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On rational 2-groups and their nilpotency classes

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On rational 2-groups and their nilpotency classes

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Abstract

A finite group G is called a rational group if all the generators of every cyclic subgroup of G are conjugate. In this article we discuss about nilpotency class of rational 2-groups and we give an upper bound for nilpotency class of a rational group G of order 2^n. Furthermore we show that an irreducible character χ of a rational 2-group G does not appear as a constituent of character χ^2 except for $\chi = 1_G$, the principal character of G .

AMS subject Classification 2010: 20C15.

Keywords: Rational group, Sylow subgroup, Nilpotency class.

1 Introduction and preliminaries

Let G be a finite group. We call G a rational group, or briefly a $\mathbb{Q}$-group, if for every $x \in G$, all the generators of $\langle x \rangle$ are conjugate in G. It is well known that a group G is rational if and only if for each irreducible complex character χ of G and every $g \in G$, the value $\chi(g)$ is a rational number. Some famous examples of rational groups are the symmetric group $\mathrm{S_n}$, the Weyl groups of complex Lie algebras and the elementary abelian 2-groups. Many mathematicians have studied rational groups, and yet there are some unsolved problem concerning them. In 1976, Gow in [3], have shown that the order of every solvable $\mathbb{Q}$-group G, has the form $2^\alpha 3^\beta 5^\gamma$. As it is proved that every finite rational group has an even order, the Sylow 2-subgroups of nontrivial $\mathbb{Q}$-groups are nontrivial, and there was a long standing conjecture that states "the Sylow 2-subgroups of rational groups are also rational". However the inaccuracy of this conjecture is proved by Isaacs and Navarro in [4], in 2012. In spite of this fact, they have proved that if a Sylow 2-subgroup P of a solvable rational group G has nilpotency class 2 then P is rational. A conjecture about the composition factors of a rational group states that, if $\mathbb{Z}_p$ is a composition factor of a rational group then $p \leq 5$. Relating to this conjecture in [9], J.G. Thompson in 2008, has proved that $p \leq 11$. Darafsheh and Sharifi in [2] recognized the Frobenius $\mathbb{Q}$-groups. The authors of the paper in [5]

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characterized the rational groups whose irreducible characters vanish only on involutions. Throughout the paper we assume that G is a finite group and we use the standard notations. Here we recall some properties of rational groups, that we need for our reasoning. For the proof of propositions 1.1 and 1.2, one can see [6], which is a comprehensive reference for studying $\mathbb{Q}$-groups.

Proposition 1.1. *Let G be a rational group. Then $Z(G)$ is an elementary abelian 2-group, moreover if $H \lhd G$ then G/H is a rational group.*

Proposition 1.2. *If G is a rational group and $x \in G$ then $N_G(\langle x\rangle)/C_G(\langle x\rangle) \cong Aut(\langle x\rangle)$.*

In order to study the nilpotency class of rational 2-groups, the work of Y. Takegahara and T. Yoshida, entitled by "Character theoritical aspects of nilpotency class", see [7], motivated us to think about the subject, from the point of view of matrix theory. In the sequel we will assign to every finite group G, a matrix T, and discuss about the nilpotency class of G via the index of nilpotency of T. Following the notation of [7], here we suppose that G is a finite group and $\chi_0 = 1_G, \chi_1, \ldots, \chi_d$ are all the irreducible complex characters of G and we set $\Lambda = \{0, 1, \cdots, d\}$ and $\Lambda^0 = \Lambda - \{0\}$. Suppose that for $i, j, k \in \Lambda$, the non-negative integers t_{ijk} are defined such that

$$\chi_i \chi_j = \sum_{k=0}^{d} t_{ijk} \chi_k.$$

For $j \in \Lambda$, it is defined $\hat{j}$ to be the member of Λ for which $\chi_{\hat{j}}(g) = \overline{\chi_j(g)}$, i.e., the complex conjugate of $\chi(g)$. Then T is defined as a $d \times d$ matrix whose rows and columns are indexed by the members of Λ^0, such that its (i, j) entry equals $t_{\hat{j}ji}$. The next result is the restatement of Theorem 1.2 of [7].

Proposition 1.3. *The following conditions are equivalent:*

(1) G is nilpotent of class r.

(2) The matrix T is nilpotent of index of nilpotence r.

2 Main Results

One knows that the maximal nilpotence class of a group of order p^n is $n - 1$.

