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On the converse of Ostrowski Theorem

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Abstract

In this paper, we prove that the converse of Ostrowski theorem is true. More precisely, we show that if $A \in M_n$ and for every positive semidefinite matrix $P \in M_n$, the inclusion $\sigma(AP) \subseteq \sigma(A)W(P)$ holds, then A is a scalar multiple of a Hermitian matrix. Here $\sigma(.)$ and $W(.)$ denote the spectrum and the numerical range of matrices. In addition, We show that the converse of Ostrowski theorem is not true for the bounded linear operators that act on an arbitrary Hilbert space.

Keywords: eigenvalue perturbation, inertia, numerical range

Mathematics Subject Classification [2010]: 47A12, 15A18

1 Introduction

Two matrices A, B in M_n, the algebra of all $n \times n$ matrices with complex entries are said to be *congruent if there is a nonsingular matrix $S \in M_n$ such that $A = S^*BS$. By Sylvester's theorem if A and B are Hermitian matrices then they are *congruent if and only if they have the same inertia. The following theorem of Ostrowski [5, Theorem 4.5.9] determines the changes of magnitudes of the eigenvalues under *congruence:

Theorem 1.1 (Ostrowski). Let $A, S \in M_n$, A be Hermitian and eigenvalues of A and SS^* be arranged in increasing order $\lambda_1(A) \leq \ldots \lambda_n(A)$ and $\lambda_1(SS^*) \leq \ldots \lambda_n(SS^*)$. For each $k = 1, 2, \ldots, n$ there exists a nonnegative real number θ_k such that $\lambda_1(SS^*) \leq \theta_k \leq \lambda_n(SS^*)$ and $\lambda_k(SAS^*) = \theta_k \lambda_k(A)$.

In this paper we prove that the converse of the Ostrowski theorem is true and the scalar multiples of Hermitian matrices are only matrices for which the following inclusion holds

$$\sigma(AP) \subseteq \sigma(A)W(P), \text{for every positive semidefinite matrix } P \in M_n.$$

In addition, we will have a short discussion about the Ostrowski theorem on an arbitrary Hilbert space. Before starting the next section, we need to introduce some basic notations.

The numerical range of $A \in M_n$ is a compact and convex region [6] of the complex plane which is defined by

$$W(A) = \{< Ax, x >: \ x \in \mathbb{C}^n, \|x\| = 1\}.$$

The numerical radius of A is displayed by $r(A)$ and is defined as follows

$$r(A) = sup\{|\lambda| : \lambda \in W(A)\}.$$

A is called a spectral matrix if $r(A) = \rho(A)$, where $\rho(A)$ denotes the spectral radius of A. The following theorem [6, Corollary 1.7.7] will be used in the next section frequently:

Theorem 1.2. Let $P \in M_n$ be a positive semidefinite matrix. Then for every $A \in M_n$ the inclusion $\sigma(AP) \subseteq W(A)\,W(P)$ holds.

In [1, 2, 3, 7] authors proved some versions of the converse of this theorem and related problems.

2 Main results

Theorem 2.1. For every $A \in M_n$ the following expressions are equivalent

(i) A is a scalar multiple of a Hermitian matrix.

(ii) For every $S \in M_n$ and $k = 1, 2, \ldots, n$ there exists a nonnegative real number θ_k such that $\lambda_1(SS^*) \leq \theta_k \leq \lambda_n(SS^*)$ and $\lambda_k(SAS^*) = \theta_k \lambda_k(A)$.

(iii) $\sigma(AP) \subseteq \sigma(A)W(P)$, for every positive semidefinite matrix $P \in M_n$.

(iv) $\sigma(AP) \subseteq \sigma(A)W(P)$, for every rank one positive semidefinite matrix P.

Proof. $(i) \to (ii)$ This is Ostrowski theorem.

$(ii) \to (iii)$ Let P be an arbitrary positive definite matrix in M_n. Then there exits an invertible matrix $S \in M_n$ such that $P = S^*S$ and so $\sigma(AP) = \sigma(AS^*S) = \sigma(SAS^*)$. If $\lambda \in \sigma(AP)$, then by (ii) there exists $1 \leq k \leq n$ such that $\lambda = \lambda_k(SAS^*) = \theta_k \lambda_k(A)$, for some nonnegative real number $\lambda_1(SS^*) \leq \theta_k \leq \lambda_n(SS^*)$. This implies that $\lambda \in W(P)\sigma(A)$ and so $\sigma(AP) \subseteq \sigma(A)W(P)$. Finally if P is a positive semidefinite matrix, substituting P with $P_\epsilon = P + \epsilon I_n$ implies $\sigma(AP_\epsilon) \subseteq \sigma(A)W(P\epsilon)$, where I_n denotes the identity matrix in M_n. Now tending ϵ to zero gives the desired result.

