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ROOT TRANSFORMATION METHOD FOR MONITORING CORRELATED VARIABLE AND ATTRIBUTE QUALITY CHARACTERISTICS

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ABSTRACT

Nowadays, quality of many products or processes is represented by two or more correlated quality characteristics. Hence, many control charts have been developed to monitor these quality characteristics in types of multivariate and multi-attribute quality characteristics, separately. However, sometimes quality of a product or a process can be characterized by combination of correlated variable and attribute quality characteristics. To the best of our knowledge there is no research in this area. In this paper a method is proposed to monitor these quality characteristics based on root transformation technique. In the proposed method we first transform the distribution of correlated variable and attribute quality characteristics to approximate multivariate normal distribution based on the reduction of skewness. Then, multivariate control charts such as T^2 are used to monitor transformed data. The performance of the proposed method is evaluated under different step shifts by using simulation studies in terms of average run length criterion. The results show that the proposed method performs satisfactory.

KEYWORDS

Statistical process control (SPC); correlated variable and attribute quality characteristics; root transformation; skewness reduction; phase II; average run length (ARL).

1. INTRODUCTION

Control charts are the most important tools of statistical process control (SPC) used to distinguish the assignable and common causes of variation and leads to improvement in the process. The first control chart is introduced by Shewhart (1931). After that, various control charts are developed by many authors for monitoring different quality characteristics under different situations. For example, refer to exponentially weighted moving average (EWMA) control chart by Roberts (1959) and cumulative sum (CUSUM) chart by Page (1954) in monitoring a variable quality characteristic. Also, see Woodall (1997) for a review on monitoring attribute quality characteristics.

Hotelling (1947) showed that monitoring two or more correlated quality characteristics separately leads to increasing in probability of Type I error in control charts. Type I error occurs when the process is in-control state and control chart gives an alarm. Therefore, researchers have developed control charts for monitoring multivariate and multi-attribute quality characteristics, separately. For example, see some of important approaches in monitoring multivariate quality characteristics in Bersimis et al. (2007). Topalidou and Psarakis (2009) have also reviewed multi-attribute control charts.

In some cases, the quality of a product or a process is represented by correlated variable and attribute quality characteristics. For instance, in plastic manufacturing companies, number of nonconformance and weight of the product are correlated. Since monitoring these quality characteristics individually leads to misleading results, these quality characteristics should be monitored simultaneously. To the best of our knowledge, there is no method in monitoring these types of quality characteristics. Only in one research confidence bound of defect rate is obtained by Kang and Brenneman (2010) for attribute and variable data, but they considered independent assumption for these quality characteristics. In this paper, we propose a method to monitor correlated variable and attribute quality characteristics. Based on the proposed method, we first transform original data by root transformation method which is proposed by Niaki and Abbasi (2007) in such a way that the marginal distribution of transformed data is approximate normal. We then apply multivariate T^2 control chart to monitor the transformed data. The performance of the proposed method is evaluated under different step shifts in mean vector of quality characteristics by using simulation studies in terms of average run length (ARL) criterion.

In the next section the problem is defined and formulated. We explain the proposed method in section 3. The performance of the root transformation method is evaluated in section 4. And finally, section 5 includes comments and some suggestions for future research.

2. PROBLEM DEFINITION

In this paper, we consider a condition in which the quality of a product or a process is represented by the combination of correlated variable and attribute quality characteristics in the $X = (x_1\ x_2)^T$ vector where x_1 is variable and follows normal distribution and x_2 is attribute quality characteristic and occurs on Poisson distribution. Also, we are in phase II of control chart, therefore, the mean vector (μ_X) and covariance matrix (Σ_X) of joint distribution of X are known based on historical data. Since shift in the mean of Poisson distribution leads to changes the variance of it, we assume that the correlation between these quality characteristics is stable. The purpose of this paper is monitoring the mean vector (μ_X) of quality characteristics.

3. PROPOSED APPROACH

As explained in the introduction section, due to correlation between correlated variable and attribute quality characteristics, monitoring each quality characteristic separately leads to increasing probability of Type I error and misleading results. In this paper, we propose root transformation method to transform quality characteristics in such a way that the joint distribution of them follows bivariate normal distribution, approximately. Then, a T^2 multivariate control chart is used to monitor the transformed data. Note that, several transformation methods have been proposed for this purpose. For example, refer to arcsin transformation by Anscombe (1948), root square transformation by Ryan (1989), Q-transformation by Quesenberry (1997), and recently, root transformation by Niaki and Abbasi (2007). It is shown that the root transformation method performs better than the other methods under monitoring multi-attribute quality characteristics (Niaki and Abbasi, 2007). Hence, we used this transformation method in our paper.

