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Saeid Jafari

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" Some Problems Concerning Finite Groups with Rational- Valued Irreducible Characters "

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Title: On connected tetravalent Cayley graphs of a non-abelian group of order $3p^2$
Speaker : Majid Abdollahi Kivi

Let G be a finite group and S be a subset of G such that $S = S^{-1}$ and $1 \notin S$. The Cayley graph of G with respect to S is denoted by $\Gamma = Cay(G, S)$ and has its vertex set G and edge set $\{\{x, sx\} |\ x \in G,\ s \in S\}$. Therefore Γ is a regular graph of valency $|S|$, and it is connected if and only if S generates G. I want to talk about tetravalent Cayley graphs of the non-abelian group of order $3p^2$, where p is a prime number greater than 3, and a Sylow p-subgroup of G is cyclic. I will show that all of these tetravalent Cayles graphs are normal. The full automorphism group of these Cayley graphs will be given and the half-transitivity and the arc-transitivity of these graphs will be investigated. I show that this group is a 5-CI-group.

Title: Some Problems Concerning Finite Groups with Rational-Valued Irreducible Characters
Speaker : Saeid Jafari

In this talk I will explain the results we found about rational groups, i.e. the groups with a character table which all its entries are rational integers. Also I will present some problems about rational groups with extraspecial Sylow 2-subgroups. By the work of Darafsheh and Sharifi on Frobenius $\mathbb{Q}$-groups (2004), when Sylow 2-subgroup of a rational group G is Q_8 then G is a $\{2, 3\}$ or $\{2, 5\}$-group and the unique involution of Q_8 acts fixed point freely on Sylow 3-subgroup or Sylow 5-subgroup of G. In the case where the Sylow 2-subgroup of G is D_8, the dihedral group of order 8, it is not hard to see that G is a $\{2, 3\}$-group and there are many evidence that at least one involution acts fixed point freely on the Sylow 3-subgroup; and this forces the Sylow 3-subgroup to be elementary abelian. Although we have not achieved the proof of this proposition yet, I am interested to present our work and get ideas of other researchers.

This is joint work with H. Sharifi

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SOME PROBLEMS CONCERNING FINITE GROUPS WITH RATIONAL-VALUED IRREDUCIBLE CHARACTERS

S. JAFARI AND H. SHARIFI

Department of Mathematics, Faculty of Science,
Shahed University, Tehran, Iran
s.jafari@shahed.ac.ir

ABSTRACT. In this talk I will explain the results that we found about rational groups, the groups with a character table which all its entries are rational integers. Also I will present some problems about rational groups with extra-special sylow 2-subgroups. By the work of Darafsheh and Sharifi on Frobenius $\mathbb{Q}$-groups (2004), when Sylow 2-subgroup of a rational group G is Q_8 then G is a $\{2,3\}$ or $\{2,5\}$-group and the unique involution of Q_8 act fixed point freely on Sylow 3-subgroup or Sylow 5-subgroup of G. In the case where the Sylow 2-subgroup of G is D_8, the dihedral group of order 8, It is not hard to see that G is a $\{2,3\}$-group and there are many evidence that at least one involution act fixed point freely on the Sylow 3-subgroup; and this forces the Sylow 3-subgroup to be elementary abelian. Although yet we did not achieve the proof of this proposition, I am interested to present our work and get the idea of the other researchers.

Keywords: Rational group, Dihedral group, Fixed point free action

1. Introduction

Let G be a finite group all whose irreducible complex characters are rational valued. Such a group G is called a **rational group** or a **$\mathbb{Q}$-group**.

1.1. **Examples and non examples.** Some examples of $\mathbb{Q}$-groups are the elementary abelian 2-groups, symmetric groups S_n, dihedral groups of orders 2, 4, 6, 8, 12 and the Weyl groups of the complex Lie algebras. The cyclic group $G = C_3$, of order 3 is a non example of minimum order, as if $G = \langle \zeta \rangle$, then both ζ and ζ^2 generate G but as G is abelian, they are not conjugate; In view of the second definition $\zeta = e^{2\pi i/3}$ is a non rational number in the character table of G. With the same reason every cyclic group of order greater than 2 is non rational.

