Optimization of probabilistic multiple response surfaces

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ABSTRACT

Response surface methodology (RSM) is a statistical–mathematical method used for analyzing and optimizing the experiments. In analysis process, experts usually face several input variables having effect on several outputs called response variables. Simultaneous optimization of the correlated response variables has become more important in complex systems. In this paper multi-response surfaces and their related stochastic nature have been modeled and optimized by Goal Programming (GP) in which the weights of response variables have been obtained through a Group Decision Making (GDM) process. Because of existing uncertainty in the stochastic model, some stochastic optimization methods have been applied to find robust optimum results. At the end, the proposed method is described numerically and analytically.

1. Introduction

Response surface methodology (RSM) is useful in improving and optimizing process by finding the analytical relationship between input and output variables considered in experiments. RSM also has the ability to produce an approximate function using a smaller amount of data and fewer numbers of experiments runs Yeh [1]. However, most previous applications based on RSM have only dealt with a single-response problem and multi-response problems have received only limited attention. Studies have shown that the optimal factor settings for one performance characteristic are not necessarily compatible with those of other performance characteristics. In more general situations, finding compromising conditions on the input variables might be considered that are somewhat favorable to all responses Koksoy [2]. More details on RSM, related designs and optimization of response surfaces are given in Kleijnen [3], Myers and Montgomery [4] and Kleijnen [5]. This paper is organized as following sections. In Sections 1.1 and 1.2 some major works on Multiple Response Optimization (MRO) and uncertainty in multi-response problems are reviewed respectively. The proposed methods and its related definitions are represented in Section 2. In order to show the efficiencies and computational steps of the proposed methods, a real case from the literature is analyzed in Section 3 and the results are compared with some previous works.

1.1. Multi-response optimization

MRO problems have been studied in several areas from different aspects and could be categorized into three general categories:

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(a) Desirability viewpoints: in this category, researchers try to aggregate information of all responses into one response and then an optimization method is performed on a single objective called total desirability function.

(b) Priority based methods: some cases have responses with different importance degrees, in such problems, we must consider most important response for optimization and if solutions were not unique, then find the best solution by comparing the status of other responses for alternative solutions and the aforesaid steps are repeated till all the responses are considered or a unique optimal solution is found.

(c) Loss function: in this category, based on Taguchi loss function, all responses values are aggregated and converted to a single one. Wide range of researches, have been studied to develop and generalize Taguchi loss function with respect to special trait of its cases.

Some earlier works in multiresponse optimization are: Derringer and Suich [6] transformed each response function of a desirability function and then maximize the geometric mean of individual desirability functions for a compromise solution. Layne [7] presented a procedure that simultaneously considers three functions – the weighted loss function, the desirability function, and a distance function – to determine the optimum parameter combination. Pignatiello [8] utilized a variance component and a squared deviation-from-target to form an expected loss function to optimize a multiple response problem. This method is difficult to implement. The first reason is that a cost matrix must be initially obtained, and the second reason is that it needs more experimental data. He et al. [9] proposed desirability based approach considering robustness which made balance between robustness and optimization for multiple response problems. Simplex search method is used to search for the most robust optimal point in the feasible operating region. Saurav et al. [10] applied a combination of Taguchi method and Principal Component Analysis (PCA) to optimize a real case in the field of submerged arc welding. They also compared the proposed approach with Gray–Taguchi method. Chang et al. [11] generalized Taguchi function to more general situations. The generalized model was represented by using a weighted convex loss function. Mak and Nebebe [12] considered parameter design in off-line quality control and the proposed approach tried to find production factors that minimized expected loss function. The expected loss function was minimized based on a distributional free procedure using the empirical distribution of the standardized residuals. Bashiri and Hejazi [13] used Multiple Attribute Decision Making (MADM) methods such as VIKOR, PROMETHEE II, ELECTRE III and TOPSIS in converting multi-response to single response to analyze the robust experimental design. The main advantage of their method was the consideration of standard deviation that contributed to robust experimental design and also because of fitting only one response regression function; the proposed method decreased statistical error. Alvarez et al. [14] applied RSM to design and optimization of a capacitive accelerometer. A faced–centered cube design was used in the experimentation and the data were obtained conducting computer simulations using finite element design techniques. Hsieh [15] used neural networks to estimate relation between control variables and responses. Tong et al. [16] used VIKOR methods in converting Taguchi criteria to single response and then found regression model and related optimal setting. Tong et al. [17] also considered correlation of responses and used PCA and TOPSIS methods to find the best variable setting. Kazemzadeh et al. [18] proposed a general framework for multiresponse optimization problems based on Goal Programming and studied some existing works and attempted to aggregate all characteristics into one approach. Shah et al. [19] Illustrated and Seemingly unrelated regressions (SUR) method for estimating the regression parameters that it can be useful when response variables in MRS problems are correlated and can lead to a more precise estimate of the optimal variable setting.

