Low-density parity-check codes with quantised messages on binary symmetric channel

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This reported work concerns the design and evaluation of low-density parity-check codes over a binary symmetric channel in the presence of a fixed-point min-sum decoder. In this case, using a discretised density evolution method together with an elegant linear programming approach, the constraint on the number of quantisation levels is incorporated in the optimisation process to derive a proper LDPC code. Simulation results demonstrate that the proposed method outperforms existing codes addressed in the literature.

Introduction: Low density parity check (LDPC) codes have received considerable attention owing to their ability to approach the capacity of communication channels [1]. There are basically various decoding methods for LDPC codes, each maintaining a balance among the complexity, speed and error probability. The most powerful decoding method is the sum-product algorithm (SPA), but it is too complex to deal with in practice. The min-sum algorithm is a low complexity decoding method with an acceptable performance. The aforementioned methods operate through passing messages between two kinds of nodes, namely the variable nodes and the check nodes. Basically, messages are in the form of log-likelihood ratio (LLR), which actually quantifies the reliability of exchange messages. Most of the previous work has focused on designing LDPC codes using the so-called density evolution (DE) algorithm on an AWGN channel (e.g. [1–3]).

The main goal of the current work is to design a practical code for binary systematic channel (BSC) to be simple enough for hardware implementation, while still having an acceptable performance. More specifically, we concentrate on a 3 bit min-sum decoder. To this end, using the discretised DE algorithm for the min-sum decoder together with an elegant linear programming (LP) approach, a proper degree distribution for the 3 bit min-sum decoder under various code rates are devised. The results are then compared with the best existing works addressed in the literature, showing the proposed method outperforms existing works.

Background information: An LDPC code with rate $R = K/N$ is a linear block code which is fully characterised with a parity-check matrix H of dimensions $M \times N$, where $M = N - K$. The parity check matrix can be represented as a bipartite Tanner graph, including two kinds of nodes, termed check nodes (CNs) and variable nodes (VNs). Generally, an LDPC code is specified by a degree distribution pair as $\lambda(x) = \sum_i \lambda_i x^{i-1}$ and $\rho(x) = \sum_i \rho_i x^{i-1}$, where λ_i and ρ_i are the fraction of edges connected to a VN and CN of degree i, respectively. Thus, $\lambda(x)$ and $\rho(x)$ form a PMF and we have $\lambda(1) = \rho(1) = 1$. Also, the code rate is obtained from $R = 1 - \int_0^1 \rho(x)dx / \int_0^1 \lambda(x)dx$. DE [1] is a technique to investigate the performance of LDPC codes under belief propagation decoding. Chung *et al.* in [4] suggest a powerful algorithm called discretised DE (DDE) which discretises the exchange messages and tracks the evolution of the exchange messages through finding their PMF. However, the Gaussian assumption cannot be incorporated in other channels, including the BSC studied here. An EXIT chart based on message error rate is another technique which can be used to capture the evolution of exchange messages at each iteration [5]. This motivated us to make use of descretised min-sum decoding and the aforementioned EXIT chart to draw a path towards finding a proper degree distribution pair of $\lambda(x)$ and $\rho(x)$ for BSC. To this end, we make use of some elementary EXIT charts to individually capture the exchange messages to/from variable nodes of the same degree, thereby making it possible to relate the message error of the exchanged messages of an irregular code with various node degrees to a linear combination of error messages associated with variable nodes of the same degree.

Proposed method: According to the min-sum decoding algorithm, variable node-to-check node messages and check node-to-variable node messages at the ith iteration, i.e. $m_{vc}^{(i)}$ and $m_{cv}^{(i)}$, are updated from

$$m_{vc}^{(i)} = m_0 + \sum_{\pi(v)\backslash c} m_{cv}^{(i-1)}$$
$$m_{cv}^{(i)} = \min_{\pi(c)\backslash v} |m_{vc}^{(i-1)}| \times \sum_{\pi(c)\backslash v} sign(m_{vc}^{(i-1)}) \tag{1}$$

where m_0 is the initial message (LLR) corresponding to the variable node v, $\pi(v)\backslash c$ is the set of check nodes connected to the variable node v excluding node c, and similarly $\pi(c)\backslash v$ is the set of variable nodes connected to node c excluding node v.

