A Scenario-Based Objective Function for an M/M/K Queuing Model with Priority (A Case Study in the Gear Box Production Factory)

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Abstract— **In most optimization problems of an M/M/K queuing model with priority, the scenarios are defined and optimized separately. However, sometimes, the scenarios are dependent and should be optimized simultaneously. In this paper, an efficient approach is developed for this purpose. The performance of the proposed approach is evaluated through a real case study in a ware house of a Gear Box production factory.**

Key words— **Queuing model, Optimization, M/M/K with priority**

I. Introduction

THE case study investigated in this paper, is related to Gear Box production company which has a high market share of Gear Box in Iran. In this company, the queuing system for the final product warehouse is studied to determine the optimum number of docks. Now, there exists three docks available and each dock requires two operators for loading trucks. This company has two types of customers. The main customer makes up 40% of the cash flow and is considered as the most important customer among the others. The other 60% cash flow is distributed among four less priority customers. The cash flow shares and the priority of all customers are summarized in TABLE 1. If the customers are waited in the queues, the company should pay a penalty cost to customers. The top manager of the company has decided to decrease the waiting time as much as possible. As faster as the company can send products to the customer, the cash flow would increase. Hence, less wait in queue time is required even if the top manager has to omit a low priority customer in future contracts.

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TABLE 1
Customers Information

Costumer No.	Demand Share	Priority
1	40%	1
2	21%	2
3	16%	3
4	12%	4
5	11%	5

As mentioned above, top manager policy will accept to eliminate one of the customers among customers number 2 to 5 from future contract list.

The way that queuing theory models have been used in practice is categorized in two major groups. The first group contains the cases where queuing theory models have been used to obtain the value of queuing system evaluation indices such as l_q and w_q for a few decision scenarios and choosing the best optimal solution. In more advanced cases an objective function is considered in which decision variables are l_q or w_q and the value of this function is calculated under any of scenarios l_q^S and w_q^S and finally, the best alternative among available scenarios are selected.

The application of the first category was optimization of Berth Crane operations in a maritime terminal [1]. Also a few other researches have used this approach in various cases to improve queuing systems. This approach has shown a good efficiency in optimization of complex queuing networks [2]. By combining queuing theories with scheduling knowledge, optimizing the loading and unloading operations in container terminal is considered in [3]. Another example is the usage of this approach for selecting the best configuration for a manufacturing system in which queuing networks existed [4].

The other category that can be called optimization based application of queuing theory models focuses on defining an objective function in which the decision variables are l_q or w_q and the cost or benefit function are computed based on l_q or w_q. For the second category, an optimization procedure for finding the optimal value of N in an N policy $M/E_K/1$ queuing system with a removable service station is considered in [5]. A recursive method to control an $M/G/1$ queuing system is proposed by [6]. The optimal policy of an $M/E_k/1$ while the service station was removable is given in [7]. An $M/G/1$ queuing model was studied in [8] to determine the optimal control of the queuing model while the capacity of the queue was definite.

Accordingly, there are some researches which improved the second approach for problems with fuzzy input parameters. As an example, an $M/M/K$ queuing model with fuzzy input parameters was studied in [9]. In this paper the number of servers is optimized based on the degree of customer's satisfaction. The research for computing queue length in a $MAP/G/1/N$ model [10] as well as an N policy $G/M/1$queuing system can be notable examples [11] in this category. Furthermore, examples of queuing networks with finite capacity queues can be seen in [12]. Defining the service rate while parameters of the queuing model are fuzzy is another example of this category [13].

The rest of the paper is organized as follows: In section II, proposed approach is explained. In section III, the problem is modeled and the queuing system is analyzed. Moreover, the objective function is defined. In section IV, the optimization procedure is presented and the results of optimization are illustrated. In the final section conclusion is discussed.