1.2. Uncertainty in MRO problems

There are two main approaches for characterizing uncertainty in design of experiments including fuzzy theory, and probabilistic analysis. Fuzzy theory facilitates uncertainty analysis of systems where uncertainty is due to vagueness or fuzziness rather than randomness alone and can be applied to uncertainty analysis with imprecise observation or with verbal expressions. Probabilistic analysis is the most widely used method for characterizing uncertainty in real systems and can describe uncertainty arising from stochastic disturbances, variability and has capability for risk considerations, especially when estimates of input variables probability distributions are available.

Nan [20] combined some of the well-established methods in multiresponse optimization such as genetic algorithm, Artificial Neural Network (ANN), Taguchi loss function and desirability function to optimize stencil printing operations. In the proposed approach, fuzzy quality loss function was applied for analyzing qualitative measures. Chiao and Hamada [21] considered experiments with correlated multiple responses whose means, variances, and correlations depend on experimental factors. Analysis of these experiments consists of modeling distributional parameters in terms of the experimental factors and finding factor settings which maximize the probability of being in a specification region, i.e., all responses are simultaneously meeting their respective specifications. Amiri et al. [22] used genetic algorithm to find best solution of multiresponse problem in fuzzy environment. Their proposed method was a combination of simulation approach, fuzzy Goal Programming and genetic algorithm. Khuri and Conlon [23] and Khuri and Cornell [24] considered multivariate RSM and the stochastic character when optimizing several response surfaces simultaneously, in which case they applied the minimax technique to analyze the stochastic aspect and the distance based method was applied to solve the multiple objective problem. Díaz–García et al. [25] studied application of some stochastic programming methods in response surface optimization. They assumed normal distribution for coefficients of response surface and compared the results of some methods such as E-Model, V-Model, and Lexicography. Hejazi et al. [26] represented a novel method based on goal programming to find the best combination of factors and stochastic covariates to optimize multiresponse-multicovariate surfaces with consideration of location and dispersion effects. The method also
considered covariate probable values as an objective function which should be maximized. Table 1 shows a summary of MRO studies that considered uncertain and probabilistic aspects.

2. Proposed approach

In this paper, multi-response surfaces with probabilistic coefficients and responses weights have been modeled and optimized so as to ensure robustness in results. For this purpose, several steps are considered through the integrated approach. The proposed approach has the following steps:

(a) Identifying the experimental variables such as responses and factors.
(b) Applying a proper design, running experiments and fitting the response surfaces.
(c) Getting information about the importance weights of response variables from an expert team through a GDM process.
(d) Forming a multiresponse Goal Programming model subjected to the technical limitations and robustness consideration.
(e) Solving the model using some stochastic programming methods.

Comparisons of the proposed model with other techniques and related studies are summarized in Table 2. Table 2 shows that the proposed model has some capabilities to analyze multi-response model in which most of the major work’s characteristics are considered simultaneously. For instance, multiple response surfaces according to the proposed model considered the assumptions mentioned in Díaz-García et al. [25] in addition to some modifications listed below:

- Multiple response variables are analyzed and optimized simultaneously.
- Correlations between response variables are considered.
- Correlations between response regression coefficients $\beta_j$ are considered.
- Group Decision Making approach is applied in identifying the importance of response variables.

2.1. Model description

In this section the proposed approach will be explained by using mathematical formulations and related assumptions will be described. The aim of this section is to develop stochastic GP model in terms of controllable variables.
2.1.1. Notation

