field treatment of a phase transition in critical adsorption or in the modeling of clusters of galaxies. So far the modification of the HPM has been carried out by using He's polynomials and the Laplace transform method [41,42]. The modified HPM is shown to be able to handle the nonlinear term in PDEs. The homotopy perturbation transform method (HPTM) [43] is a combination of the homotopy perturbation method and Laplace transform method that is used to solve various types of linear and nonlinear systems of partial differential equations. In this method (HPTM) He's polynomials are used to decompose the nonlinear terms in partial differential equations. The

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modified homotopy perturbation transform method (MHPTM) [44] is based on the application of Laplace transform to solve the boundary layer equations in fluid mechanics. The nonlinear terms are handled by using He's polynomials on the semi-infinite domain.

The basic motivation of the present work is to develop a hybrid procedure combining the Fourier transform and the modification of homotopy perturbation method, the modified HPM, to handle the singularity and nonlinearity of differential equations, which cannot be solved with standard semi-analytical methods. He's polynomial incorporated into the homotopy perturbation method combined with the Fourier transform is used to solve the linear, nonlinear and singular partial differential equations. In the present work, three different singular and nonlinear Lane–Emden equations are solved using the modified HPM. The closed form solutions for two problems are similar to the existing exact solutions reported in the literature. The solution of the third problem is expressed in series form; therefore, the Pade approximants are used to demonstrate the trend of monotonic convergence for the solution.

2. The idea under laying the homotopy perturbation method [18,19]

The homotopy perturbation method (HPM) originally developed by He [18,19] is a combination of the classical perturbation technique and the homotopy technique. The basic idea of the HPM for solving nonlinear differential equations is delineated next. Consider the following differential equation:

$$E(u) = g(x). \tag{1}$$

With boundary conditions

$$B(u, \partial u/\partial n) = 0, \tag{2}$$

where $E(u)$ is any differential operator and B is a boundary operator. Usually the operator E can be decomposed into two parts, a linear operator L and a nonlinear operator N, and expressed as

$$L(u) + N(u) = g(x). \tag{3}$$

We now construct a homotopy as follows:

$$H(v, p) = (1 - p)(L(v) - L(u_0)) + p(E(u) - g(x)) = 0 \tag{4}$$

where u_0 is an initial approximation of Eq. (1). It is clear that when p is equal to zero then $H(u, 0) = L(v) - L(u_0) = 0$, and when p is equal to 1, then $H(u, 1) = E(u) - g(x) = 0$. It is worth noting that as the embedding parameter p increases monotonically from zero to unity, the zero order solution v_0 continuously deforms into the original problem $E(u) = g(x)$. The embedding parameter, $p \in [0, 1]$, is considered as an expanding parameter [18]. In the homotopy perturbation method the embedding parameter p is used to elaborate a series expansion for the solution:

$$v = \sum_{i=0}^{\infty} p^i v_i = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots. \tag{5}$$

When $p \to 1$, then Eq. (5) becomes the approximate solution to Eq. (1) written as

$$u = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{6}$$

The series in Eq. (6) is a convergent series and the rate of convergence depends on the nature of Eq. (1) [18,19].

3. Basic idea behind the modified HPM

The general forms of one-dimensional nonlinear partial differential equations are considered to illustrate the basic idea of the modified HPM. Consider the following differential equation:

$$E(u(x, t)) = 0 \quad x \geq 0,\ t \geq 0. \tag{7}$$

Normally the operator E can be decomposed into two parts, a linear operator L and a nonlinear operator N,

$$L(u(x, t)) + N(u(x, t)) = g(x). \tag{8}$$

We construct a homotopy as follows:

$$L(u(x, t)) + pN(u(x, t)) = g(x). \tag{9}$$

Taking the Fourier transform from both sides of Eq. (9) we get

$$\mathcal{F}\{L(u(x, t))\} + \mathcal{F}\{N(u(x, t))\} = \mathcal{F}(g(x)). \tag{10}$$

Focusing on the linear operator L in Eq. (9) the concept of the homotopy perturbation method with embedding parameter p is used to generate a series expansion for $L(u(x,t))$ as follows:

$$u(x,t)=\sum_{i=0}^{\infty}p^i v_i$$
$$L(u(x,t))=L\left(\sum_{i=0}^{\infty}p^i v_i\right). \tag{11}$$

