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On a Graph Associated to a $\mathbb{Q}$-Group

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Abstract

A finite group whose irreducible complex characters are rational valued is called a $\mathbb{Q}$-group. First, we define an integer valued function α on the set $\Omega(G) \times \Omega(G)$, where $\Omega(G)$ is the set of conjugacy classes of G. The conjugacy class of the identity element of G is denoted by 1. Then, we classify $\mathbb{Q}$-groups G such that there exists a unique $k(s) \in \Omega(G)$ with the property $\alpha(1, k(s)) = p$, where p is a prime number. This classification covers all finite $\mathbb{Q}$-groups in the case of $p = 2$, but if p is an odd prime our classification covers only solvable groups. .

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1 Introduction

A finite group G is called a rational or a $\mathbb{Q}$-group if every irreducible complex character of G is rational valued. In particular, the symmetric group S_n and the Weyl groups of the complex Lie algebras are $\mathbb{Q}$-group, see [2]. Some recent works about classification of these groups can be found in [3, 4, 9].

Let G be a $\mathbb{Q}$-group and $\Omega(G) = \{k(s) | s \in G\}$ be the set of conjugacy classes of G with $k(s)$ a conjugacy class with representative s. We define the relation $\Re$ on $\Omega(G)$ by setting $k(s)\Re k(t)$ whenever for $k(s), k(t) \in \Omega(G)$ and $s \sim t^n$ for some integer n. The relation $\Re$ is a well-defined partial ordering on $\Omega(G)$. We associate with $(\Omega(G), \Re)$ a labeled graph $\Gamma(G)$ as follows. The vertices of $\Gamma(G)$ are the elements in $\Omega(G)$, $k(s)$ is joined to $k(t)$, if $k(s)\Re k(t)$ and there is no $k(u)$ such that $k(s)\Re k(u)$ and $k(u)\Re k(t)$. In this case, the edge $k(s)k(u)$ is labeled n, where n is the least positive integer for which $s \sim t^n$.

" $A : B$ " as semidirect product of two groups A and B, Q_8 is the quaternion group of order 8, C_n is the cyclic group of order n and $E(p^n)$ is the elementary abelian p-group of order p^n.

We mention some well-known facts about $\mathbb{Q}$-group (see [1] and [10]).

Theorem A. A group G is a $\mathbb{Q}$-group if and only if for each $x \in G$, $\frac{N_G(<x>)}{C_G(<x>)} \simeq Aut(<x>)$.

Theorem B. If P is Sylow 2-group of a $\mathbb{Q}$-group then $Z(P)$, the center of P, is an elementary abelian 2-group.

2 Some Results

Lemma 2.1. *The label on every edge of* $\Gamma(G)$ *is a prime number.*

Proof. Let $k(s)$ and $k(t)$ be connected in $\Gamma(G)$ with label n. By definition, $k(s)\Re k(t)$, and there is no $k(u) \in \Omega(G)$ such that $k(s)\Re k(u)$ and $k(u)\Re k(t)$.

Suppose that $n = ab$, where $1 < a, b < n$. Then $k(s)\Re k(t^a)$ and $k(t^a)\Re k(t)$ since $s \sim t^n = (t^a)^b$. If $k(t^a) = k(t)$, then $t^a \sim t$ and hence $s \sim t^n = t^{ab} \sim t^b$ where $b < n$. This contradiction violates the minimality of n. On the other hand, if $k(s) = k(t^a)$, then $s \sim t^a$ where $a < n$, and this again violates the minimality of n. Therefore $k(s)\Re k(t^a)$ and $k(t^a)\Re k(t)$, which is not the case. It follows that n does not have a proper factorization. □

Let $k(s), k(t) \in \Omega(G)$ and $k(s)\Re k(t)$, then there is a least positive integer n such that $s \sim t^n$. Let $\alpha(s,t)$ stand for this integer. The integers $\alpha(s,t)$ may be used to define a function $\alpha : \Omega(G) \times \Omega(G) \longrightarrow \mathbb{Z}$ as follows: If $k(s)$ and $k(t)$ are not related by $\Re$, then put $\alpha(k(s), k(t)) = 0$, otherwise, put $\alpha(k(s), k(t)) = p$ where $s \sim t^p$. For simplicity in notation, we denote $\alpha(k(s), k(t))$ by $\alpha(s,t)$.

Corollary 2.2. *(a)* $\alpha(1, s) = o(s)$ *for every* $s \in G$.
(b) If $k(s)\Re k(u)$ *and* $k(u)\Re k(t)$*, then* $\alpha(s,u) \times \alpha(u,t) = \alpha(s,t)$.

Proof. (a) is obvious.