Let \( N \) be the number of experimental runs and \( p \) be the number of response variables which can be measured for each setting of a group of \( n \) coded variables \( x_1, x_2, \ldots, x_n \). We assume that the response variables can be modeled by second order polynomials regression model in terms of \( x_j^2 \). Hence, the \( k \)th response model can be written as

\[
R_k = X\beta_k + \epsilon_k,
\]

where \( R_k \) is an \( N \times 1 \) vector of observations on the \( k \)th response, \( X \) is an \( N \times q \) matrix of rank \( q \) termed design or regression matrix, \( q = 1 + n + n(n + 1)/2 \), \( \beta_k \) is a \( q \times 1 \) vector of unknown constant parameters, and \( \epsilon_k \) is a random error vector associated with the \( k \)th response. Note that (1) can be represented as

\[
\mathbb{R} = X\beta + \epsilon,
\]

where \( \mathbb{R} = [R_1; R_2; \ldots; R_k] \), \( \beta = [\beta_1; \beta_2; \ldots; \beta_k] \) and \( \mathbb{E} = [\epsilon_1; \epsilon_2; \ldots; \epsilon_k] \), such that \( \mathbb{E} \sim \mathcal{N}_{Np}(0, I_N \otimes \Sigma) \) i.e., \( \mathbb{E} \) has an \( N \times p \) matrix variate normal distribution with \( E(\mathbb{E}) = 0 \) and \( \text{Cov}(vec\mathbb{E}) = I_N \otimes \Sigma \), where \( \Sigma \) is \( p \times p \) positive definite matrix, \( \text{vec}\mathbb{T} = \{T_1, T_2, \ldots, T_p\} \) with \( T = \{T_1; T_2; \ldots; T_p\} \) and \( \otimes \) is a symbol for the direct (or Kronecker) product of matrices, see Muirhead \[34, \text{Theorem 3.2.2, p. 79}\].

- \( x = (x_1, x_2, \ldots, x_n) \): Vector of controllable variables.
- \( \beta = [\beta_1; \beta_2; \ldots; \beta_k] \): The least squares estimator of \( \beta \) given by \( \hat{\beta} = (X'X)^{-1}X'R \), form where \( \hat{\beta}_k = (X'X)^{-1}X'R_k \), \( k = 1, 2, \ldots, p \).
- Moreover, under the assumption that \( \mathbb{E} \sim \mathcal{N}_{Nq}(0, I_N \otimes \Sigma) \), then \( \mathbb{E} \sim \mathcal{N}_{q,p}(\beta, (XX')^{-1} \otimes \Sigma) \), with \( \text{Cov}(\text{vec}\mathbb{E}) = (XX')^{-1} \otimes \Sigma \).
- \( \mathbb{Z}(x) = (1, x_1, x_2, \ldots, x_n, x_1^2, x_2^2, \ldots, x_n^2, x_1 x_2, x_1 x_3, \ldots, x_{n-1}x_n) \): Response surface or predictor equation at the point \( x \) for \( k \)th response variable.
- \( \mathbb{R}(x) = (\hat{R}_1(x), \hat{R}_2(x), \ldots, \hat{R}_p(x)) = \hat{\beta} \mathbb{Z}(x) \): Response surface or predicted response vector at the point \( x \).
- \( \mathbb{W} = (w_1, w_2, \ldots, w_p) \): Vector of response weights.
- \( \Sigma_w \): Variance–covariance matrix of weights.
- \( \Sigma = \text{vec}(XX'XX') \): Estimator of variance–covariance matrix \( \Sigma \) such that \( (N - q)\Sigma \) has a Wishart distribution with \( (N - q) \) freedom degrees and parameter \( \Sigma \); denoted this fact as \( (N - q)\Sigma \sim \mathcal{W}_p(N - q, \Sigma) \). Here, \( I_m \) denotes a identity matrix of order \( m \).
- \( l = (l_1, l_2, \ldots, l_n) \): Vector of lower bounds of factors.
- \( u = (u_1, u_2, \ldots, u_n) \): Vector of upper bounds of factors.
- \( t = (t_1, t_2, \ldots, t_p) \): Vector of target values for response vector.

The MRO problem in general is proposed as

\[
\min_x \mathbb{R}(x) = \min_x \begin{pmatrix} \hat{R}_1(x) \\ \hat{R}_2(x) \\ \vdots \\ \hat{R}_p(x) \end{pmatrix}
\]

subject to

\( l < x < u \),

which is a nonlinear problem of the multi-objective optimization, see Steuer [35], Rios et al. [36] and Miettinen [37]; and where \( l < x < u \) denotes \( l_i < x_i < u_i, i = 1, 2, \ldots, n \).