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1.2. **Some facts about $\mathbb{Q}$-groups.** One can find some important properties of $\mathbb{Q}$-groups in [5].

Theorem 1.1. *Gow(1976) The order of every solvable $\mathbb{Q}$-group G, has the form $2^{\alpha}3^{\beta}5^{\gamma}$.*

Theorem 1.2. *Every finite $\mathbb{Q}$-group has an even order.*

In view of the above theorem the Sylow 2-subgroup of a nontrivial rational group is nontrivial.

Theorem 1.3. *Let G be a rational group. Then $Z(G)$ is an elementary abelian 2-group, moreover every quotient of a rational group and the direct product of finitely many rational groups is also rational.*

Theorem 1.4. *If G is a rational group and $x \in G$ then $N_G(\langle x\rangle)/C_G(\langle x\rangle) \cong Aut(\langle x\rangle)$.*

1.3. **An old conjecture.** The Sylow 2-subgroup of a rational group is rational.

Counterexamples: In 2012, Isaacs and Navarro have shown that, there are two rational groups of order 1536, with non rational Sylow 2-subgroups of nilpotence class 3; In spite of this fact, they have proved that if a Sylow 2-subgroup P of a solvable rational group G has nilpotency class 2 then P is rational.

2. Main Results

In addition to some other results we have proved the following three theorems in [4].

Theorem 2.1. *Let G be a solvable rational group and $P \in Syl_2(G)$ with $cl(P) \leq 2$ and $K \in Syl_3(G)$. If G' is nilpotent, then G is a $\{2,3\}$-group and $G \cong K \rtimes P$ and we have the following*

- (a) *If G' is abelian, then $G \cong E(3^k) \rtimes P$ for some k, and G contains a normal elementary abelian 2-subgroup H such that $G/H \cong E(3^m) \rtimes E(2^n)$, for some $m \geq 0$ and $n \geq 0$.*
- (b) *G' is non-abelian if and only if K is non-abelian.*

By M we mean the Markel group of order 200, i.e. the semidirect product of $\mathbb{Z}_5 \times \mathbb{Z}_5$ and the Sylow 2-subgroup of $SL(2,5)$ (which is isomorphic to Q_8, the quaternion group of order 8) via the natural action; This group is indexed as $SmallGroup(200,44)$ in the GAP-system [2].

Theorem 2.2. *Let G be a solvable rational group and $P \in Syl_2(G)$ with $cl(P) \leq 2$. If G' is not nilpotent, then we have the following*

- (a) *If G is a $\{2,5\}$-group, then $G/O_2(G) \cong \prod_{i=1}^{k} M_i$, in which, for every $i \in \{1,\cdots,k\}$, M_i is a copy of Markel group M.*
- (b) *If G is a $\{2,3\}$-group, $K \in Syl_3(G')$ and $H \in Syl_2(G')$ with $H \leq P$, then $K \lhd G'$ if and only if $P' = H$.*

Theorem 2.3. *Let G be a non-solvable rational group and $P \in Syl_2(G)$ with $cl(P) \leq 2$. Then every non-cyclic composition factor of G is isomorphic to A_n for $n \in \{5, 6, 7\}$.*

Let $R_{\mathbb{Q}}(G)$ and $P(G)$ denote the ring of $\mathbb{Z}$-linear combinations of rationally represented and permutation characters of a finite group G, respectively. We call the exponent of $R_{\mathbb{Q}}(G)/P(G)$, the Artin exponent of G, and we use the notation $\gamma(G)$ for it. Indeed $\gamma(G)$ is the minimal number $d \in \mathbb{N}$ such that $d\chi \in P(G)$ for all $\chi \in R_{\mathbb{Q}}(G)$. By a theorem of Artin, $\gamma(G)$ divides $|G|$. Yet there is no a characterization for the groups satisfying $\gamma(G) = 1$.