II. Proposed approach

A. Modeling the Problem

Modeling a queuing system includes estimating λ and μ, mathematical definition of queue policies as well as recognition the type of queue in the case that it matches any of available models. For estimating λ and μ, a sampling procedure should be done to calculate the average time between interval and the average time of serving. The distribution time of arrival and service time in queuing systems should be determined in advance. Statistical methods such as goodness-of-fit test or graphical methods such as probability plots can be used for this purpose.

B. Evaluation of the Current System

In this step by knowing the type of queuing model the performance measures of the current system are calculated.

C. Cost and Benefit Terms

Before defining the objective function, an initial step is to define the cost terms affecting the objective which are in accordance with l_q or w_q. The benefits of serving a customer with a lower wait in queue are defined through a proportion of cash flow per customer.

D. Objective Function

Using the terms of costs and benefits, objective function is easily defined by subtracting costs from benefits.

E. Optimization Procedure

To optimize the objective function, an iterative algorithm is proposed in which, in every step by changing decision variables, the value of objective function is obtained until the time no more improvement of the objective function is achieved.

F. Sensitivity Analysis

Finally a sensitivity analysis can be helpful for evaluating possible changes of input parameters of the model.

III. Illustrative Exaple

This paper investigates the application of an $M/M/K$ with priority model to optimize an objective function for the daily benefits for a warehouse of Gear Box production company. The full description of the case is given in the introduction section.

A. Modeling the Problem

To model the problem, estimations of λ *and* μ are required. For this purpose, a dataset of a 5 working days period is gathered. The dataset for one of these days is shown in TABLE 2.

For determining the distribution of time between arrivals, plotting the histogram of the dataset in TABLE 2 can be helpful. The histogram of the dataset in TABLE 2 is shown in Fig. 1.

TABLE 2
Data Set For The Time Between Intervals

Demand No	Time of Arrival	Time between arrivals(Minute)
1	08:16	-
2	08:30	14
3	08:32	2
4	08:39	7
5	09:03	24
6	09:19	16
7	09:25	6
8	09:42	17
9	09:52	10
10	10:30	38
11	11:01	31
12	11:41	40
13	12:22	41
14	13:20	48
15	14:39	79
16	15:31	52
17	15:55	24

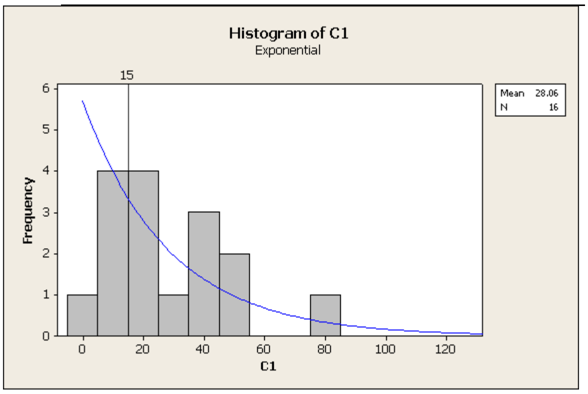

Fig. 1. Histogram of Time between arrivals

As shown in the histogram of Fig. 1, the dataset follow an exponential distribution, but for an accurate statistical judgment, a goodness-of-fit test approves the assumption of exponential distribution for the dataset. Minitab results for this test are illustrated in Fig. 2.

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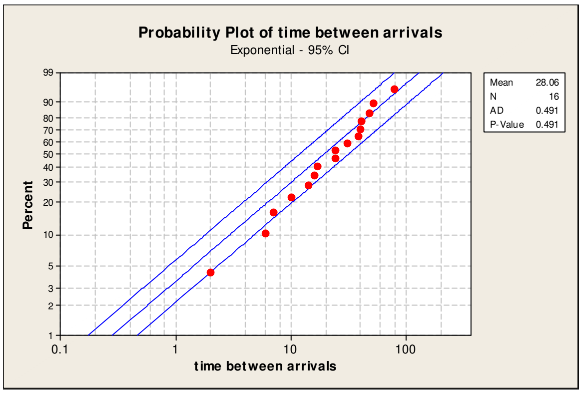

Fig. 2. A-D test for time between arrivals

Some descriptive statistics and the P-value of Anderson-Darling(AD) Test for the dataset in TABLE 2 are summarized in TABLE 3.

