Asymptotic capacity of multi-band decentralised wireless networks under Gaussian code book assumption

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A communication network composed of M sub-channel and N transmitter-receiver pairs, each of unit transmit power constraint, is considered. The capacity of this network under Gaussian code book assumption for the case of a decentralised single-hop and one-shot game is investigated, showing the corresponding sum-rate capacity behaves like $M \log(N) - \theta(\log\log(N))$.

Introduction: Bandwidth partitioning has been regarded as one of the main concerns in wireless networks. This is due to the limited amount of bandwidth and more demands, arising from emerging wireless technologies. Spectrum sharing is a key solution to this need [1].

Generally, there are two main power allocation methods, centralised against decentralised approaches. The former makes use of a central node to set the transmit power of each link with a view to increasing the network throughput (e.g. [2]). The latter, however, assumes each link sets the corresponding power selfishly, while considering the network benefits in terms of increasing the network throughput [3]. To this end, most of the decentralised schemes attempt to define a proper utility function for each link, which aims at increasing the network throughput [4].

The main contribution of the current work is the introduction of the optimum decentralised power allocation method for the case of having several independent frequency bands, when each transmitter makes use of a Gaussian code book. This optimum power allocation method is first addressed in [5], however its optimality is not proved there.

The rest of this Letter is organised as follows. The following Section presents the mathematical model in detail. Accordingly, the optimum power allocation strategy, leading to the sum-rate capacity is investigated in the subsequent Section. The final Section summarises the findings.

Throughout the Letter, for any function $f(n)$ and $g(n)$, $f(n) = O(g(n))$, $f(n) = o(g(n))$ and $f(n) = w(g(n))$ are equivalent, respectively, to $\lim_{n\to\infty} |f(n)/g(n)| < \infty$, $\lim_{n\to\infty} |f(n)/g(n)| = 0$ and $\lim_{n\to\infty} |f(n)/g(n)| = \infty$.

System model: We consider N transmitter-receiver pairs, communicating over M I.I.D. sub-channels. It is assumed that the number of links (N) as well as the number of sub-channels (M) are the common information at the transmitters. Also, each transmitter only knows the direct channel gains of M sub-channels to the affiliated receiver, and is restricted to transmit with unit power.

The corresponding channel gain between the jth receiver and the ith transmitter in the kth sub-channel is denoted by a random variable $g_{ji}^{(k)}$, for $i, j \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$. Assuming a quasi-static Rayleigh fading channel, thus the channel strength $h_{ji}^{(k)} = |g_{ji}^{(k)}|^2$ has an exponential distribution, i.e. $Pr_{h_{ij}^{(k)}}(x) = e^{-x}$ for $x \geq 0$.

We also assume the average transmit power of the ith user in the kth sub-channel is represented by $p_i^{(k)} \in [0, 1]$. Thus, to have the unit transmit power constraint, it follows $\sum_{k=1}^{M} p_i^{(k)} \leq 1$ for $i = 1, \ldots N$.

Optimum power allocation strategy: The following theorem states the optimum power allocation strategy.

Theorem: Assuming there are N links, communicating over M I.I.D. sub-channels, such that each link has only access to its direct channel gains over M sub-channels, then the optimum power allocation strategy is to communicate over the best sub-channel if the corresponding channel strength exceeds a certain threshold, otherwise the transmitter should remain silent.

Proof: The received signal of the ith user from M sub-channels can be formulated as the following matrix notation,

$$\mathbf{y}_i = \mathbf{G}_{ii}\mathbf{x}_i + \sum_{\substack{j=1\\ j\neq i}}^{N} \mathbf{G}_{ij}\mathbf{x}_j + \mathbf{n} \tag{1}$$

where $\mathbf{x}_i = [x_i^{(1)}\ x_i^{(2)} \ldots x_i^{(M)}]^T$ is the transmitted vector of the ith user over M sub-channels. $\mathbf{y}_i = [y_i^1\ y_i^2\ \ldots\ y_i^M]^T$ represents the received vector, and $\mathbf{G}_{ij}$ denotes the diagonal channel matrix between the jth transmitter and the ith receiver, i.e. $\mathbf{G}_{ij} = \mathrm{diag}[g_{ij}^{(1)}, g_{ij}^{(2)}, \ldots, g_{ij}^{(M)}]$, where $g_{ij}^{(k)}$ denotes the corresponding channel gain between the jth transmitter and the ith receiver in the kth sub-channel.