Switching to the nonlinear operator N in Eq. (9) we use He's polynomial, H_n, as follows:

$$N(u(x,t))=\sum_{n=0}^{\infty}p^n H_n \tag{12}$$

where He's polynomial [45,46], H_n, is defined as

$$H_n(u_0,\ldots,u_n)=\frac{1}{n!}\frac{d^n}{dp^n}N\left(\sum_{i=0}^{n}p^i u_i\right)_{p=0}. \tag{13}$$

Substituting Eqs. (12) and (11) into Eq. (10) we obtain

$$\mathcal{F}\left\{L\left(\sum_{i=0}^{\infty}p^i v_i\right)\right\}+\mathcal{F}\left\{\sum_{i=0}^{\infty}p^{i+1}H_i\right\}=\mathcal{F}(g(x)), \tag{14}$$

where He polynomials [45,46], H_n, are expressed by

$$\begin{aligned}
H_0&=N(u_0),\\
H_1&=\frac{d}{dp}N\left(\sum_{i=0}^{1}p^i u_i\right)_{p=0},\\
H_2&=\frac{1}{2!}\frac{d^2}{dp^2}N\left(\sum_{i=0}^{2}p^i u_i\right)_{p=0},\\
H_3&=\frac{1}{3!}\frac{d^3}{dp^3}N\left(\sum_{i=0}^{3}p^i u_i\right)_{p=0},
\end{aligned} \tag{15}$$

and so on. On the other hand, Eq. (14) can be rewritten in the following form:

$$\sum_{i=0}^{\infty}p^i\mathcal{F}\{L(v_i)\}+\sum_{i=0}^{\infty}p^{i+1}\mathcal{F}\{H_i\}=\mathcal{F}\{g\}. \tag{16}$$

Using Eq. (16) we introduce the recursive relation:

$$\begin{aligned}
&\mathcal{F}\{L(v_0)\}=\mathcal{F}\{g\}\\
&\sum_{i=1}^{\infty}p^i\mathcal{F}\{L(v_i)\}+\sum_{i=0}^{\infty}p^{i+1}\mathcal{F}\{H_i\}=0.
\end{aligned} \tag{17}$$

Alternatively, the recursive Eq. (17) can be rewritten as

$$\begin{aligned}
p^0&:\mathcal{F}\{L(v_0)\}=\mathcal{F}\{g\},\\
p^1&:\mathcal{F}\{L(v_1)\}+\mathcal{F}\{H_0\}=0,\\
p^2&:\mathcal{F}\{L(v_2)\}+\mathcal{F}\{H_1\}=0,\\
p^3&:\mathcal{F}\{L(v_3)\}+\mathcal{F}\{H_2\}=0,\\
&\vdots\\
p^k&:\mathcal{F}\{L(v_k)\}+\mathcal{F}\{H_{k-1}\}=0.
\end{aligned} \tag{18}$$

Using the Maple symbolic code, the first part of Eq. (18), p^0, gives the value of $\mathcal{F}\{v_0\}$. First, applying the inverse Fourier transform to $\mathcal{F}\{v_0\}$ gives the value of v_0, that will define He's polynomial, H_0 using the first part of Eq. (19). In the second part of Eq. (18), p^1, the He polynomial H_0 will enable us to evaluate $\mathcal{F}\{v_1\}$. Second, applying the inverse Fourier transform to $\mathcal{F}\{v_1\}$ gives the value of v_1, that will define He's polynomial H_1 using the second part of Eq. (15) and so on. This in turn will lead to the complete evaluation of the components of v_k, $k\geq 0$ upon using different corresponding parts of Eqs. (18) and (15). Therefore the series solution follows immediately after using the first part of Eq. (11) with embedding parameter $p=1$.