Note that for 3 bit min-sum decoding, the LLR messages are in the range of $(-4, 3)$, meaning all messages with values more than 3 and less than -4 are quantised to 3 and -4, respectively. Considering coded bits with alphabet $\{0, 1\}$ are mapped to $c_k \in \{1, -1\}$, the initial LLR messages received from a binary symmetric channel with crossover probability p become $\text{LLR}(c_k) = \ln(pr(c_k = 1|r_k)/(pr(c_k = -1|r_k)) = r_k \ln(1 - p/p)$. In this work, due to the use of min-sum decoding, we can simply scale the LLR values to -1 and $+1$ as the exchange messages in the min-sum decoding are a multiple of initial messages, thus this scaling does not hinder performance. In other words, this conversion has no effect on VNs and CNs update equation and merely decreases the intensity associated with the reliability of received bits. Now, we express the DDE for the above decoding algorithm. Without loss of generality, it is assumed an all zero codeword is transmitted. As is described in [1], the PMF of initial messages (P_0) in BSC with crossover probability p is $P_0 = p\delta(x + \ln 1 - p/p) + (1 - \rho)\delta(x - \ln 1 - p/p)$, where δ denotes the Dirac delta function. (The Dirac delta function is zero everywhere except at the origin, where it is infinite. Moreover, this function should satisfy the constraint $\int_{-\infty}^{+\infty} \delta(x)dx = 1$.) As is argued earlier, due to the scaling of initial messages, P_0 simplifies to $P_0 = p\delta(x + 1) + (1 - p)\delta(x - 1)$. This PMF is considered as initial messages sent from VNs to their neighboring CNs according to (1). To get the PMF of messages coming out of CNs, we concentrate on pairwise computation of this PMF. To this end, referring to (1) for the update equation at the CNs and noting $y = \min_{i=1,\ldots,N} |X_i| \cdot \prod_{i=1,\ldots,N} sign(X_i) = \min(\min(\min(|X_1|, |X_2|), |X_3|), \ldots, |X_n|) \cdot \prod_{i=1,2} sign(X_i)$, one can simply divide the computation of PMF at the output of CNs into $n - 1$ steps of pairwise computations. On the other hand, one can readily verify that when $Y = \min(|X_i|, |X_j|). \prod_{i=1,2} sign(X_i)$ the PMF of Y, i.e. $p_Y(i) = pr(Y = i)$, is related to the probability mass function (PMF) and cumulative density function (CDF) of X_i and X_j as

$$\begin{aligned} P_Y(i) &= p_{X_1}(i)(1 - P_{X_2}(i)) + p_{X_2}(i)(1 - P_{X_1}(i)) \\ &\quad + p_{X_1}(j)P_{x2}(j) + p_{X_2}(j)P_{X_1}(j) + p_{X1}(i)p_{X_2}(i) \\ &\quad - p_{x1}(j)p_{X_2}(j) \quad i > 0 \\ P_Y(i) &= p_{X_1}(i)(1 - P_{X_2}(j)) + p_{X_2}(i)(1 - P_{X_1}(j)) \\ &\quad + p_{X_1}(j)P_{X_2}(i) + p_{X_2}(j)P_{X_1}(i) \quad i < 0 \end{aligned} \tag{2}$$

where $P_{X_j}(i) = pr(X_j \leq i)$ for $j = 1, 2$. Finding the PMF of outgoing messages from variable nodes is simply computed from the convolution of the PMF of incoming messages and that of the initial message, according to the first equation of (1). Finally, the error probability at the VNs is the sum of the negative tail of output messages from VNs.

Now, we are going to present our proposed algorithm in an attempt to get proper degree distribution pair $\lambda(x)$ and $\rho(x)$ at a desired rate. To this end, for a given $\rho(x)$ and input crossover probability p_{in}, we get the output error probability p_{out} at the VNs after one iteration, for a range of variable node degrees up to d_v. Assuming $P_{out} = f_i(p_{in})$ is the resulting output error probability at a variable node with degree i when the input crossover probability is set to p_{in}, one can form the following LP problem:

$$\begin{aligned} &\mathit{Maximise} && \sum_{i \geq 2} \lambda_i / i \\ &\mathit{subject\ to} && \lambda_i \geq 0, \quad \sum_{i \geq 2} \lambda_i = 1, \mathit{and} \\ & && \forall p_{\text{in}} \in [0, p_0]: \quad \sum_{i \geq 2} \lambda_i f_i(p_{in}) < p_{in} \end{aligned} \tag{3}$$

where the objective in (3) aims at maximising the code rate, since we have $R = 1 - \int_0^1 \rho(x)/\int_0^1 \lambda(x) = 1 - \sum_{i \geq 2 \rho_i/i} / \sum_{i \geq 2} \lambda_i / i$ and it is assumed $\rho(x)$ is fixed. (Throughout the simulations, it is assumed check nodes have the same degree, i.e. $\rho(x) = x^i$. Accordingly, the best value of i is numerically derived.) The constraint $\sum_{i \geq 2} \lambda_i = 1$ ensures $\lambda(x)$ forms a PMF. Finally, the constraint $\sum_{i \geq 2} \lambda_i f_i(p_{in}) < p_{in}$ for $p_{in} \epsilon [0, p_0]$ is to make sure the message error rate decreases at each iteration.