Considering $\mathbf{I}_{ij} = \mathbf{G}_{ij}\mathbf{x}_j$ as the interfering signal arising from the jth transmitter at the ith receiver, one can readily compute the corresponding correlation matrix of this interfering signal as follows

$$\begin{aligned}\mathbb{E}[\mathbf{I}_{ij}\mathbf{I}_{ij}^{\dagger}] &= \mathbb{E}_{\mathbf{G}_{ij}}[\mathbf{G}_{ij}\mathbb{E}_{\mathbf{x}_j}[\mathbf{x}_j\mathbf{x}_j^{\dagger}]\mathbf{G}_{ij}^{\dagger}] \\ &= \mathbb{E}_{\mathbf{G}_{ij}}[\mathbf{G}_{ij}\Lambda_j\mathbf{G}_{ij}^{\dagger} =^{(a)} \mathrm{diag}(\Lambda_j) = \Sigma_j\end{aligned} \tag{2}$$

where in (2) it is assumed $\mathbb{E}[\mathbf{x}_j\mathbf{x}_j^{\dagger}] = \Lambda_j$ for $j = 1, \ldots, N$. Also, $\Sigma_j = \mathrm{diag}(\Lambda_j)$ is a matrix whose diagonal elements are the same as that of Λ_j, while its off diagonal entries are zero. Also, owing to the the unit transmit power constraint, $\mathrm{trac}(\Lambda_j) \leq 1$, hence it follows trace $\Sigma_j \leq 1$. Again, noting the power allocation strategy of the jth link merely depends on the direct channel gain matrix $\mathbf{G}_{jj}$, thus Σ_j is a function of $\mathbf{G}_{jj}$, hence, one can easily verify that

$$\begin{aligned}\mathbb{E}[\mathbf{I}_i\mathbf{I}_i^{\dagger}] &= \sum_{j=1, j\neq i}^{j=N} \mathbb{E}[\mathbf{I}_{ij}\mathbf{I}_{ij}^{\dagger}] \\ &=^{(a)} \sum_{j=1, j\neq i}^{j=N} \Sigma_j \simeq^{(b)} (N-1)\mathbb{E}_{\mathbf{G}_{ii}}[\Sigma_i] \simeq (N-1)\overline{\Sigma}\end{aligned} \tag{3}$$

where (a) comes from the fact that $\mathbb{E}[\mathbf{I}_{ij}\mathbf{I}_{ij}^{\dagger}]$ for $j = 1, \ldots, M$ are independent. Also, referring to the central limit theorem and noting the aforementioned interfering signals arising from different transmitters are drawn from the same distribution, the sum of $N-1$ identically distributed random variables can be approximated using a Gaussian distribution. Thus, assuming the kth diagonal entry of $\mathbb{E}[\mathbf{I}_i\mathbf{I}_i^{\dagger}]$, is $i^{(k)}$ it follows

$$\begin{aligned}&Pr\{(N-1)\overline{\mu}^{(k)}(1-\delta) < i^{(k)} < (N-1)\overline{\mu}^{(k)}(1+\delta)\} \\ &\qquad \simeq 1 - 2Q\left(\frac{(N-1)\overline{\mu}^{(k)}\delta}{\sqrt{(N-1)\sigma^{(k)^2}}}\right)\end{aligned} \tag{4}$$

where in (4), $\overline{\mu}^{(k)}$ and $\sigma^{(k)^2}$ denote, respectively, the kth diagonal entry of $\overline{\Sigma}$ and the average interference power of the kth sub-channel. It can be verified that by setting $\delta = O(\sqrt{\frac{\ln N}{N}})$ and noting the approximation $Q(x) = \frac{1}{\sqrt{2\pi x}}e^{x^2/2}$ for $x \gg 1$, $\mathbb{E}[\mathbf{I}_i\mathbf{I}_i^{\dagger}]$ with probability approaching one tends to the following,

$$\mathbb{E}[\mathbf{I}_i\mathbf{I}_i^{\dagger}] = (N-1)\left(1 + O\left(\sqrt{\frac{\ln N}{N}}\right)\right)\overline{\Sigma} \tag{5}$$

This justifies (b) in (3). Thus it follows,

$$\mathbb{E}[\mathbf{y}_i\mathbf{y}_i^{\dagger}|\mathbf{G}_{ii}] \simeq \mathbb{E}_{\mathbf{X}_i}[\mathbf{G}_{ii}\mathbf{x}_i\mathbf{x}_i^{\dagger}\mathbf{G}_{ii}^{\dagger}] + (N-1)\overline{\Sigma} + \sigma_n^2\mathbf{I} \tag{6}$$