Lemma 2.1. *Let G be a nonabelian rational group of order 2^n. Then G is of maximal class if and only if $n = 3$ and G is isomorphic to the dihedral group D_8 or quaternion group Q_8.*

Proof. Obviously, the nilpotence class of each of two groups D_8 and Q_8 is 2 and they are of maximal class. For the reverse, by [1, Corrolary 1.7], G is isomorphic to one of the groups D_{2^n}, Q_{2^n} or SD_{2^n}. But every group of these types is metacyclic and so G has an element x such that $G/\langle x\rangle$ is cyclic, hence by Proposition 1.1, $G/\langle x\rangle \cong \mathbb{Z}_2$. Since G is nonabelian, $|G| \geq 8$, and hence $o(x) > 2$. As $Z(G)$ is elementary abelian, $x \notin Z(G)$ and since the index of $\langle x\rangle$ in G is 2, we see that $C_G(\langle x\rangle) = \langle x\rangle$. Now by Proposition 1.2, we have

$$Aut(\langle x\rangle) \cong \frac{N_G(\langle x\rangle)}{C_G(\langle x\rangle)} \cong \frac{G}{\langle x\rangle} \cong \mathbb{Z}_2 \qquad (*)$$

Suppose that $o(x) = 2^a$, then $a \geq 2$ and we have $Aut(\langle x \rangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$ and then by $(*)$ we have $a = 2$, i.e. $o(x) = 4$ and $|G| = 8$. But all the nonabelian rational groups of order 8 are D_8 and Q_8 and the proof is complete. □

Using the GAP system [8], there are exactly two nonabelian rational group of order 2^4, which are isomorphic to $D_8 \times \mathbb{Z}_2$ and $Q_8 \times \mathbb{Z}_2$. Obviously these two groups have nilpotence class 2. The number of nonabelian rational groups of order 2^5 is 9, among them two group is of nilpotence class 3 and others are of class 2. Among the groups of order 2^n, $n \geq 6$, we didn't find a rational group of nilpotence class $n-2$. This motivated us to prove the following assertion.

Theorem 2.2. *Suppose that G is a $\mathbb{Q}$-group of order 2^n and $n > 5$. Then the nilpotency class of G is at most $n-3$.*

Proof. For $n = 6$, the assertion holds according to the information in GAP system [8]. So let $n \geq 7$ and suppose that every rational group of order 2^m, with $m \in \{6, \cdots, n-1\}$, satisfies the above mentioned claim. Now let $cl(G) = k$ and suppose that $N = G^{k-1}$ is the last nontrivial term in lower central series of G. Then for some s we have $|N| = 2^s$. Obviously $s \geq 1$. Now we know that G/N is a rational group of order 2^{n-s} and $n - s \leq n - 1$. Then if $n - s \geq 6$, by the induction hypothesis $k - 1 = cl(G/N) \leq n - s - 3$, so $k \leq n - s - 2 \leq n - 3$. Therfore we suppose that $n - s < 6$; but if this is the case then again using the GAP system we have $k - 1 = cl(G/N) \leq 3$, so $k \leq 4$. But by the hypothesis $n \geq 7$; Therefore $n - 3 \geq 4$ and so $k \leq n - 3$ and the proof is complete. □

Theorem 2.3. *Let G be a rational 2-group and χ be a non-principal irreducible character of G. Then χ is not a constituent of χ^2.*

Proof. Suppose that d, Λ, Λ^0 and $t_{\hat{j}ji}$ be as introduced in the introduction section. As G is rational all its characters are rational valued. So $\chi_{\hat{j}}(g) = \chi_j(g)$ for every $g \in G$ and $j \in \Lambda$. Therefore $\hat{j} = j$ for every $j \in \Lambda$. Then by the definition of t_{ijk} we have

$$\chi_j^2 = \sum_{i=0}^{d} t_{jji}\, \chi_i.$$

But then as G is nilpotent, by Proposition 1.3, T is a nilpotent matrix and since all the entries of T are non-negative, all its diagonal entries are 0. It means that $t_{jjj} = 0$ for all $j \in \Lambda^0$ and this completes the proof. □

It is worthwhile to mention that, one can generalize the assertion of the obove theorem to every group G, having all the characters real valued and in this case, the above proof works properly. By the way the claim of Theorem 2.3 does not hold for every group. For an example one can check the symmetric group S_3, for which the unique non-linear irreducible character χ, is a constituent of its squared. Every finite abelian group satisfies the claim of the above theorem, because all its irreducible characters are linear. However we could not find a finite nilpotent group without the above property. Now the following question arises.

Question 2.4. Let G be a finite nilpotent group and χ be a non-principal irreducible character of G. Is that correct to say, χ is not a constituent of χ^2?

In our studying the rational 2-groups from the matrix theory point of view, as we mentioned above, using the GAP system [8], we found that the $d \times d$ matrix T, associated with the rational 2-group G, is always an upper triangular matrix, with all the diagonal entries 0. Having Theorem 2.2 in mind, we surmise that when n become greater, the upper bound of nilpotency class of G will become smaller. Now the following question can arise.

Question 2.5. Is it possible to find a better upper bound for nilpotency class of a $\mathbb{Q}$-group G of order 2^n, via studying the index of nilpotence of its associated matrix T?

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