Theorem 2.4. *If G is a $\mathbb{Q}$-group with a Sylow 2-subgroup isomorphic to Q_8 then $\gamma(G) = 1$.*

Theorem 2.5. *If G is a $\mathbb{Q}$-group with a Sylow 2-subgroup isomorphic to D_8 and an abelian normal Sylow 3-subgroup then $\gamma(G) = 1$.*

One of our goals was studying non-nilpotent rational groups with extra-special Sylow 2-subgroups. In our work on the subject the next problems arise

Problem 1 : Let G be a non-nilpotent rational group with an extra-special Sylow 2-subgroup and a nontrivial center. Does there exist such a group?

Problem 2 : Is that correct to say, there is no a non-nilpotent rational group G with an extra-special Sylow 2-subgroup P such that $|P| > 8$?

Problem 3 : Let G be a rational group with a Sylow 2-subgroup isomorphic to D_8 and $K \in Syl_3(G)$. Is that correct to say K is abelian?

Using GAP system [2], we conjecture that the answers to the above problems are No, Yes and Yes, respectively. However yet we could not prove these claims.

Here is the main theorem of [1], by Darafsheh and Sharifi:

Theorem 2.6. *If G is a Frobenius $\mathbb{Q}$-group, then exactly one of the following occurs*

(i) *We have $G \cong E(3^n) \rtimes \mathbb{Z}_2$, where $n \geq 1$ and $\mathbb{Z}_2$ acts on $E(3^n)$ by inverting every nonidentity element.*

(ii) *We have $G \cong E(3^{2m}) \rtimes \mathbb{Q}_8$, where $m \geq 1$ and $E(3^{2m})$ is a direct sum of m copies of the 2-dimensional irreducible representation of $\mathbb{Q}_8$ over the field with 3 elements.*

(iii) *We have $G \cong E(5^2) \rtimes \mathbb{Q}_8$, where $E(5^2)$ is the 2-dimensional irreducible representation of $\mathbb{Q}_8$ over the field with 5 elements.*

It is proved that every non-nilpotent rational group with quaternion group as a Sylow 2-subgroup, is a Frobenius group (Berkovich). In view of the previous theorem one can see that if the Sylow 2-subgroup of a $\mathbb{Q}$-group is Q_8, then G is a $\{2, 3\}$ or $\{2, 5\}$-group and the unique involution of Q_8 act fixed point freely on Sylow 3-subgroup or Sylow 5-subgroup of G.

In the case where the Sylow 2-subgroup of G is D_8, the dihedral group of order 8, It is not hard to see that G is a $\{2,3\}$-group and there are many evidence that at least one involution act fixed point freely on the Sylow 3-subgroup; and this forces the Sylow 3-subgroup to be elementary abelian. Although yet we did not achieve the proof of this proposition, I am interested to get the idea of the other researchers.

In our investigation using *GAP*-system [2], to find rational groups with D_8 as Sylow 2-subgroup, we saw that, there is only one group of order $2^3 \times 3$, (which is $\mathbb{S}_4$), two groups of order $2^3 \times 3^2$, i.e. $SmallGroup(72, i)$, $i \in \{40, 43\}$, only one group of order $2^3 \times 3^3$, $SmallGroup(216, 165)$, two groups of order $2^3 \times 3^4$, i.e. $SmallGroup(648, i)$, $i \in \{725, 738\}$ and only one group of order $2^3 \times 3^5$ which is $SmallGroup(1944, 3946)$. In all of these groups the Sylow 3-subgroup is elementary abelian and it is normal in some cases.

Here we mention that, when G is a non-nilpotent $\mathbb{Q}$-group with a dihedral Sylow 2-subgroup, our conjecture is that the Sylow 3-subgroup is elementary abelian and more precisely, G has one of the following forms:

(1) $((E(3^k) : \mathbb{Z}_2) \times (E(3^k) : \mathbb{Z}_2)) : \mathbb{Z}_2$
(2) $(E(3^k) \times A_4) : \mathbb{Z}_2$,

in which k is a positive integer and A_4 is the alternating group of degree 4. Also the Sylow 3-subgroup of G in (1) is normal and in (2) is not.

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