Asymptotic capacity of multi-band decentralised wireless networks under Gaussian code book assumption

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A communication network composed of $M$ sub-channel and $N$ transmitter-receiver pairs, each of unit transmit power constraint, is considered. The capacity of this network under Gaussian code book assumption for the case of a decentralised single-hop and one-shot game is investigated, showing the corresponding sum-rate capacity behaves like $M \log(N) - \Theta(\log\log(N))$.

Introduction: Bandwidth partitioning has been regarded as one of the main concerns in wireless networks. This is due to the limited amount of bandwidth and more demands, arising from emerging wireless technologies. Spectrum sharing is a key solution to this need [1]. Generally, there are two main power allocation methods, centralised against decentralised approaches. The former makes use of a central node to set the transmit power of each link with a view to increasing the network throughput (e.g. [2]). The latter, however, assumes each link sets the corresponding power selfishly, while considering the sub-channel is represented by $x$.

Sub-channels, such that each link has only access to its direct channel assumption for the case of a decentralised single-hop and one-shot game is investigated, showing the corresponding sum-rate capacity behaves like $M \log(N) - \Theta(\log\log(N))$.

Theorem: Optimum power allocation strategy:

$$
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$$

$$
\text{Theorem: Assuming there are } N \text{ links, communicating over } M \text{ I.I.D. sub-channels. It is assumed that the number of links (as well as the number of sub-channels) are the common information at the transmitters. Also, each transmitter only knows the direct channel gains of } M \text{ sub-channels to the affiliated receiver, and is restricted to transmit with unit power.}

The corresponding channel gain between the $j$th receiver and the $i$th transmitter in the $k$th sub-channel is denoted by a random variable $g_{ik}$, for $i,j \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$. Assuming a quasi-static Rayleigh fading channel, thus the channel strength $h_{ik} = |g_{ik}|^2$ has an exponential distribution, i.e. $P_{h_{ik}}(x) = e^{-x/\alpha}$ for $x \geq 0$.

We also assume the average transmit power of the $i$th user in the $k$th sub-channel is represented by $P_{ik} = E[h_{ik}] = \frac{\alpha}{N}$.

Opportunity power allocation strategy: The following theorem states the optimum power allocation strategy.

Proof: The received signal of the $i$th user from $M$ sub-channels can be formulated as the following matrix notation,

$$
y_i = G_i x_i + \sum_{j \neq i} G_{ij} x_j + n_i
$$

where $x_i = [x_{i1}^T, x_{i2}^T, \ldots, x_{iM}^T]^T$ is the transmitted vector of the $i$th user over $M$ sub-channels. $y_i = [y_{i1}^T, y_{i2}^T, \ldots, y_{iM}^T]^T$ represents the received vector, and $G_i$ denotes the diagonal channel matrix between the $i$th transmitter and the $i$th receiver, i.e. $G_i = \text{diag}(g_{i1}^M, g_{i2}^M, \ldots, g_{iM}^M)$, where $g_{ij}^M$ denotes the corresponding channel gain between the $j$th transmitter and the $i$th receiver in the $k$th sub-channel.

Considering $x_i = G_i x_i$ as the interfering signal arising from the $j$th transmitter at the $i$th receiver, one can readily compute the corresponding interference matrix of this interfering signal as follows

$$
E[y_i y_i^T] = E_{i} G_i G_i^T = \text{diag}(\lambda_i)
$$

where in (2) it is assumed $E[y_i y_i^T] = \lambda_i$ for $i = 1, \ldots, N$. Also, $\lambda_i = \text{diag}(\lambda_i)$ is a matrix whose diagonal elements are the same as that of $\lambda_i$, while its off-diagonal entries are zero. Also, owing to the unit transmit power constraint, $\text{tr}(\lambda_i) \leq 1$, hence it follows trace $\lambda_i \leq 1$.

Again, noting the power allocation strategy of the $k$th link merely depends on the direct channel gain matrix $G_i$, thus $\lambda_i$ is a function of $G_i$, hence, one can easily verify that

$$
E[y_i y_i^T] = \sum_{j=1}^{N} E[y_{ij} y_{ij}^T] = \sum_{j=1}^{N} \lambda_{ij}
$$

where $(a)$ comes from the fact that $E[y_i y_i^T] = \lambda_i$ for $i = 1, \ldots, M$ are independent. Also, referring to the central limit theorem and noting the aforementioned interfering signals arising from different transmitters are drawn from the same distribution, the sum of $N-1$ identically distributed random variables can be approximated using a Gaussian distribution. Thus, assuming the $\lambda$th diagonal entry of $E[y_i y_i^T]$, is $\lambda_i$ it follows