Based on the root transformation method, for i^{th} quality characteristic, r_i is determined in such a way that the marginal distribution of r^{th} root transformation ($y_i=x_i^{r}$) has zero skewness. Moreover, the mean vector (μ_Y) and covariance matrix (Σ_Y) of transformed data ($Y=(y_1\ y_2)^T$) change and are estimated based on the transformed historical data using moment method.

In order to find appropriate transformation, a bisection method is employed based on historical data to find the power (r) of the root transformation for each quality characteristic.

The bisection method is based on the opposite signs of function in two side of initial interval. The sign of function is evaluated at the central point of an interval and the central point is replaced instead of limit which has the same sign. The bisection method will be continued until the value of function becomes less than predefined value.

For example, to find a root of $f(r) = 0$ in the interval of (a_0, b_0) where $f(a_0)\, f(b_0) < 0$, we define desired stopping value (ε) and then apply the following algorithm:

$k = 0$,

while $|f(r_{k+1})| > \varepsilon$

$r_{k+1} = (a_k+b_k)/2$

if ($f(r_{k+1})\, f(a_k) < 0$), then

$a_{k+1} = a_k$ and $b_{k+1} = r_{k+1}$

else

$b_{k+1} = b_k$ and $a_{k+1} = r_{k+1}$

end if

$k = k+1$

end while

$r^* = r_k$

In the root transformation method, we consider $f(r)$ as the value of skewness on the r^{th} root transformation of quality characteristics (x_i^{r}). The purpose is finding r such that $f(r)$ becomes near to zero. Therefore, we want to find a root for $f(r) \approx 0$ in the initial interval (0, 1).

Since the variable quality characteristic has normal distribution (based on assumption) we always set $r_1=1$ for it. After finding a proper root (r_2) for transforming attribute data, mean vector and covariance matrix of the transformed data is estimated. In each sampling first transformed quality characteristics are calculated then we can use T^2 control chart to monitor the transformed data. The statistic of control chart is $T^2 = n\,(Y - \mu_Y)^T\, \Sigma_Y^{-1}\, (Y - \mu_Y)$ where n is the number of samples in each subgroup. The upper control limit (UCL) is

$\chi^2_{2,\alpha}$ where α is the desired probability of Type I error in control chart. In the control chart when the control statistic becomes greater than upper control limit, it is considered as a signal. For more information about T^2 control chart, see Mason and Young (2002). Summarized steps are as follows:

- Find a proper root transformation by using bisection method for each quality characteristic based on historical data (Note that for variable quality characteristic the power of root transformation is always set equal to 1).
- Estimate the mean vector and covariance matrix of transformed data by using moment method.
- Calculate the transformed vector for each sample and monitor transformed data in T^2 control chart.

In the next section, the performance of root transformation method for correlated variable and attribute quality characteristics is evaluated by using simulation studies.

4. SIMULATION STUDIES

In this section, performance of the proposed method is evaluated using a numerical example. In this example, quality of a product or a process is characterized by a combination of correlated variable and attribute quality characteristics, where the variable quality characteristic follows a normal distribution with mean 3 and variance 4 and the attribute quality characteristic follows Poisson distribution with parameter 4, and the correlation between them is equal to 0.35.

In the first subsection, the performance of root transformation is investigated using normality test by Jarque and Bera (1987). In the second subsection, the performance of control chart in monitoring transformed quality characteristics is evaluated in terms of out-of-control average run length (ARL_1) when the mean vector changes in different scenarios.