TABLE 3
Descriptive Statistics

N	N^*	Mean	St Dev
16	0	28.0625	20.6832
Median		Minimum	Maximum
24		2	79
Goodness of Fit Test:			
Distribution		AD	P
Exponential		0.491	0.491

Based on the number of the orders for a four months period given in TABLE 4, λ, the arrival rate, is estimated by using Equation (1).

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad (1)$$

TABLE 4
Data Set Needed For Estimating λ

Month	Number of received orders	Average Number of orders in a day (x_i)
March	546	21
April	468	18
May	390	15
June	520	20

The number of orders in a day summarized in TABLE 4 is based on the assumption that the number of working days in month is 26.

The estimation of λ is computed as

$$\hat{\lambda} = \frac{21+18+15+20}{4} = 18.5 \text{ orders par day}$$

In Table 5, the needed time for serving the customer at each dock are shown. The A-D test results which approves exponentially distribution for service rate per any servers (docks) is given in TABLE 6. Also the results of goodness-of-fit test for dataset in TABLE 5 are illustrated in TABLE 6.

TABLE 5
Data Set Needed For Estimating μ

	Service Time Dock1 (Min)	Service Time Dock2 (Min)	Service Time Dock3 (Min)
1	51	90	93
2	20	32	74
3	45	96	43
4	35	63	62
5	72	50	65
6	163	30	23
7	51	75	169
8	72	145	46
9	39	-	60

TABLE 6
Goodness of Fit Test for Exponential Distribution of service time

Dock	AD	P
1	1.116	0.071
2	1.015	0.093
3	1.222	0.053

The procedure for estimation of μ which represents the rate of serving each customer (loading trucks) is roughly similar to estimating λ.

For a more accurate estimation of μ , the service time for ten days was recorded and finally the average of service times was fitted for μ.

$$\frac{1}{\mu} = 72.28 \text{ minutes}$$

Results in

$$\mu = 6.46 \text{ orders per day}$$

Hence, the service rate for all three servers (docks) is equal to 6.46 orders per day.

Based on the calculations of this section, the time between arrivals and service time for all docks follow exponential distribution with $\lambda = 26$ and $\mu = 6.46$ orders per day, respectively. There are three servers, and each customer has a specific priority, an $M/M/K$ model with considering priorities of customers is suitable for this case. Note that preemption of serving is not allowed in our case. i.e., when serving a customer starts, it cannot be interrupted in any ways.

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B. Calculation of queuing system evaluation indices

To calculate evaluation indices, first w_q is calculated by equations (2), (3), and (4). Then, l_q is obtained by equation (5). Finally, using equations (6), (7), w and l are calculated, respectively. The notations are as follows:

l_q : The average number of customers waiting in queue per day.

w_q : The average time of waiting in a queue per customer.

w: The average time that a customer should spend in a system. (Average time of waiting in the queue plus the average time of being served)

l: The average number of customers waiting in the system

$$A = m!(m\mu-\lambda)\left(\frac{\mu}{\lambda}\right)^m \sum_{j=1}^{m-1} \frac{\left(\frac{\lambda}{\mu}\right)^j}{j!} + m\mu \qquad (2)$$

$$B_0 = 1 \; and \; B_i = 1 - \frac{\sum_{j=1}^{i} \lambda_j}{m\mu}, \quad \text{i=1,2,3,4,5} \qquad (3)$$

where m is the number of sever.

$$w_{qi} = \frac{1}{AB_{i-1}B_i}, \quad i = 1,2,3,4,5 \qquad (4)$$

$$l_{qi} = \lambda w_{qi} \qquad (5)$$

$$w_i = w_{qi} + \frac{1}{\mu} \qquad (6)$$

$$l_i = \lambda w \qquad (7)$$

The results of evaluation indices by using equations (4), (5), (6) and (7) for each customer in current status are shown in TABLE 7.