Numerical results: We assume $\lambda(x)$ is of degree $d_v = 10$. Accordingly, for $\rho(x) = x^4$, the message error probability of different variable node degrees for a range of p_{in} starting from 0 to $p_{in}^{max} = 0.075$ are depicted in Fig. 1. Note that the objective is to linearly combine the message error curves with scaling values $0 \le \lambda_i \le 1$ ($\sum_{i=1}^{i=10} \lambda_i = 1$) such that the resulting curve resides below the curve $p_{out} = p_{in}$ for $p_{in} \in [0, p_{in}^{max}]$. This is achieved through solving (3). Note that increasing p_{in}^{max} results in a smaller code rate. Thus, there is a balance between p_{in}^{max} and the code rate R. For instance for $p_{in}^{max} = 0.075$ the code rate becomes $R = 0.33$ and the resulting $\lambda(x)$ becomes $\lambda(x) = 0.85x^2 + 0.15x^8$. Table 1 is provided to give the best pair of $\lambda(x)$ and $\rho(x)$. Note that the impact of quantisation levels is reflected in message error curves $f_i(p_{in})$. Finally, Fig. 2 compares the probability of error against crossover probability of the proposed codes with some existing LDPC codes addressed in the literature [6], considering the code length is 2304.

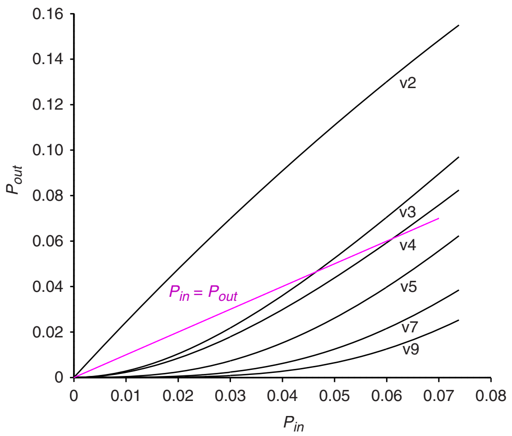

Fig. 1 *Elementary EXIT charts for different variable degrees on BSC when $d_c = 5$*

Table 1: Threshold of the best code ensemble with different rates on BSC

	Code1	Code2	Code3	Code4	Code5	Code6
Rate	0.1666	0.25	0.3333	0.4	0.5	0.75
Threshold	0.1273	0.1069	0.07386	0.0611	0.0394	0.00824
(λ, ρ)	$\lambda(x) = 0.85x^2 + 0.15x^8$ $\rho(x) = x^3$	$\lambda(x) = x^2$ $\rho(x) = x^3$	$\lambda(x) = 0.85x^2 + 0.15x^8$ $\rho(x) = x^4$	$\lambda(x) = x^2$ $\rho(x) = x^4$	$\lambda(x) = x^2$ $\rho(x) = x^5$	$\lambda(x) = 0.81x^2 + 0.19x^4$ $\rho(x) = x^{12}$

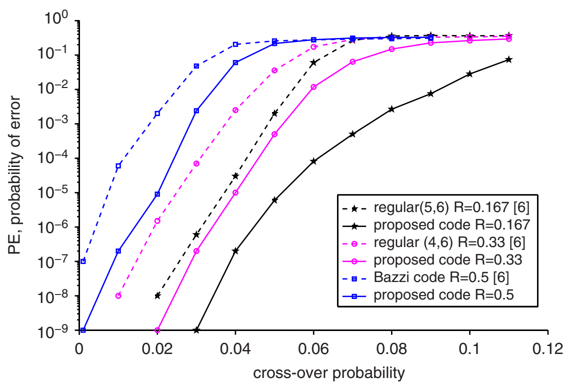

Fig. 2 *Comparison results*

Conclusion: This Letter aimed at finding proper LDPC codes under a min-sum decoding algorithm with quantised messages. Accordingly, the impact of quantisation levels is reflected in message error rates associated with each variable node degree, then it is incorporated in a LP problem to find a proper degree distribution pair of $\lambda(x)$ and $\rho(x)$.

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One or more of the Figures in this Letter are available in colour online.

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