Note that there is a dependency between $\mathbf{x}_i$ and $\mathbf{G}_{ii}$, as it is assumed each transmitter has access to its channel gain matrix. On the other hand, using Hadamard inequality, one can argue that when dealing with network throughput, the diagonal entries of matrix $\mathbf{x}_i\mathbf{x}_i^{\dagger}$ merely affect the result. The Hadamard inequality states that for a given square matrix $\mathbf{K}$ of dimension n and diagonal entries k_{ii} for $i = 1, \ldots, n$, we have $det(\mathbf{K}) \geq \prod_{i=1}^{n} k_{ii}$. Moreover, the equality holds if $\mathbf{K}$ is a diagonal matrix.

Let us define the kth diagonal entry of matrix $\mathbf{x}_i\mathbf{x}_i^{\dagger}$ as $f_k(.)$ which in fact is a function of direct channel gain matrix $\mathbf{G}_{ii}$, i.e. $f_k(g_{ii}^{(1)}, g_{ii}^{(2)}, \ldots, g_{ii}^{(M)})$. The problem is to find the set of functions $f = \{f_1(.), \ldots, f_M(.)\}$ for which the network throughput is maximised. Noting the above and using (3) and (6), the achievable throughput under Gaussian code book assumption becomes

$$\overline{R}_i = \mathbb{E}_{\mathbf{G}_{ii}}\Bigg[\max_{\sum_{k=1}^{M} f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})\le 1} \log\det[\pi e(\mathbf{G}_{ii}\mathbf{x}_i\mathbf{x}_i^{\dagger}\mathbf{G}_{ii}^{\dagger} + (N-1)\overline{\Sigma} + \sigma_n^2\mathbf{I}] - \log_2\det[\pi e((N-1)\overline{\Sigma} + \sigma_n^2\mathbf{I})]\Bigg]$$
$$\le^{(a)} \mathbb{E}_{\mathbf{G}_{ii}}\Bigg[\max_{\sum_{k=1}^{M} f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})\le 1} \sum_{k=1}^{M}\log \times\left(1+\frac{|g_{ii}^{(k)}|^2 f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})}{(N-1)\overline{\mu}^{(k)}+\sigma_n^2}\right)\Bigg] \tag{7}$$

where (a) is a direct consequence of applying Hadamard inequality. It should be noted that if one simply sets the off diagonal entries of matrix $\mathbf{x}_i\mathbf{x}_i^{\dagger}$ to zero, the inequality in (a) can be replaced with equality. Also, $\overline{\mu}^{(k)}$ represents the kth diagonal entry of diagonal matrix $\overline{\Sigma}$. Owing to the symmetrical properties of sub-channels, it can be concluded that $\overline{\mu}^{(k)} = \mathbb{E}[f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})] = \overline{\mu}$ for the optimum power allocation strategy. Noting the above and using the fact that the interference is strong enough, i.e. $(N-1)\overline{\mu}^{(k)} = (N-1)\mathbb{E}[f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})] = \omega(1)$. Thus considering $\log(1+x)\simeq x$ for $x \ll 1$, the aforementioned problem can be cast as the following optimisation problem,

$$\max_{\sum_{k=1}^{M} f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})\le 1} \sum_{k=1}^{M} \mathbb{E}_{\mathbf{G}_{ii}}[|g_{ii}^{(k)}|^2 f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})]$$
$$\text{s.t. } \mathbb{E}[f_k(g_{ii}^{(1)},\ldots,g_{ii}^{(M)})] = \frac{w(1)}{N-1},\ k\in\{1,M\} \tag{8}$$

Let us define $|g_{ii}^{(k)}|^2 \triangleq h_{ii}^{(k)}$ as the channel strength of the kth sub-channel which has the exponential distribution. Thus, the real function $f_k(.)$ can be considered as a function of $h_{ii}^{(k)}$ for $k = 1, \ldots, M$. As a result, using the method of Lagrange multipliers, (8) can be converted to

$$L(f_1(.),\ldots,f_k(.),\lambda_1,\ldots,\lambda_M)$$
$$=\Bigg[\sum_{k=1}^{M}\left(\int\cdots\int_0^{\infty} h_{ii}^{(k)} f_k(h_{ii}^{(1)},\ldots,h_{ii}^{(M)})e^{-\sum_{m=1}^{M} h_{ii}^{(m)}}dh_{ii}^{(1)}\ldots dh_{ii}^{(M)}\right) + \lambda_k\Big(\frac{w(1)}{N} - \int\cdots\int_0^{\infty} f_K(h_{ii}^{(1)},\ldots,h_{ii}^{(M)})e^{-\sum_{m=1}^{M} h_{ii}^{(m)}}dh_{ii}^{(1)}\ldots dh_{ii}^{(M)}\Big)\Bigg]$$
$$\text{s.t. } \sum_{k=1}^{M} f_K(h_{ii}^{(1)},\ldots,h_{ii}^{(M)}) \le 1 \tag{9}$$