(b) Let $s \in G$ and let n be a positive integer. Then $s^n \sim s^{(n,o(s))}$. $(n, o(s)) = an + bo(s)$ for some integers a, b and $(a, \frac{o(s)}{(n,o(s))}) = (a, o(s^n)) = 1$. Hence $s^{(n,o(s))} = s^{an} \sim s$. It follows that, if $\alpha(s,t) = m$, then $t^{(m,o(t))} \sim t^m \sim s$ and hence $(m, o(t)) = m$ by the minimality of m. Hence m divides $o(t)$. Therefore

$$\alpha(1,t) = \frac{o(t)}{m} m = o(t^m)m = o(s)m = \alpha(1,s) \times \alpha(s,t)$$

Now, suppose that $k(s)\Re k(u)\Re k(t)$. Then

$$\alpha(1,s) \times \alpha(s,u) \times \alpha(u,t) = \alpha(1,u) \times \alpha(u,t) = \alpha(1,t) = \alpha(1,s) \times \alpha(s,t).$$

It follows that $\alpha(s,u) \times \alpha(u,t) = \alpha(s,t)$. □

Corollary 2.3. *Suppose that $k(s)\Re k(t)$. Then every path from $k(s)$ to $k(t)$ has the same cardinality.*

Proof. It suffices to show that every path from $k(1)$ to $k(t)$ has the same cardinality. Let C_1 and C_2 be two such paths and let $o(t) = q_1^{n_1}...q_m^{n_m}$ where the q_i are prime. It follows from Lemma 2.1 that every edge in both C_1 and C_2 must be labeled with one of the primes q_i. Let

$$c_i = \text{the number of edges in } C_1 \text{ labeled } q_i,$$
$$d_i = \text{the number of edges in } C_2 \text{ labeled } q_i.$$

Applying Result 2.2 to both paths, we have

$$o(t) = \alpha(1,t) = q_1^{c_1}...q_m^{c_m} = q_1^{d_1}...q_m^{d_m},$$

which implies that $c_i = d_i$ for each $i = 1, 2, ..., m$. Hence

$$\text{length } C_1 = c_1 + ... + c_m = d_1 + ... + d_m = \text{length } C_2.$$

It follows that all chains from $k(s)$ to $k(t)$ have the same cardinality. □

Theorem 2.4. *Suppose that G is a $\mathbb{Q}$-group and there is only one $k(s) \in \Omega(G)$ such that $\alpha(1,s) = p$, where p is prime. Then one of the following holds:*

(a) if $p = 2$, then a Sylow 2-subgroup of G is isomorphic to C_2 or Q_8 and G is isomorphic to one of the following groups:

(i) $G \simeq G' : C_2$ where G' is the commutator subgroup of G and a Sylow 3-subgroup of G as well

(ii) $G \simeq E(p^n) : Q_8$ where $n \in \mathbb{N}$ and $p = 3, 5$

(b) if $p \neq 2$ and G is solvable, then a Sylow p-subgroup of G is abelian and $p = 3$ or 5.

Proof. Suppose that $p = 2$. If $k(s)$ is the only vertex that is connected to $k(1)$ by an edge with label 2, then there is a unique conjugacy class of involutions. Therefore a Sylow 2-subgroup P of G is the cyclic group of order 2^n or the generalized quaternion group of order 2^n. If $P \simeq C_{2^n}$, then $n = 1$ since $Z(P)$ is elementary abelian by Theorem B. Otherwise, $P = Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1}\rangle$. Hence $\mid \frac{N_G(<a>)}{C_G(<a>)} \mid = \varphi(o(a)) = 2^{n-2}$ and therefore $\mid N_G(a)_2 \mid = 2^{n-2} \mid C_G(a)_2 \mid \geq 2^{n-2}o(a) = 2^{n-3}$. But Q_{2^n} is a Sylow 2-subgroup of G. Therefor $2n - 3 \leq n$. It follows that $n = 3$ and $P = Q_8$. If $P \simeq C_2$ then by [7] case(i) holds and if $P \simeq Q_8$ then by ([10], p. 35) case(ii) holds.

Now, Suppose that $p \neq 2$. In the case $\alpha(1,s) = p$, all elements of odd prime order p are conjugate and if G is solvable, then its Sylow p-subgroup is abelian, due to a consequence of a result of Gaschütz and Yen [5] (see also Theorem 8.7, p. 512 of [8]). On the other hand, if G is a solvable $\mathbb{Q}$-group, then $\pi\{G\} \subseteq \{2, 3, 5\}$ (see [6]). The proof is done. □

Remark 2.5. *It is well known theorem in character theory (for an example see [1] page 227), that if character* $\chi \in Ch(G)$ *(characters ring) takes values in* $\mathbb{Q}$ *and* $s, t \in G$ *such that p-parts s and t be conjugate, then* $\chi(s) \equiv \chi(t)$ *(mod p), where p is a prime number. Therefore, if G is a* $\mathbb{Q}$*-group and* $\alpha(s, t) = p$ *for* $k(s), k(t) \in \Omega(G)$*, then for every character* $\chi \in Ch(G)$, $\chi(s) \equiv \chi(t)$ *(mod p).*

The converse of the theorem is not always true. For example, consider the Weyl group G of the exceptional Lie algebra F_4*. It is easy to see that there are two elements s and t in G such that,* $\chi(s) \equiv \chi(t)$ *(mod 2) for every* $\chi \in Ch(G)$*, but* $\alpha(s, t) \neq 2$*. The following problem is reasonable to be asked:*

Problem. *Suppose that G is a* $\mathbb{Q}$*-group for which every* $\chi \in Ch(G)$ *satisfies* $\chi(s) \equiv \chi(t)$ *(mod p) if and only if* $k(s), k(t) \in \Gamma(G)$*. Then, what is the structure of G?*

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