The implication $(iii) \to (iv)$ is obvious.

$(iv) \to (i)$ Let x be a unit vector in $\mathbb{C}^n$ and consider the rank one matrix $P = xx^*$. Using (iii) we will have

$$x^*Ax \in \sigma(AP) \subseteq W(P)\sigma(A) = [0, 1]\sigma(A).$$

Taking union over all unit vectors $x \in \mathbb{C}^n$, we will have $W(A) \subseteq [0, 1]\sigma(A)$ and since $W(A)$ is a convex set, this inclusion implies that $W(A)$ should lies on a line which pass through the origin. Therefore A is a scalar multiple of a Hermitian matrix. □

Theorem 2.2. Let $A \in M_n$. Then the following conditions are equivalent.

(i) A is spectral.

(ii) For every positive semidefinite matrix $P \in M_n$, the inequality $\rho(AP) \leq \rho(A)\rho(P)$ holds.

(ii) The inequality $\rho(AP) \leq \rho(A)\rho(P)$ holds, for every rank one positive semidefinite matrix P in M_n.

Proof. $(i) \Rightarrow (ii)$ Let P be an arbitrary positive semidefinite matrix in M_n. By Theorem 1.2 we have

$$\rho(AP) \leq r(A)\ r(P) = \rho(A)\rho(P).$$

The implication $(ii) \Rightarrow (iii)$ is evident.

$(iii) \Rightarrow (i)$ Let x be a unit vector in $\mathbb{C}^n$ and $|x^*Ax| = r(A)$. Setting $P = xx^*$ implies that

$$|x^*Ax| = \rho((Ax)x^*) = \rho(AP) \leq \rho(A)\rho(P) = \rho(A).$$

Taking suprimum over all unit vectors x in $\mathbb{C}^n$, we get the desired result. □

Note that in general Theorem 2.1 is not true if we substitute $\mathbb{C}^n$ with an arbitrary Hilbert space. The right unilateral shift operator is an example of a non Hermitian operator that satisfies the conditions (iii) and (iv) of this theorem. Characterizing those operators for which the condition (iii) of Theorem 2.1 remains valid seems a good but not easy question. However we have the following characterization of those operators which satisfy the condition (iv) of Theorem 2.1. The following theorem is a direct result of [4, Theorem 2.4-1]. It is an extension of Theorem 1.2 and will be used in Theorem 2.4. Note that the numerical range of an operator in $B(H)$ is a convex set but in general it is not a closed set.

Theorem 2.3. Let $P \in B(H)$ be a positive operator. Then for every $T \in B(H)$ the inclusion $\sigma(TP) \subseteq \overline{W(T)}\ \overline{W(P)}$ holds.

Theorem 2.4. For every $T \in B(H)$ the following expressions are equivalent

(i) $\overline{W(T)} \subseteq [0, 1]\sigma(T)$.

(ii) $\sigma(TP) \subseteq \sigma(T)\overline{W(P)}$, for every non invertible positive operator $P \in B(H)$.

(iii) $\sigma(TP) \subseteq \sigma(T)\overline{W(P)}$, for every rank one positive operator $P \in B(H)$.

Proof. $(i) \to (ii)$ By Theorem 2.3 we have

$$\sigma(TP) \subseteq \overline{W(T)}\ \overline{W(P)} \subseteq \sigma(T)[0, 1][0, \|P\|] = \sigma(T)[0, \|P\|] = \sigma(T)\overline{W(P)}.$$

The implication $(ii) \to (iii)$ is obvious.

$(iii) \to (i)$. Let S_H be the unit sphere of H, consider $x \in S_H$ and set $P = x \otimes x$, where $P(y) =< y, x > x$. Then we have

$$< Tx, x > \in \sigma(TP) \subseteq \overline{W(P)}\sigma(T) = [0,1]\sigma(T).$$

Taking union over all $x \in S_H$ gives the desired result. □

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