4. A case study involving the Lane–Emden equation

In this section, we solve three differential equations, the so-called Lane–Emden equation, to demonstrate the effectiveness and the validity of the present method, the modified HPM, in the entire range of the problem domain and in the singular problems. The Lane–Emden equation has been used to formulate several phenomena in mathematical physics and astrophysics. This equation encounters wide applications in the modeling of the thermal behavior of a spherical cloud of gas acting under a mutual attraction of its molecules and subject to the classical laws of thermodynamics. The general form of the Lane–Emden equation is written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\frac{\partial u}{\partial x} = -u^n. \tag{19}$$

Example 1. The Lane–Emden equation is obtained by assigning $n = 1$ in Eq. (19) as follows:

$$\begin{aligned} &u_{xx} + (2/x)u_x = -u, \quad x \geq 0, \\ &u(0) = 1, \qquad u_x(0) = 0. \end{aligned} \tag{20}$$

The Fourier transform of Eq. (20) is

$$\omega^2 \frac{d}{d\omega}\hat{u}(\omega) = i + \frac{d}{d\omega}\hat{u}(\omega). \tag{21}$$

Substituting the recursive equation, Eq. (17), into Eq. (21) leads to the following equation:

$$\omega^2 \frac{d}{d\omega}\sum_{n=0}^{\infty} p^n \hat{u}_n(\omega) = i + \frac{d}{d\omega}\sum_{n=0}^{\infty} p^{n+1}\hat{u}_n(\omega). \tag{22}$$

The sequence of recursive equations deduced from Eq. (22) can be written in the following manner:

$$\begin{aligned} &p^0 : \omega^2 \frac{d}{d\omega}\hat{u}_0(\omega) - i = 0, \qquad \hat{u}_0(0) = 0, \\ &p^1 : \omega^2 \frac{d}{d\omega}\hat{u}_1(\omega) - \frac{d}{d\omega}\hat{u}_0(\omega) = 0, \qquad \hat{u}_1(0) = 0, \\ &p^2 : \omega^2 \frac{d}{d\omega}\hat{u}_2(\omega) - \frac{d}{d\omega}\hat{u}_1(\omega) = 0, \qquad \hat{u}_2(0) = 0, \\ &p^3 : \omega^2 \frac{d}{d\omega}\hat{u}_3(\omega) - \frac{d}{d\omega}\hat{u}_2(\omega) = 0, \qquad \hat{u}_3(0) = 0, \end{aligned} \tag{23}$$

and so on. In view of the foregoing, the solution of the recursive equation, Eq. (23), can be written as follows:

$$\begin{aligned} &\hat{u}_0(\omega) = -i/\omega, \\ &\hat{u}_1(\omega) = -i/3\omega^3, \\ &\hat{u}_2(\omega) = -i/5\omega^5, \\ &\hat{u}_3(\omega) = -i/7\omega^7, \end{aligned} \tag{24}$$

and so on. Using the Maple symbolic code for the inverse Fourier transforms of Eq. (24) delivers

$$\begin{aligned} &u_0(x) = 1, \\ &u_1(x) = -x^2/3!, \\ &u_2(x) = x^4/5!, \\ &u_3(x) = -x^6/7!, \end{aligned} \tag{25}$$

and so on. Consequently, the solution of Eq. (20) in a series form is given by

$$u(x) = 1 - (x^2/3!) + (x^4/5!) - (x^6/7!) + \cdots. \tag{26}$$

The Taylor series expansion for $(\sin(x)/x)$ is written as

$$\sin(x)/x = 1 - (x^2/3!) + (x^4/5!) - (x^6/7!) + \cdots. \tag{27}$$

By comparing Eqs. (27) and (26), the latter Eq. (26) can be reduced to

$$u(x) = \sin(x)/x. \tag{28}$$

Notwithstanding this is precisely the exact solution of Eq. (20).

Example 2. By assigning $n = 5$ in Eq. (19) the corresponding Lane–Emden equation is

$$\begin{aligned} &u_{xx} + (2/x)u_x = -u^5, \quad x \geq 0, \\ &u(0) = 1, \qquad u_x(0) = 0. \end{aligned} \tag{29}$$

The Fourier transform of Eq. (29) is

$$\omega^2 \frac{d}{d\omega}\hat{u}(\omega) = i + \frac{d}{d\omega}\mathcal{F}\{u^5\}. \tag{30}$$

Substituting the recursive equation, Eq. (17), into Eq. (30) results in

$$\omega^2 \frac{d}{d\omega}\sum_{n=0}^{\infty} p^n \hat{u}_n(\omega) = i + \frac{d}{d\omega}\sum_{n=0}^{\infty} p^{n+1} H_n(\omega). \tag{31}$$