In the RMS context, observe that in multi-objective optimization problems, there rarely exists a point \( x^* \) which is considered as an optimum, i.e., few cases satisfy the requirement that \( \mathbb{R}(x) \) is minimum for all \( k = 1, 2, \ldots, p \). From a point of view of multi-objective optimization, this justifies the following notion of the Pareto point, which is more weakly defined than an optimum point:

We say that \( \mathbb{R}(x) \) is a Pareto point of \( \mathbb{R}(x) \), if there is not other point \( \mathbb{R}^1(x) \) such that \( \mathbb{R}^1(x) \leq \mathbb{R}(x) \), i.e., for all \( k \). \( \hat{R}_k(x) \leq \hat{R}_k(x) \) and \( \hat{R}_k(x) \neq \hat{R}_k(x) \).

Existence criteria for Pareto points of a multi-objective optimization problem are established in Rios et al. [36] and Miettinen [37]. In particular we have:

Given \( \mathbb{R}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^p \) and let us consider a nonempty compact \( \mathcal{N} \subset \mathbb{R}^n \) such that \( \mathcal{N} \) is the set of all possible values of \( x \) determined by the restrictions in (3). If \( \mathbb{R}(x) \) is an upper semi-continuous for each \( k = 1, \ldots, p \), then the problem (3) has a Pareto optimal solution.

On the other hand, Steuer [35], Rios et al. [36] and Miettinen [37] studied the extension of scalar optimization (Kuhn-Tucker’s conditions) to the vectorial case. In particular, they proposed necessary conditions for Pareto solutions, which become sufficient conditions if: \( \mathcal{N} \) is convex; the functions \( \hat{R}_k(x), k = 1, \ldots, p \) are convex; and the Lagrange generalized multipliers \( \delta_k \), associated with each function \( \hat{R}_k(x) \), are positive, \( \delta_k > 0 \) for all \( k \).
Methods for solving a multi-objective optimization problem are based on the information possessed about a particular problem. There are three possible scenarios; when the investigator possesses: complete, partial or null information, see Rios et al. [36], Miettinen [37] and Steuer [35]. In an RMS context, complete information means that, the investigator knows the population in such a way that it is possible to propose a value function reflecting the importance of each response variables, where

\[
\text{Value function is a function } \phi : \mathbb{R}^n \to \mathbb{R} \text{ such that } \min_x \hat{R}(x) < \min_x \tilde{R}(x) \iff \phi(\hat{R}(x)) < \phi(\tilde{R}(x)), \quad x' \neq x.
\]

In partial information, the investigator knows the main response variable of the study very well and this is sufficient support for the research. Finally, under null information, the researcher only possesses information about the estimators of the parameter of response surfaces, and with this material an appropriate solution can be found.

Then, observe that the three general categories described in Section 1.1 are particular cases of the method of value function. In particular we have the following alternative approach in terms of GP model of the multi-objective problem (3)

\[
\begin{align*}
\min_x F(x) &= \min_x \sum_{k=1}^p w_k (d_k + d_k') \\
\text{subject to} & \\
\tilde{R}_k(x) + d_k - d_k' &= t_k, \quad k = 1, 2, \ldots, p, \\
1 < x < u,
\end{align*}
\]

where

\[
\begin{align*}
d_k &= \frac{1}{2} \left( \left| \tilde{R}_k(x) - t_k \right| + \left( \tilde{R}_k(x) - t_k \right) \right), \\
d_k' &= \frac{1}{2} \left( \left| \tilde{R}_k(x) - t_k \right| - \left( \tilde{R}_k(x) - t_k \right) \right).
\end{align*}
\]

Consider multi-response model in form of second order polynomial function

\[
\tilde{R}_k(x) = \tilde{\beta}_0 + \tilde{\beta}_i x_i + \tilde{\beta}_{ii} x_i^2 + \sum_{i=1}^n \sum_{j=1}^n \tilde{\beta}_{ij} x_i x_j = z'(x) \tilde{\beta}_k \quad \forall k = 1, 2, \ldots, p.
\]

Hence, see Khuri and Cornell [24, p. 289]

\[
E(\tilde{R}(x)) = E(\tilde{B} z(x)) = \tilde{B} z(x)
\]

and

\[
\text{Cov}(\tilde{R}(x)) = z'(x)(X'X)^{-1}z(x) \Sigma.
\]

An unbiased estimator of \(\text{Var}(\tilde{R}(x))\) is given by

\[
\text{Cov}(\tilde{R}(x)) = z'(x)(X'X)^{-1}z(x) \Sigma.
\]

Finally, observing that \(\text{Var}(\tilde{R}_k(x) + d_k - d_k') = 0\) and taking into account that \(\Sigma \) and \(\Sigma^*\) is a random matrix and assuming that \(w\) is an independent random vector, we have the following stochastic optimization problem version of (4)

\[
\begin{align*}
\min_x F(x) &= \min_x \sum_{k=1}^p w_k (d_k + d_k') \\
\text{subject to} & \\
\tilde{R}_k(x) + d_k - d_k' &= t_k, \quad k = 1, 2, \ldots, p, \\
1 < x < u
\end{align*}
\]

\(w \in \Omega_w, \quad E(w) \quad \text{and} \quad \text{Cov}(w) = \Sigma_w\) and their estimators are known.

