**On Solvable Circulant Graph**

Hesam Sharifi
*Department of Mathematics, Faculty of Science, Shahed University, Tehran, Iran.*
*hsharifi@shahed.ac.ir*

Kobra Ali Mohammdi
*Department of Mathematics, Faculty of Science, Shahed University, Tehran, Iran.*
*kalmo-sh@yahoo.com*

**Abstract:** Suppose that $n$ is a positive integer and $(n, \varphi(n)) = 1$. Using concept of solvable finite groups, we prove that, if every eigenvalue of the adjacency matrix of vertex-transitive graph of order $n$ is simple, then the graph is isomorphic to a circulant graph of order $n$.

*Keywords*: solvable finite group; circulant graph.

*Mathematics Subject Classification (2000)*: 05C25; 20D45.

## 1. Introduction

For any natural number $n$, we use $\mathbb{Z}_n$ to denote the additive cyclic group of integers modulo $n$. For any set $S$ of integers, let $Cay(\mathbb{Z}_n; S)$ be the digraph whose vertex set is $\mathbb{Z}_n$, and in which there is an arc from $v_i$ to $v_j$, whenever $v_i v_j^{-1} \in S$. A digraph is circulant if it is (isomorphic to) $Cay(\mathbb{Z}_n; S)$, for some choice of $n$ and $S$.

The *adjacency* matrix $A = A(\Gamma)$ of a labelled digraph $\Gamma$ is the $n \times n$ matrix $[a_{ij}]$ with $a_{ij} = 1$ if $v_i\ v_j$ is an arc of $\Gamma$ and 0 otherwise. The characteristic polynomial of $\Gamma$ is written as

$$p(\Gamma) = p(\Gamma, x) = det(xI - A) = \sum_{i=0}^{n} a_i x^{n-i}.$$

The sequence $\lambda_0, ..., \lambda_{n-1}$ of the roots of $p(\Gamma)$ is called the spectrum of $\Gamma$, denoted by $Spec(\Gamma)$. In general, $Spec(\Gamma)$ contains both real and complex eigenvalues. The spectrum of circulant digraph is given by [1,4]

$$Spec(\Gamma(\mathbb{Z}_n, S)) = (\lambda_0, ..., \lambda_{n-1}),$$

where $\lambda_r = \sum_{s \in S} \omega_n^{rs}$with $0 \leqslant r \leqslant n-1$ and $\omega_n = e^{2\pi i/n}$. Note that $\{\omega_n^r : 0 \leqslant r \leqslant n-1\}$ are all (complex) solutions of the equation $x^n = 1$, and they are called the $n$th roots of unity.

The isomorphism problem for circulant graphs has been extensively investigated and studied over the past 30 years, by some mathematicians and there are some unsolved problems about them [1,3]. The methods used in this area range from group theory, including the solvable finite group, through to algebraic graph theory.

## 2. Main Results

Before stating our main theorem, we will mention some well-known Lemma about automorphisms of a graph, which can be found in [1].

**Lemma 1.** *If all of the eigenvalues of adjacency matrix* $\Gamma$ *are simple, then every elements* $Aut(\Gamma)$ *are of order 2.*

**Corollary 2.** *With the assumption in Lemma 2.1,* $Aut(\Gamma)$ *is nilpotent, therefore it should be solvable.*

In this case, the graph is called *solvable graph*.

**Theorem 3.** *[2] Let* $n$ *be positive integer with condition* $gcd(n, \varphi(n)) = 1$. *A vertex-transitive graph* $\Gamma$ *of order* $n$ *is isomorphic to a circulant digraph of order* $n$ *if and only if there exists a solvable subgroup of* $Aut(\Gamma)$.

**Theorem 4.** *With the assumption in Theorem 2.1, if all of the eigenvalues of adjacency matrix of related to graph* $\Gamma$ *are simple, then the graph* $\Gamma$*, isomorphic to a circulant digraph of order* $n$.

## References

[1] N. Biggs: *Algebraic Graph theory*, North-Holland, Amsterdam, 1985.
[2] E. Dobson: *On Solvable Groups and Circulant Graphs*, Europ. J. Combinatorics (2000), No. **21**, pp. 881-885.
[3] J. F. Lazenby: *Circulant Graphs and Their Spectra*, A Thesis Presented to The Division of Mathematics and Natural Sciences Reed College In Partial, 2008.
[4] C. Godsil, G. Royle: *Algebraic Graph Theory*, Springer-Verlag Inc, New York, 2001.