# On Rational Valued Character of Finite Groups

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**Abstract:** A finite group whose irreducible complex characters are rational valued is called a rational group or $\mathbb{Q}$-group. We prove that if $G$ is a finite group that the number of kernels is equal to the number of classes, then every rationally represented character is a generalized permutation character.

*Keywords*: Rational group; Artin exponent; Involution.

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## 1. Introduction

Let $Char_{\mathbb{Q}}(G)$ and $P(G)$ denote the ring of $\mathbb{Z}$-linear combination of rationally represented characters and permutation characters of a finite group $G$, respectively. It is easy to see that $P(G)$ a is subring of $Char_{\mathbb{Q}}(G)$. By a theorem of Artin, $\mid G \mid \chi \in P(G)$ for all $\chi \in Char_{\mathbb{Q}}(G)$. The minimal number $d \in \mathbb{N}$ such that $d\chi \in P(G)$ for all $\chi \in Char_{\mathbb{Q}}(G)$, is called the Artin exponent of $G$ and is denoted by $\gamma(G)$. Indeed, $\gamma(G)$ is the exponent of $Char_{\mathbb{Q}}(G)/P(G)$. A nice description of $\mathbb{Q}$-groups and Artin exponent can be found in [1,4]. The Artin exponent induced from cyclic subgroups of finite groups was studied extensively by T. Y. Lam in [3]. He proved that $A(G)$ = exponent $(\frac{Char_{\mathbb{Q}}(G)}{P(G)_{cyclic}})$=1 if and only if $G$ is cyclic. One can show that $\gamma(G)$ divides the $A(G)$ and therefore divides $\mid G \mid$. There is a fundamental distinction between $\gamma(G)$ and $A(G)$. While groups satisfying $A(G) = 1$ have been characterized, there is no such characterization for groups satisfying $\gamma(G) = 1$. In this paper, we prove that if a $\mathbb{Q}$-group that the number of kernels is equal to the number of classes, then $\gamma(G/O_2(G)) = 1$.

## 2. Main Theorem

The definition and next theorem can be found in [4].

**Definition 1.** *An involution $a$ in a group $G$ is called irreducible, if $a$ can not be factored as a product of two involutions or $a$ is only involution in its centralizer $C_G(a)$.*

**Theorem 2.** *Let $G$ be a $\mathbb{Q}$-group. Then $G$ is counting an irreducible involution if* *and only if a Sylow 2-subgroup of $G$ is either $\mathbb{Z}_2$ or $Q_8$, where, $\mathbb{Z}_2$ is the cyclic group of order 2 and $Q_8$ is the quaternion group of order 8.*

***Theorem 3.*** *[2] Let $G$ be a $\mathbb{Q}$-group having a irreducible involution. Then $G$ is isomorphic to one of the following groups:*

- $G \simeq G' : \mathbb{Z}_2$
- $G \simeq E(p^n) : Q_8$

*where $E(p^n)$ is an elementary abelian $p$-group of odd order $p^n$, $G'$ is the commutator subgroup of $G$ and ":" is semidirect product of two groups.*

***Corollary 4.*** *If $G$ is a $\mathbb{Q}$-group counting an irreducible involution, then $\gamma(G) = 1$.*

*By a theorem in [5], we can prove that:*

***Theorem 5.*** *Let $G$ be a finite group that the number of kernels is equal to the number of classes. Then $\gamma(G/O_2(G)) = 1$.*

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