$$
Pr[(N-1)\eta^{\lambda_i} (1-\delta) < \lambda_i < (N-1)\eta^{\lambda_i}(1+\delta)] \\
\approx 1 - 2O\left(\frac{(N-1)\eta^{\lambda_i} (1+\delta)}{\sqrt{(N-1)\eta^{\lambda_i}}}\right)
$$

where in (4), $\eta^{\lambda_i}$ and $\delta^{\lambda_i}$ denote, respectively, the $\lambda$th diagonal entry of $\Sigma$ and the average interference power of the $k$th sub-channel. It can be verified that by setting $\delta = \Theta(\frac{\log N}{N})$ and noting the approximation $O(x) = \frac{x^{1/2}}{2} \sqrt{x}$ for $x \gg 1$, $E[y_i y_i^T]$ with probability approaching one tends to the following,

$$
E[y_i y_i^T] = (N-1)\left(1 + O\left(\frac{\log N}{N}\right)\right)
$$

This justifies $(b)$ in (3). Thus it follows,

$$
E[y_i y_i^T] \approx E_{i} G_i G_i^T + (N-1)\Sigma + \Sigma^{\text{diag}}
$$

Note that there is a dependency between $x_i$ and $G_i$, as it is assumed each transmitter has access to its channel gain matrix. On the other hand, using Hadamard inequality, one can argue that when dealing with network throughput, the diagonal entries of matrix $xx^T$ merely affect the result. The Hadamard inequality states that for a given square matrix $K$ of dimension $n$ and diagonal entries $k_{ii}$ for $i = 1, \ldots, n$, we have $det(K) \geq \prod_{i=1}^{n} k_{ii}$. Moreover, the equality holds if $K$ is a diagonal matrix.

Let us define the $\lambda$th diagonal entry of matrix $xx^T$ as $f_{\lambda}(.)$ which in fact is a function of direct channel gain matrix $G_i$, i.e. $f_{\lambda}(g_{i1}^M, g_{i2}^M, \ldots, g_{iM}^M)$. The problem is to find the set of functions $f_{\lambda}(.)$ for which the network throughput is maximised. Noting the above and using (3) and (6), the achievable throughput under Gaussian code
book assumption becomes
\[
\mathbf{P}_k = \mathbb{E}_c \left[ \max_{\sum_k h_k^{(1)}, \ldots, h_k^{(M)} \leq 1} \log \det (\mathbf{G}_k x_k \mathbf{G}_k^T) \right] + (N - 1) \mathbb{E}_c [\sigma_k^2] - \log \det (\mathbf{G}_k \mathbf{G}_k^T) \]

\[= \mathbb{E}_c \left[ \max_{\sum_k h_k^{(1)}, \ldots, h_k^{(M)} \leq 1} \sum_{\ell=1}^M \log \left( 1 + \frac{\nu_k^{(1)} f_k(h_k^{(1)}), \ldots, h_k^{(M)})}{(N - 1) \mathbb{E}_c [\sigma_k^2]} \right) \right] \]

(7)

where (a) is a direct consequence of applying Hadamard inequality. It should be noted that if one simply sets the off diagonal entries of matrix \( \mathbf{G}_k \) to zero, the inequality in (a) can be replaced with equality. Also, \( \mathbf{P}_k^{(k)} \) represents the \( k \)-th diagonal entry of diagonal matrix \( \mathbf{P}_k \). Owing to the symmetrical properties of sub-channels, it can be concluded that \( \mathbf{P}_k^{(k)} = \mathbb{E}[f_k(g_k^{(1)}, \ldots, g_k^{(M)})] = \mathbf{P}_k \) for the optimum power allocation strategy. Noting the above and using the fact that the interference is strong enough, i.e. \((N - 1)P_k^{(k)} = (N - 1) \mathbb{E}[f_k(g_k^{(1)}, \ldots, g_k^{(M)})] = \Theta(1)\). Thus considering \( \log(1 + x) \geq x \) for \( x < 1 \), the aforementioned problem can be cast as the following optimisation problem,

\[
\max_{\sum_k h_k^{(1)}, \ldots, h_k^{(M)} \leq 1} \sum_{k=1}^M \mathbb{E}_c [\log \left( 1 + \frac{\nu_k^{(1)} f_k(h_k^{(1)}), \ldots, h_k^{(M)})}{N - 1} \right) ]
\]

s.t. \( \mathbb{E}[f_k(g_k^{(1)}, \ldots, g_k^{(M)})] = \nu_k^{(1)}/N \) for \( k \in [1, M] \).