TABLE 7
Evaluation Indexes For Different Customers

Customer	λ	w_q	w	l	l_q
1	7.400	0.077	0.232	1.716	0.569
2	3.885	0.184	0.339	1.318	0.716
3	2.960	0.431	0.586	1.736	1.277
4	2.220	1.206	1.361	3.022	2.678
5	2.035	7.245	7.400	15.059	14.743

C. Objective function

To model the objective function, note that building a new dock imposes cost of construction as well as cost of hiring two operators for each new dock. The notations of the objective function are defined as following:

$$W_i = \begin{cases} 1 & \text{customer } i\text{th is remain} \\ 0 & \text{customer } i\text{th is eliminated} \end{cases}$$

C: A set of customers

c: Number of customers

m_0: Number of existing docks

TS_i: Average of daily cash flow with *i*th customer

PR_i: The percentage of profit for dealing with *i*th *customer*

W_{qi}: The expected time of waiting in queue for *i*th customer

LR_i: The penalty percentage for lateness in serving *ith* customer

TCD: The daily costs for constructing a new dock (cost in this case is considered for ten years life and hiring 2 operator for each dock)

The objective function is as follows:

$$Z = \left(\sum_{i=1}^{c} W_i \left(TS_i \; PR_i - W_{qi} \; TS_i \; LR_i\right)\right) - (m - m_0)\,TCD \qquad (8)$$

Equation (8) considers total profit, lateness penalty and the cost of new dock, respectively. All of terms are determined based on a day. m and w_i are decision variables. m_0 and c in this case are 3 and 5, respectively. Also, note that, for i=1($i \in C$), W_1 equals one and it is not a decision variable.

IV. Optimization Procedure

According to W_q in TABLE 7, the initial total profit is equal to Z_0=3922300. Note that currency of Iran is Rial.

Since each term in the objective function affects on each other, iterative optimization is suggested for this problem. For this purpose, MATLAB software is applied to find the best solution. Note that m and W_i's are decision variables and should be reported to top manager, in addition to the value of the objective function. The best decision variables for company are shown in TABLE 8.

TABLE 8
Result of Optimization

Best customer for eliminating	Customer 5 (w_5=0)
Number of requirement servers (docks)	3,(m=3)
Initial Total profit per day(Rials)	3.9223e+006
Optimized total profit per day (Rials)	6.6445e+006

In Fig. 3 and Fig. 4, neighborhood solutions are presented for selecting the best decision variables.

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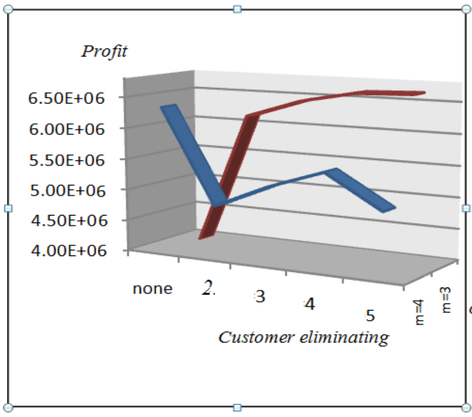

Fig. 3. Profit chart according to customer eliminating and number of docks(m=3 or m=4)

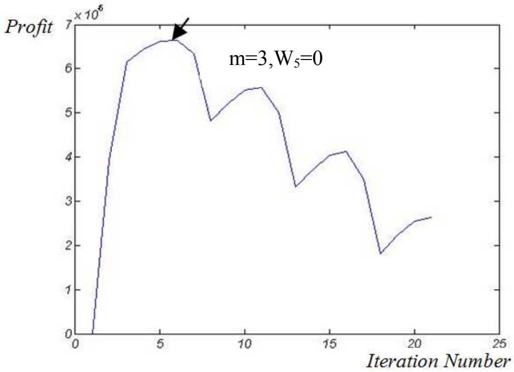

Fig. 4. Iteration plot for selecting the best solution

As shown in Fig. 4. , the company does not need to increase the number of docks. In addition, customer 5 is not profitable for company. Hence, if the customer 5 eliminated, benefit would be maximum.