The recursive equation deduced from Eq. (31) can be written as follows:

$$\begin{aligned} &p^0 : \omega^2 \frac{d}{d\omega}\hat{u}_0(\omega) = i \qquad \hat{u}_0(0) = 0, \\ &p^1 : \omega^2 \frac{d}{d\omega}\hat{u}_1(\omega) = \frac{d}{d\omega}\hat{H}_0(\omega), \qquad \hat{u}_1(0) = 0, \\ &p^2 : \omega^2 \frac{d}{d\omega}\hat{u}_2(\omega) = \frac{d}{d\omega}\hat{H}_1(\omega), \qquad \hat{u}_2(0) = 0, \\ &p^3 : \omega^2 \frac{d}{d\omega}\hat{u}_3(\omega) = \frac{d}{d\omega}\hat{H}_2(\omega), \qquad \hat{u}_3(0) = 0, \end{aligned} \tag{32}$$

and so on. The solution of the recursive equation, Eq. (32), can be compactly written as

$$\begin{aligned} &\hat{u}_0(\omega) = -i/\omega, \\ &\hat{u}_1(\omega) = -i/3\omega^3, \\ &\hat{u}_2(\omega) = 1/i\omega^5, \\ &\hat{u}_3(\omega) = 25/3i\omega^7, \end{aligned} \tag{33}$$

and so on. Using the Maple symbolic code the inverse Fourier transform of Eq. (33) is

$$\begin{aligned} &u_0(x) = 1, \\ &u_1(x) = -x^2/6, \\ &u_2(x) = x^4/24, \\ &u_3(x) = -5x^6/432, \end{aligned} \tag{34}$$

and so on. Consequently, the solution of Eq. (29) in a series form is given by

$$u(x, t) = 1 - (x^2/6) + (x^4/24) - (5x^6/432) + \cdots. \tag{35}$$

Owing that the Taylor series expansion for $(1/\sqrt{1 + \frac{x^2}{3}})$ is

$$1/\sqrt{1 + \frac{x^2}{3}} = 1 - (x^2/6) + (x^4/24) - (5x^6/432) + \cdots \tag{36}$$

by comparing Eqs. (36) and (35), Eq. (35) can be ultimately reduced to

$$u(x) = 1/\sqrt{1 + \frac{x^2}{3}}. \tag{37}$$

In fact, this is the exact solution of the problem, Eq. (29).

Example 3. The other form of the Lane–Emden equation for an isothermal gas sphere can be written as

$$\begin{aligned} &u_{xx} + (2/x)u_x = -e^u, \quad x \geq 0, \\ &u(0) = 0, \quad u_x(0) = 0. \end{aligned} \tag{38}$$

This is a model that describes isothermal gas sphere where the temperature remains constant. By applying the Fourier transform to Eq. (38) we obtain the following equation:

$$\omega^2 \frac{d\hat{u}}{d\omega} = \frac{d}{d\omega}\mathcal{F}\{e^u\}. \tag{39}$$

Substituting the recursive equation, Eq. (17), into Eq. (39) supplies the following equation:

$$\omega^2 \frac{d}{d\omega}\sum_{n=0}^{\infty} p^n \hat{u}_n(\omega) = \frac{d}{d\omega}\sum_{n=0}^{\infty} p^{n+1}\hat{H}_n(\omega). \tag{40}$$

The recursive equation deduced from Eq. (40) can be written as

$$\begin{aligned}
&p^0 : \omega^2 \frac{d}{d\omega}\hat{u}_0(\omega) = 0 \quad \hat{u}_0(0) = 0,\\
&p^1 : \omega^2 \frac{d}{d\omega}\hat{u}_1(\omega) = \frac{d}{d\omega}\hat{H}_0(\omega), \quad \hat{u}_1(0) = 0,\\
&p^2 : \omega^2 \frac{d}{d\omega}\hat{u}_2(\omega) = \frac{d}{d\omega}\hat{H}_1(\omega), \quad \hat{u}_2(0) = 0,\\
&p^3 : \omega^2 \frac{d}{d\omega}\hat{u}_3(\omega) = \frac{d}{d\omega}\hat{H}_2(\omega), \quad \hat{u}_3(0) = 0,
\end{aligned} \tag{41}$$