\(B, \Sigma \in \Omega_{B, \Sigma}\) and

\[
(\tilde{B}, \tilde{\Sigma}) \sim N_q,p(B, (X'X)^{-1} \Sigma), \quad \text{and};
\]

\[
(N - q) \Sigma \sim W_q(N - q, \Sigma),
\]

where \(\Omega_w \) and \(\Omega_{B, \Sigma}\) are definition regions of response variables.

In addition, for incorporating weights in the model and developing the GP form (Eq. (4)), some other notations about multiplication of probabilistic variables have to be considered. Following relations show expected value and variance of goal function.

**Lemma 1.** Assume that \(Y\) and \(X\) are random variables (not necessarily independent). If \(Z_1\) and \(Z_2\) are defined as \(Z_1 = X + Y\) and \(Z_2 = XY\). Then

\[
\begin{align*}
E(Z_1) &= E(X) + E(Y), \\
\text{Var}(Z_1) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y), \\
E(Z_2) &= E(X)E(Y) + \text{Cov}(X, Y).
\end{align*}
\]

\[
\text{Var}(Z_2) = E^2(Y)\text{Var}(X) + E^2(X)\text{Var}(Y) + 2E(X)E(Y)\text{Cov}(X,Y) - \text{Cov}^2(X,Y) + E[(X - E(X))^2(Y - E(Y))^2]
+ 2E(Y)E[(X - E(X))^2(Y - E(Y))] + 2E(X)E[(X - E(X))(Y - E(Y))^2].
\] (13)

According to Eqs. (10)–(13) written based on Taylor series in several variables [38], the general GP model (9) could be converted into deterministic model by use of some stochastic programming techniques termed deterministic equivalent model, Díaz-García et al. [25]. Furthermore, it is reasonable to assume that the weights and responses regression coefficients are independent so equivalent problem for GP model could be developed in the next step.

### 2.2. Deterministic equivalent problem

In context of stochastic programming there are some deterministic equivalents for stochastic optimization models such as E-model, modified V-model and Lexicographic method, see Charnes and Cooper [39]. A brief description of these methods is represented below (for more details see Díaz-García et al. [25]).

Also it is reasonable to assume that the weights and responses regression coefficients are independent then equivalent problem for GP model could be developed as follow.

#### 2.2.1. Modified E-model

In this method a linear combination of expected value and standard deviation of objective function is used to model and analyze the stochastic nature of (9), see Prekopa [40].

\[
\begin{align*}
\min & \quad r_1 \hat{E}(F(x)) + r_2 \sqrt{\text{Var}(F(x))} \\
\text{subject to} & \\
\hat{R}_k(x) + d_k - d'_k = t_k, & k = 1, 2, \ldots, p, \\
1 < x < u,
\end{align*}
\] (14)

where \( r_1 \) and \( r_2 \) are non-negative constants representing importance of the expectation and variance, which in general are such that \( r_1 + r_2 = 1 \); we are denoted \( \hat{E}(F(x)) \equiv E(F(x)) \) and \( \text{Var}(F(x)) \equiv \text{Var}(F(x)) \) and

\[
\hat{E}(F(x)) = \sum_{k=1}^{p} \hat{E}(w_k \hat{E}(d_k + d'_k)) + \text{Cov}[w_k, (d_k + d'_k)].
\] (15)

\[
\text{Var}(F(x)) = \sum_{k=1}^{p} \text{Var}[w_k (d_k + d'_k)] + 2 \sum_{i=1}^{p} \sum_{j=i+1}^{p} \text{Cov}[w_i (d_i + d'_i), w_j (d_j + d'_j)].
\] (16)

Note that the type of \( F(x) \) in this E-model is Smaller-The-Better (STB).