V. Conclusion

In this paper, we analyzed a queuing system in Gear Box Production Company. The main problem of this company was busy docks for serving customers which had a penalty cost for management. To analyze the problem, we proposed an approach to compare between increasing docks or eliminating the secondary customers simultaneously. This approach determined the best number of docks in addition to candidate customer for eliminating as decision variables and led to profit for company.

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References

[1] P. Canonaco, P. Legato, R. M. Mazza, R.Musmanno, "A Queuing Network Model for the Management of Berth crane operations," *Computers & Operations Research*, vol. 35, no. 8, pp. 2432–2446, 2008.

[2] D. R. Bingham, Y. C. Hung , G. Michailidis, "Developing efficient simulation methodology for complex queueing networks," in *Proceedings of the winter simulation conference*, pp. 512–519, 2003.

[3] E. K. Bish, "A Multiple-Crane-constrained Scheduling problem in a Container Terminal," *European Journal of Operational Research*, vol. 144, no. 1, pp. 83–107, 2003.

[4] P. Canonaco, P. Legato, F. Vocaturo, "Steady state Simulation of Queuing Network Models of Manufacturing Systems for Purposes of ranking and selection of the best Configuration," in *Proceedings of the AIRO XXXV annual conference*, 2004.

[5] W. L. Pearn, Y. C. Chang, "Optimal Management Of the N policy M/E_K/1 Queuing system with a Removable Service Station a sensitivity Investigation," *Computers and Operations Research,* vol. 31, no.7 pp. 83-107, 2004.

[6] K. H. Wang, J. C. Ke, "A recursive method to the optimal control of an M/G/1 queuing system with Definite capacity and indefinite capacity," *Applied Mathematical Modeling*, vol. 24, no.12, pp. 899–914, 2000.

[7] K. H. Wang, H. M. Huang, "Optimal control of an M/E_k /1 queuing system with a removable service station". *Microelectronics and Reliability*, vol. 35, no. 7, pp. 1023–1030, 1995.

[8] Jr. J. Teghem, "Optimal control of a removable server in an M/G/1 queue with Definite capacity," *European Journal of Operational Research*, vol. 31, no. 3, pp. 358–367, 1987.

[9] M. J. Pardo, D. Fuente , "Design of a fuzzy finite capacity queuing model based on the degree of customer satisfaction: Analysis and fuzzy optimization," *Fuzzy Sets and Systems*, vol. 159, no.24, pp. 3313-3332, 2008.

[10] U. C. Gupta, K. Sikdar, "Computing Queue length distributions in MAP/G/1/N queue under single and multiple vacation," *Applied Mathematics and Computations*, vol. 17, no. 2, pp. 1498–1525, 2006.

[11] J.C. Ke, K.H. Wang, "A recursive method for the N policy G/M/1 queuing system with finite capacity," *European Journal of Operational Research*, vol. 142, no. 3, pp. 577–594, 2002.

[12] S. Balsamo, V. D. Persone, P. Inverardi, "A review on queuing network models with finite capacity queues for software architectures performance prediction," *Performance Evaluation,* vol. 51, no. 2, pp. 269–288, 2003.

[13] M. J. Pardo, D. de la Fuente, "Optimal selection of the service rate for a finite input source fuzzy queuing system", *Fuzzy Sets and Systems,* vol. 159, no. 3, pp. 325–342, 2008.