Let us define \( g_k^{(1)}, \ldots, g_k^{(M)} \) as the channel strength of the \( k \)-th sub-channel which has the exponential distribution. Thus, the real function \( f_k(.) \) can be considered as a function of \( h_k^{(1)}, \ldots, h_k^{(M)} \) for \( k = 1, \ldots, M \). As a result, using the method of Lagrange multipliers, (8) can be converted to

\[
L(f_1(\ldots), f_k(\ldots), \lambda_1, \ldots, \lambda_M) = \sum_{k=1}^M \left( \int_0^{\infty} h_k^{(1)} f_k(h_k^{(1)}, \ldots, h_k^{(M)}) e^{-\sum_{j=1}^M h_j^{(j)}} dh_k^{(1)} \ldots dh_k^{(M)} \right) + \lambda_k \left( \nu_k^{(1)}/N - 1 \right)
\]

s.t. \( \sum_{k=1}^M f_k(h_k^{(1)}, \ldots, h_k^{(M)}) \leq 1 \), \( \lambda_k > 0 \)

(9)

Again, owing to the symmetrical properties of sub-channels, it can be verified that \( \lambda_k \) for \( k = 1, \ldots, M \) are the same. Thus, (9) simplifies to

\[
L(f_1(\ldots), f_k(\ldots), \lambda) = \int_0^{\infty} \sum_{k=1}^M \left( f_k(h_k^{(1)}, \ldots, h_k^{(M)}) - \lambda f_k(h_k^{(1)}, \ldots, h_k^{(M)}) \right) dh_k^{(1)} \ldots dh_k^{(M)} e^{-\sum_{j=1}^M h_j^{(j)}} + \frac{\lambda M \nu_k^{(1)}}{N}
\]

s.t. \( \sum_{k=1}^M f_k(h_k^{(1)}, \ldots, h_k^{(M)}) \leq 1 \), \( \lambda > 0 \)

(10)

Considering \( \lambda \) is fixed, this problem can be thought of as a definite integral of weighted linear combination of unknown functions \( f_k(\cdot) \) subject to the linear affine constraint \( \sum_k f_k \leq 1 \). Noting the corresponding weight of \( f_k(\cdot) \) is \( (\lambda f_k(h_k^{(1)}, \ldots, h_k^{(M)}) - \lambda f_k(h_k^{(1)}, \ldots, h_k^{(M)}) \), it can be easily verified that for each realisation of \( h_k^{(1)} \) for \( k = 1, \ldots, M \), the aforementioned

optimisation problem is maximised when the function corresponding to the maximum positive weight is set to one and others take zero values. Also, if all weights are negative, i.e. \( h_k^{(1)} W_k^{(1)} < \lambda \), all functions should take zero values.

As a result, \( f_k^{(1)}(h_k^{(1)}, \ldots, h_k^{(M)}) = 1 \) if \( h_k^{(1)} = \max(h_k^{(1)}, \ldots, h_k^{(M)}, \lambda) \), otherwise it sets to zero. Thus, for the optimum set of functions \( f_k^{(1)}(.) \) for \( k = 1, \ldots, M \), it follows,

\[
L(f_1^{(1)}(\ldots), \ldots, f_M^{(1)}(\ldots), \lambda) = M \int_0^{\infty} \left( h_1^{(1)} - \lambda \right) e^{-h_1^{(1)}}
\]

\[
\times \left( 1 - e^{-h_1^{(1)}} \right) M - 1 + \lambda M \nu_1^{(1)}
\]

(11)

According to the method of Lagrange multipliers, the optimum value of \( \lambda \) (\( \lambda^{(1)} \)) can be obtained through setting the derivation of \( L(f_1^{(1)}(\ldots), \ldots, f_M^{(1)}(\ldots), \lambda) \) to zero,

\[
\frac{d}{d\lambda} L(f_1^{(1)}(\ldots), \ldots, f_M^{(1)}(\ldots), \lambda) = (1 - e^{-h_1^{(1)}}) M - \lambda M \nu_1^{(1)} = 0
\]

As a result, for a large value of \( N \), and noting \( \lambda^{(1)} \) should take a large value, it follows

\[
M \log(1 - e^{-\lambda^{(1)}}) \approx - \frac{M \nu_1^{(1)}}{N}
\]

(12)

Conclusion: The optimum power allocation strategy for multi-band decentralised wireless networks under Gaussian code book assumption is studied, showing the optimal solution is a threshold based sub-channel selection together with an on-off scheme, inferring cooperation among sub-channels does not increase the network throughput.

References

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