#### 2.2.2. Modified V-model

Under this technique the deterministic equivalent model is

\[
\begin{align*}
\min & \quad \sqrt{\text{Var}(F(x))} \\
\text{subject to} & \\
\hat{R}_k(x) + d_k - d'_k = t_k, & k = 1, 2, \ldots, p, \\
\hat{E}(F(x)) \in \{\text{desired region}\} \\
1 < x < u,
\end{align*}
\] (17)

By Díaz-García et al. [25] we have that the solutions given by the modified E-model and modified V-model techniques are also solution of the following deterministic multi-objective problem

\[
\begin{align*}
\min & \quad \left[ \frac{\hat{E}(F(x))}{\sqrt{\text{Var}(F(x))}} \right] \\
\text{subject to} & \\
\hat{R}_k(x) + d_k - d'_k = t_k, & k = 1, 2, \ldots, p, \\
1 < x < u,
\end{align*}
\] (18)

These equivalences allow us the use of other multi-objective optimization techniques for solved (18) and propose these as stochastic solutions of (9).
In the next section, a numerical example is analyzed by some aforesaid methods and results are compared with other previous works.

3. Numerical example

In this section a real case from the literature is analyzed by the proposed procedure steeply. The case is analyzed step by step in order to illustrate the proposed method more intelligible.

Step a: In this example there are two response variables $R = \{R_1, R_2\}$ with 4 replicates and three setting variables $x = (x_1, x_2, x_3)$. The experimental data is shown in Table 3. In this example it is assumed that the targets of the responses are 103 and 73 and that the specification regions are (97,109), (70,76) for $R_1$, $R_2$, respectively, see Pignatiello [8].

Step b: Next, Eqs. (19) and (20) are response surfaces for $R_1$ and $R_2$.

$$\hat{R}_1(x) = 104.86 - 3.147x_1 - 0.142x_2 - 0.199x_3 + 2.379x_1x_2 - 0.35x_1x_3 - 0.106x_2x_3,$$

$$\hat{R}_2(x) = 70.45 - 0.348x_1 + 3.59x_2 + 0.28x_3 + 0.323x_1x_2 - 0.45x_1x_3 + 0.614x_2x_3.$$  

(19) and (20)

From where the multi-response optimization problem is given as

$$\min_x \mathcal{R}(x) = \min_x \left( \frac{\hat{R}_1(x)}{\hat{R}_2(x)} \right)$$

subject to

$$-1 < x_1, x_2, x_3 < 1.$$  

(21)

Step c: In this section, the importance of each response variable must be assessed from the decision makers’ viewpoint. Table 4 shows the hypothetical importance of two response variables from the viewpoint of 20 decision makers.

Step d: Now the optimization model can be developed in form of Goal Programming as in (4).

$$\min_x F(x) = \min_x [w_1(d_1 + d_1') + w_2(d_2 + d_2')]$$

subject to

$${\hat{R}}_1(x) + d_1 - d_1' = t_1,$$

$${\hat{R}}_2(x) + d_2 - d_2' = t_2,$$

$$-1 < x_1, x_2, x_3 < 1.$$  

Now from Table 4 we have

$$\hat{E}(\mathbf{w}) \equiv \mu_\mathbf{w} = \begin{bmatrix} 0.285 \\ 0.715 \end{bmatrix}$$

and $$\hat{\text{Cov}}(\mathbf{w}) \equiv \Sigma_\mathbf{w} = \begin{bmatrix} 0.0073 & -0.0073 \\ -0.0073 & 0.0073 \end{bmatrix}$$

and from (19) and (20)

$$\hat{\boldsymbol{B}} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}' = \begin{bmatrix} 104.86 & -3.147 & -0.142 & -0.199 & 2.379 & -0.35 & -0.106 \\ 70.45 & -0.348 & 3.59 & 0.28 & 0.323 & -0.45 & 0.614 \end{bmatrix}'$$

Also,

Table 3

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<td>6</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>98.73</td>
<td>99.36</td>
<td>102.84</td>
<td>94.24</td>
<td>67.10</td>
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<td>2</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>100.19</td>
<td>99.63</td>
<td>100.27</td>
<td>100.60</td>
<td>67.03</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>103.15</td>
<td>106.96</td>
<td>107.62</td>
<td>103.44</td>
<td>71.68</td>
</tr>
<tr>
<td>3</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>106.08</td>
<td>105.64</td>
<td>105.67</td>
<td>103.39</td>
<td>72.94</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>113.52</td>
<td>111.12</td>
<td>112.85</td>
<td>106.67</td>
<td>68.29</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>109.90</td>
<td>109.76</td>
<td>110.70</td>
<td>109.77</td>
<td>67.70</td>
</tr>
</tbody>
</table>

