Sparsity Order Estimation under Constrained Budget

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***Abstract*—In many applications of compressed sensing it is either necessary or beneficial to have a priori information about the sparsity order $\|\mathbf{x}\|_0$ of an unknown sparse vector $\mathbf{x}$. As shown in [1], $\frac{\|\mathbf{x}\|_1}{\|\mathbf{x}\|_2}$ provides a more reliable measure of compressibility than the sparsity order. The latter could also be estimated by observing the vector through random linear measurements following Cauchy and Gaussian distributions. In this paper, we determine the optimal number of Cauchy and Gaussian measurements to minimize the estimation error under limited measurement budget.**

I. Background

In various applications such as some of the greedy recovery techniques in compressed sensing, a prior knowledge or an upperbound on the ℓ_0 (pseudo-)norm or sparsity order of an unknown vector $\mathbf{x}_{n\times 1}$ is required. In some applications such as cognitive radio, instead of recovering a sparse vector from compressed measurements, the goal is to first determine whether the vector is sparse, and second to identify its support [2]. In this application, the full radio spectrum plays the role of an unknown and high-dimensional vector $\mathbf{x}_{n\times 1}$ in which one wishes to detect vacant frequency intervals for temporary wireless usage.

Since sparsity order estimation might be a preprocessing task for full recovery of the signal, it is beneficial to base the estimation technique on data sets that could be reused in the recovery task. Inline with this argument, it is shown in [3] that if a sparse recovery method is applied to linear measurements of the form $\mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\mathbf{x}_{n\times 1}$ with small m, although the recovered signal is generally invalid, its sparsity level is a fair indicative of the original sparsity level. For robust estimation of the sparsity order, numerical results indicate that m could be set around 70% of the full recovery requirement.

In [4], [5], the sparsity order estimation has been modeled as a source enumeration problem. The classical solutions in this context consist of estimating the covariance matrix and distinguishing between noise and signal subspaces. The latter could be related to the eigenvalues of the covariance matrix. Another example that connects sparsity order of the vector to the eigenvalues of the covariance matrix is presented in [6].

The estimation of numerical sparsity, S_n, defined as

$$S_n = \frac{\|\mathbf{x}\|_1}{\|\mathbf{x}\|_2},$$

is proposed in [1] as an alternative to sparsity order ($\|\mathbf{x}\|_0$) estimation. It is even shown that S_n is a superior measure than $\|\mathbf{x}\|_0$ in noisy scenarios.

II. Preliminaries

For estimating S_n, it is proposed to separately estimate $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$. These estimates are obtained by observing $\mathbf{x}$ through symmetric α-stable (SαS) random vectors with $\alpha = 1, 2$. The random variable φ is a SαS with scale parameter σ if its characteristic function $\widehat{p}_\varphi(\omega)$ satisfies

$$\widehat{p}_\varphi(\omega) := \mathbb{E}\big\{\mathrm{e}^{\mathrm{j}\omega\varphi}\big\} = \exp\big(-|\sigma\,\omega|^\alpha\big), \quad \alpha \in (0, 2]. \tag{1}$$

Zero-mean Cauchy and Gaussian random variables are special examples of the SαS family with $\alpha = 1$ and $\alpha = 2$, respectively. If $c_1, \ldots, c_l$ are arbitrary real numbers and $\varphi_1, \ldots, \varphi_l$ are independent SαS random variables with unit scale parameter, then $\sum_{i=1}^{l} c_i\varphi_i$ is again a SαS random variable with scale parameter $\sigma = \sqrt[\alpha]{\sum_{i=1}^{l} |c_i|^\alpha}$ [7]. Thus, if $\boldsymbol{\varphi}_1^{(\alpha)}, \ldots, \boldsymbol{\varphi}_m^{(\alpha)}$ are independent n-dimensional vectors with i.i.d. entries distributed according to the SαS with $\sigma = 1$, then, $\big\{\langle \boldsymbol{\varphi}_i^{(\alpha)}, \mathbf{x}\rangle\big\}_{i=1}^{m}$ are i.i.d. SαS random variables with $\sigma = \|\mathbf{x}\|_\alpha$. Hence, estimating $\|\mathbf{x}\|_\alpha$ for $\alpha = 1, 2$ requires linearly measuring the vector $\mathbf{x}$ via zero-mean Cauchy and Gaussian random vectors, and empirically estimating the scale parameter of the outcomes:

$$\widehat{S_n} = \frac{\mathrm{median}_{1\le i\le m_1}\{|\langle \boldsymbol{\varphi}_i^{(1)}, \mathbf{x}\rangle|\}}{\sqrt{\frac{1}{m_2}\sum_{i=1}^{m_2} |\langle \boldsymbol{\varphi}_i^{(2)}, \mathbf{x}\rangle|^2}}, \tag{2}$$

where m_1 and m_2 are the number of Cauchy and Gaussian measurements, respectively.

III. Main Result

Here, we seek the optimal strategy for setting m_1 and m_2 when the overall budget $L = m_1 + m_2$ is fixed.

Theorem 1. *The estimator $\widehat{S_n}$ in (2) is biased. For the bias-compensated version $\widehat{S}_{n,U}$ we can show that*

- *the estimation error is minimized when*

$$m_1 = \frac{\pi}{\pi + \sqrt{2}}(L - 2) \approx 0.69L,$$

- *the relative error with the above choice is independent of n (ambient dimension) and is given by*

$$\frac{\mathrm{Var}(\,\widehat{S}_{n,U}\,)}{S_n} = \frac{(\pi + \sqrt{2})^2}{L - 2},$$

- *compared to the 50%-50% strategy, the optimal setting of m_i results in 19% reduction of the estimation error.*

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