and so on. The solution of the recursive equation, Eq. (41), can be expressed by

$$\begin{aligned}
&\hat{u}_0(\omega) = 0,\\
&\hat{u}_1(\omega) = -i/3\omega^3,\\
&\hat{u}_2(\omega, t) = 1/5i\omega^5,\\
&\hat{u}_3(\omega, t) = 8/21i\omega^7,
\end{aligned} \tag{42}$$

and so on. Using the Maple symbolic code, the inverse Fourier transform of Eq. (42) is

$$\begin{aligned}
&u_0 = 0,\\
&u_1 = -x^2/3.2!,\\
&u_2 = x^4/5.4!,\\
&u_3 = -x^6/21.6!,
\end{aligned} \tag{43}$$

and so on. Consequently, the solution of Eq. (38) in a series form is given by

$$u(x) = -\frac{1}{3.2!}x^2 + \left(\frac{1}{5.4!}x^4\right) - \left(\frac{8}{21.6!}x^6\right) + \left(\frac{122}{81.8!}x^8\right) - \left(\frac{61.67}{495.10!}x^{10}\right) + \cdots. \tag{44}$$

Using the Maple symbolic code, the Pade approximants of [3, 3], [5, 5] and [7, 7] are given by

$$[3, 3] = -\frac{1}{6}\frac{x^2}{1 + \frac{1}{20}x^2}, \qquad [5, 5] = \frac{-\frac{1}{6}x^2 - \frac{431}{55080}x^4}{1 + \frac{89}{918}x^2 + \frac{43}{25704}x^4},$$

$$[7, 7] = \frac{-\frac{1}{6}x^2 - \frac{15587041}{319406280}x^4 - \frac{72143758}{27668569005}x^6}{1 + \frac{912438}{2661719}x^2 + \frac{78025283}{2635101810}x^4 + \frac{3508303919}{5691819909600}x^6}.$$

Fig. 1 shows the trend of convergence of the solution for the three Pade approximants of [3, 3], [5, 5] and [7, 7]. Shown in Fig. 2 are the variations of the results versus space coordinates using logarithmic scales. It is clear that as $x \to \infty$ the results approach 1. This turns out to be an important property such as the thermal equilibrium of $u(x)$.

5. Conclusions

In this paper, a method of incorporating He's polynomial into the HPM combined with the Fourier transform is proposed. The nonlinear partial differential equations related to the Lane–Emden problem are solved. The validity and the effectiveness of the method are shown in a systematic fashion. The proposed solutions of the three singular linear and nonlinear differential equations of Lane–Emden problem using the method have verified the physical properties of the equilibrium of Lane–Emden problem, as $x \to \infty$ the solution monotonically approaches a constant. This is indeed an important physical property of the Lane–Emden problem known as the equilibrium state.

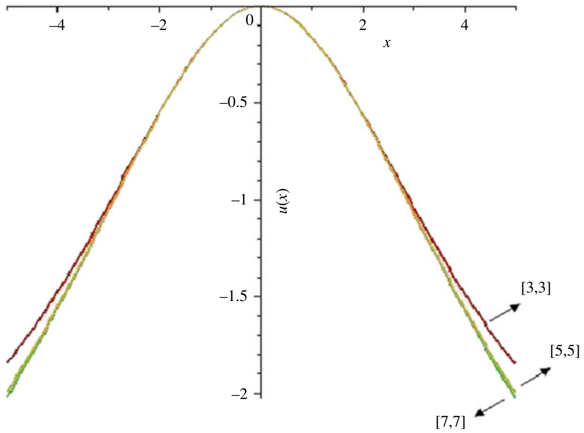

Fig. 1. Plots of the [3, 3], [5, 5] and [7, 7] Pade approximant in Eq. (44).

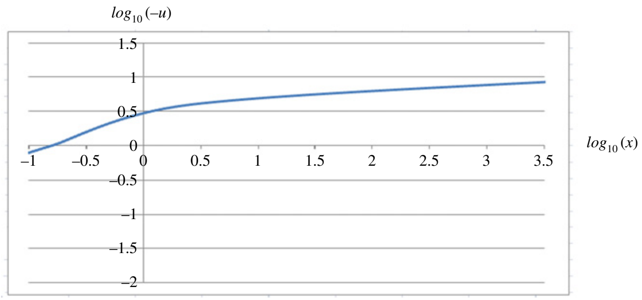

Fig. 2. Variations of the computed results using the modified HPM solution of Eq. (38).

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