There are several methods in stochastic programming that can solve such problems. In this paper and according to Eqs. (41x50) following objective and constraints:

\[ \begin{aligned}
\min_{\mathbf{x}} & \quad \mathbf{1}^\top \mathbf{E}(\mathbf{F}(\mathbf{x})) \\
\text{subject to} & \quad \hat{R}_1(\mathbf{x}) + d_1 - d'_1 = 103, \\
& \quad \hat{R}_2(\mathbf{x}) + d_2 - d'_2 = 103, \\
& \quad -\mathbf{1} < \mathbf{x} < \mathbf{1}
\end{aligned} \]

where

\[ \begin{align*}
\mathbf{E}(\mathbf{F}(\mathbf{x})) &= 0.285 \hat{R}_1(\mathbf{x}) - 103 + 0.715 \hat{R}_2(\mathbf{x}) - 73 \\
\text{and} & \\
\mathbf{Var}(\mathbf{F}(\mathbf{x})) &= [0.285]^2 \mathbf{Var}(\hat{R}_1(\mathbf{x})) + 0.0073 \mathbf{Var}(\hat{R}_1(\mathbf{x}) - 103)^2 + 0.0073 \mathbf{Var}(\hat{R}_1(\mathbf{x}) - 103) + 0.715]^2 \mathbf{Var}(\hat{R}_2(\mathbf{x})) + 0.0073 \mathbf{Var}(\hat{R}_2(\mathbf{x}) - 73)^2 + 0.0073 \mathbf{Var}(\hat{R}_2(\mathbf{x})) - 0.0073 \mathbf{Var}(\hat{R}_1(\mathbf{x}) - 103) \mathbf{Var}(\hat{R}_2(\mathbf{x}) - 73) + (0.285) \\
& \times (0.715) \hat{\rho}_{\hat{R}_1(\mathbf{x}), \hat{R}_2(\mathbf{x})} \left( \sqrt{\mathbf{Var}(\hat{R}_1(\mathbf{x})) \mathbf{Var}(\hat{R}_2(\mathbf{x}))} \right)
\end{align*} \]

As mentioned in Section 2.2 there are several methods in stochastic programming that can solve such problems. In this paper another method is also represented according to the context of robust optimization. Minimax approach is one of the well-established methods in robust optimization (Matos [41]) so the multiobjective model in the case study is solved by combination of modified E-model and Minimax approach. The robust E-model could represent aforesaid multiobjective problem in following objective and constraints:

\[ \begin{aligned}
\min_{\mathbf{x}} & \quad \mathbf{1}^\top \mathbf{E}(\mathbf{F}(\mathbf{x})) + z \sqrt{\mathbf{Var}(\mathbf{F}(\mathbf{x}))} \\
\text{subject to} & \quad \hat{R}_k(\mathbf{x}) + d_k - d'_k = t_k, \quad k = 1, 2, \ldots, p, \\
& \quad \mathbf{1} < \mathbf{x} < \mathbf{u}.
\end{aligned} \]

\[ \text{(22)} \]
As represented above, the model (22) tries to minimize upper bound of \((1 - \alpha)\%\) confidence interval for expected value of the goal function. Table 5 shows optimal results for proposed approach in comparison with some other methods. It is considerable that because of some existing different between the parameters of those methods, the model has been analyzed without consideration of the GDM phase and the weights of the response variables assumed to be equal and the results have been compared with some other methods.

As shown in Table 5, the proposed method has considered both location and dispersion effects in presence of correlated responses. One other benefit of the proposed method is to analyze the response surfaces with stochastic terms.

For incorporating the GDM phase into the analysis process, by using the model represented in step e, the case is analyzed and optimized too. Table 6 shows the results of proposed approach with consideration of probabilistic weights (see Table 4).