Again, owing to the symmetrical properties of sub-channels, it can be verified that λ_k for $k = 1, \ldots, M$ are the same. Thus, (9) simplifies to

$$L(f_1(.),\ldots,f_M(.),\lambda) = \int\cdots\int_0^{\infty}\sum_{k=1}^{M}(h_{ii}^{(k)}-\lambda)f_k(h_{ii}^{(1)},\ldots,h_{ii}^{(M)}) \times e^{-\sum_{m=1}^{M} h_{ii}^{(m)}}dh_{ii}^{(1)}\ldots dh_{ii}^{(M)} + \frac{\lambda M w(1)}{N}$$
$$\text{s.t. } \sum_{k=1}^{M} f_k(h_{ii}^{(1)},\ldots,h_{ii}^{(M)}) \le 1,\ \lambda \ge 0 \tag{10}$$

Considering λ is fixed, this problem can be thought of as a definite integral of weighted linear combination of unknown functions $f_k(.)$ subject to the linear affine constraint $\sum_{k=1}^{M} f_k \le 1$. Noting the corresponding weight of $f_k(.)$ is $(h_{ii}^{(k)}-\lambda)e^{-(h_{ii}^{(1)+\ldots+h_{ii}^{(M)}})}$, it can be easily verified that for each realisation of $h_{ii}^{(k)}$ for $k = 1, \ldots, M$, the aforementioned optimisation problem is maximised when the function corresponding to the maximum positive weight is set to one and others take zero values. Also, if all weights are negative, i.e. $h_{ii}^{(max)} < \lambda$, all functions should take zero values. As a result, $f_k^{opt}(h_{ii}^{(1)},\ldots,h_{ii}^{(M)}) = 1$ if $h_{ii}^{(k)} = max(h_{ii}^{(1)},\ldots,h_{ii}^{(M)},\lambda)$, otherwise it sets to zero. Thus, for the optimum set of functions $f_k^{opt}(.)$ for $k = 1, \ldots, M$, it follows,

$$L(f_1^{opt}(.),\ldots,f_M^{opt}(.),\lambda) = M\int_{\lambda}^{\infty}(h_{ii}^{(1)}-\lambda)e^{-h_{ii}^{(1)}} \times (1-e^{-h_{ii}^{(1)}})^{M-1}dh_{ii}^{(1)} + \frac{\lambda M w(1)}{N} \tag{11}$$

According to the method of Lagrange multipliers, the optimum value of λ (λ^{opt}) can be obtained through setting the derivation of $L(f_1^{opt}(.),\ldots,f_M^{opt}(.),\lambda)$ to zero,

$$\frac{\partial}{\partial\lambda}L(f_1^{opt}(.),\ldots,f_M^{opt}(.),\lambda) = (1-e^{-\lambda})^M + \frac{Mw(1)}{N} - 1 = 0$$

As a result, for a large value of N, and noting λ^{opt} should take a large value, it follows

$$M\log(1-e^{-\lambda^{opt}}) = \log\left(1-\frac{Mw(1)}{N}\right) \simeq -\frac{Mw(1)}{N}$$
$$-Me^{-\lambda^{opt}} \simeq -\frac{Mw(1)}{N} \Rightarrow \lambda^{opt} =^{(a)} \log(N) - \theta_N(\log\log(N))$$

where (a) comes from the fact that according to the definition of $w(1)$, one can replace it with $\log(N)$. In [5], it is proved that the achievable sum-rate of this strategy scales as

$$\bar{R} \simeq M\log N - \theta_N(\log\log(N)) \tag{12}$$

Conclusion: The optimum power allocation strategy for multi-band decentralised wireless networks under Gaussian code book assumption is studied, showing the optimal solution is a threshold based sub-channel selection together with an on-off scheme, inferring cooperation among sub-channels does not increase the network throughput.

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