4.1 Performance of Root Transformation Method

The historical five thousand random vectors are generated using Gaussian copula (Cherubini et al. 2004). Then mean vector and covariance matrix of quality characteristics is obtained using moment method as follows:

$$\hat{\mu}_X = \begin{pmatrix} 3.01 & 4.01 \end{pmatrix}^T \text{ and } \hat{Cov}(X) = \begin{pmatrix} 4.03 & 1.4 \\ 1.4 & 4.04 \end{pmatrix}.$$

Skewness of quality characteristics are 0.03 and 0.49 for variable and attribute quality characteristics, respectively. The normality of variable quality characteristic is accepted and the normality of attribute quality characteristic is rejected based on the Jarque and Bera normality test. In the first step, we set $r_1=1$ for variable quality characteristic ($y_1=x_1^1$) and find the proper root (r_2) for the attribute quality characteristic by using the bisection method ($y_2=x_2^{r2}$). The values of 1 and 0.72 are obtained for r_1 and r_2, respectively. The mean vector and covariance matrix of transformed data are estimated by using moment method as follows:

$$\hat{\mu}_Y = \begin{pmatrix} 3.01 & 2.63 \end{pmatrix}^T \text{ and } \hat{Cov}(Y) = \begin{pmatrix} 4.03 & 0.7 \\ 0.7 & 1.01 \end{pmatrix}.$$

The skewness values of transformed data are reduced to 0.03 and zero. The Jarque and Bera normality test approves the normality of transformed variable and attribute data with the P-values of 0.70 and 0.42, respectively.

The results show that root transformation method transforms the correlated variable and attribute quality characteristics to the approximate multivariate normal distribution.

4.2 Monitoring Transformed Data

In this subsection, the performance of the control chart in monitoring transformed correlated variable and attribute quality characteristics is evaluated using simulation studies in terms of out-of-control average run length (ARL_1) criterion. In order to monitor the transformed data, in each sampling of this simulation, one vector of correlated variable and attribute quality characteristics is simulated. Then, the sample vector is transformed based on proper root transformation obtained in previous subsection. And finally, the transformed data is monitored by T^2 control chart. Note that the mean vector and covariance matrix of transformed data which is used in control statistic are obtained in the previous subsection.

We first set the upper control limit (UCL) of the T^2 control chart equal to $\chi^2_{2,0.005} = 10.59$ to obtain the $ARL_0 = 200$. Then, this UCL is tested by simulation studies and $ARL_0 \approx 219$ is obtained after 10000 replications. In order to evaluate the performance of control chart in detecting out-of-control state, we consider different step shift scenarios in the mean vector of quality characteristics. Then, out-of-control state is simulated to obtain out-of-control run length, these simulations are replicated 10000 times for each scenario and the results of ARL_1's are summarized in Table 1. Note that the number of samples in each subgroup is considered equal to 1 ($n=1$). Meanwhile, in each scenario the (θ_1 θ_2) vector shows the magnitude of shift in the mean vector of quality characteristics in the unit of its standard deviation (σ). For example (2σ 2σ) shows that the mean vector of quality characteristics is changed to ($\mu_1+2\sigma_1$ $\mu_2+2\sigma_2$).

The results of ARL_1 for different shifts considered in the paper are summarized in Table 1 as follows:

Table 1:
ARL_1 values for step shifts in the mean vector of quality characteristics

$(\sigma \quad 0)$	$(0 \quad \sigma)$	$(2\sigma \quad 0)$	$(0 \quad 2\sigma)$	$(3\sigma \quad 0)$	$(0 \quad 3\sigma)$
37.98	30.59	5.72	6.43	1.87	2.6
$(\sigma \quad \sigma)$	$(2\sigma \quad 2\sigma)$	$(3\sigma \quad 3\sigma)$	$(-\sigma \quad 0)$	$(0 \quad -\sigma)$	$(-\sigma \quad -\sigma)$
23.64	4.18	1.63	37.38	75.58	41.61

The results of Table 1 show that the changes in the mean vector of the original data can be detected soon in all shift scenarios considered in this paper which shows effectiveness of the proposed method in monitoring correlated variable and attribute quality characteristics.

5. CONCLUSIONS AND FUTURE RESEARCHES

In this paper, we considered a skewness reduction approach to monitor correlated variable and attribute quality characteristics. For this purpose, we proposed root transformation method to transform the correlated variable and attribute quality characteristics in such a way that the transformed data follows approximate multivariate normal distribution. We then used multivariate T^2 control chart for monitoring the transformed quality characteristics. The performance of the proposed method was evaluated based on average run length (ARL) criterion by using simulation studies. The results showed the effectiveness of the proposed method. As a future research, other transformation techniques can be addresses by researchers for monitoring correlated variable and attribute quality characteristics. Moreover, other multivariate control charts such as MEWMA can be used to monitor these transformed data.

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