---

**Table 5**
Comparison between the results of the proposed model and other methods.

<table>
<thead>
<tr>
<th>Stochastic programming method</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(\hat{R}_1(x))</th>
<th>(\hat{R}_2(x))</th>
<th>(\text{Var}(\hat{R}_1(x)))</th>
<th>(\text{Var}(\hat{R}_2(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kazemzadeh et al. [18]</td>
<td>-</td>
<td>-0.265</td>
<td>-1.000</td>
<td>108.504</td>
<td>69.367</td>
<td>0.185</td>
<td>0.130</td>
</tr>
<tr>
<td>Chiao and Hamada [21]</td>
<td>1.000</td>
<td>1.000</td>
<td>-1.000</td>
<td>104.612</td>
<td>73.574</td>
<td>0.012</td>
<td>0.448</td>
</tr>
<tr>
<td>Vining [42]</td>
<td>0.848</td>
<td>0.659</td>
<td>0.333</td>
<td>103.247</td>
<td>72.806</td>
<td>0.355</td>
<td>0.394</td>
</tr>
<tr>
<td>Proposed approach</td>
<td>0.953</td>
<td>0.709</td>
<td>0.407</td>
<td>103.247</td>
<td>73.000</td>
<td>0.428</td>
<td>0.476</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>104.612</td>
<td>73.574</td>
<td>0.131</td>
<td>0.146</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>104.865</td>
<td>70.453</td>
<td>0.131</td>
<td>0.146</td>
</tr>
</tbody>
</table>

* Squared euclidean distance.

**Table 6**
Optimum results of the proposed model and other models.

<table>
<thead>
<tr>
<th>Stochastic programming method</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(\hat{R}_1(x))</th>
<th>(\hat{R}_2(x))</th>
<th>(\text{Var}(\hat{R}_1(x)))</th>
<th>(\text{Var}(\hat{R}_2(x)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed approach</td>
<td>0.836</td>
<td>0.703</td>
<td>0.369</td>
<td>103.325*</td>
<td>73.000</td>
<td>0.372</td>
<td>0.414</td>
</tr>
<tr>
<td>Distance based</td>
<td>0.832</td>
<td>0.703</td>
<td>0.368</td>
<td>103.322</td>
<td>73.000</td>
<td>0.370</td>
<td>0.412</td>
</tr>
<tr>
<td>Robust E-model</td>
<td>1.000</td>
<td>0.707</td>
<td>0.483</td>
<td>103.000</td>
<td>73.000</td>
<td>0.470</td>
<td>0.523</td>
</tr>
<tr>
<td>Lexicographic (First (\hat{E}(F(x))))</td>
<td>0.113</td>
<td>0.250</td>
<td>0.039</td>
<td>104.533</td>
<td>71.333</td>
<td>0.141</td>
<td>0.156</td>
</tr>
<tr>
<td>Lexicographic (First (\text{Var}(F(x))))</td>
<td>0.418</td>
<td>0.693</td>
<td>0.213</td>
<td>104.053</td>
<td>73.000</td>
<td>0.238</td>
<td>0.265</td>
</tr>
<tr>
<td>Modified V-model</td>
<td></td>
<td></td>
<td></td>
<td>104.533</td>
<td>71.333</td>
<td>0.141</td>
<td>0.156</td>
</tr>
</tbody>
</table>

* Bold values represent the better results.

**Fig. 1.** Sensitivity analysis on location and dispersion effects for \(\hat{R}_1(x)\): \(x = (x_1, x_3 = 0.693, x_3)\).

As represented above, the model (22) tries to minimize upper bound of \((1 - \alpha)\%\) confidence interval for expected value of the goal function. Table 5 shows optimal results for proposed approach in comparison with some other methods. It is considerable that because of some existing different between the parameters of those methods, the model has been analyzed without consideration of the GDM phase and the weights of the response variables assumed to be equal and the results have been compared with some other methods.

As shown in Table 5, the proposed method has considered both location and dispersion effects in presence of correlated responses. One other benefit of the proposed method is to analyze the response surfaces with stochastic terms.

For incorporating the GDM phase into the analysis process, by using the model represented in step e, the case is analyzed and optimized too. Table 6 shows the results of proposed approach with consideration of probabilistic weights (see Table 4).
Fig. 1 provides a graph in order to represent the confliction in considering the location and dispersion effect (the dimensions are homogenized).

4. Conclusions

RSM is an important method in process improvements. It is based on statistical and mathematical concepts so its stochastic aspects have to be considered in analysis process. In this paper multiple correlated response surfaces have been studied with the stochastic nature. In addition, in multiple response problems, the analyst must consider the relative preference of the responses. In this study importance degree of each response variable has been obtained by Group Decision Making process. The weights and regression coefficients in the proposed model have been considered with probabilistic view so fuzzy analysis of such problem in modeling and optimization phase could be included in future research. At the end, through a numerical example, it was shown that the proposed method would produce more reliable results that those of other major methods and also it has a capability to consider different output weights by sampling from expert population.

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