Bulletin of the Iranian Mathematical Society Vol. 37 *No. 3* (*2011*), *pp 291-304.*

LINEAR SPHERICITY TESTING OF 3-CONNECTED SINGLE SOURCE DIGRAPHS

A. DOLATI

Communicated by Hossein Hajiabolhassan

Abstract. It has been proved that sphericity testing for digraphs is an NP-complete problem. Here, we investigate sphericity of 3-connected single source digraphs. We provide a new combinatorial characterization of sphericity and give a linear time algorithm for sphericity testing. Our algorithm tests whether a 3-connected single source digraph with n vertices is spherical in $O(n)$ time.

1. **Introduction**

Graph embeddings and their generalization on surfaces have many applications such as VLSI layout, and graphical representations of a poset. Upward embedding is an important extention of graph embedding whose definition is as follows an upward embedding of a digraph (directed graph) D on an embedded surface S is an embedding of its underlying graph on the surface such that all arcs are represented by monotonic curves that point to a fixed direction; in some literature, it is called upward drawing without crossing. Here, we focus on the upward embedding on the round sphere $S = \{(x, y, z) : (x^2 + y^2 + z^2) = 1\}$. On the one hand, it is a closed, compact, orientable surface in $\mathbf{R^3}$, and thus, has a simple structure. On the other hand, from the upward embedding

MSC(2000): Primary: 5C20, 5C85; Secondary: 5C10
Keywords: Embedding, upward embedding, sphericity, single source digraph.
Received: 31 May 2009, Accepted: 23 March 2010.
∗Corresponding author

point of view, it provides more freedom than the plane. More precisely, an undirected graph has an embedding on the round sphere if and only if it is a planar graph. However, there are digraphs that have upward embedding on the round sphere but have no upward embedding on the plane [4, 7]. In general, the study of upward embedding on surfaces has been motivated by graph embedding, and topological graph theory, whose literature is extensive (cf., for example, [6] or [11]). But, there are major differences between graph embedding and upward embedding of digraphs. For instance, all genus one orientable surfaces such as horizontal and vertical tori are topologically homeomorphic and from the point of view of graph embedding are equivalent. But Dolati et al. [5] showed that for upward embedding, horizontal and vertical tori are not equivalent. That is, a digraph, with an underlying graph with genus one, may have an upward embedding on the vertical torus but may fail to have an upward embedding on the horizontal torus. Therefore, the embedding of surfaces in $\mathbf{R^3}$ is important for upward embedding.

Clearly, if a digraph has an upward embedding on a surface, then it must be acyclic. Therefore, in upward embedding, the directed acyclic graphs (dags) have been considered. A dag is called *spherical* if it has an upward embedding on the round sphere. A spherical dag and its upward embedding on the round sphere are depicted in Figure 1. Since the underlying graph of a spherical dag has an embedding on the sphere, therefore a necessary condition for a dag to be spherical is that its underlying graph to be planar. The *sphericity testing* decision problem is as follows.

Instance: Directed acyclic graph (dag) D.

Question: Is D a spherical digraph?

Sphericity testing problem for digraphs is an NP-complete problem [8]. There is an $O(n + r^2)$ time algorithm for sphericity testing of a 3-connected single source digraph (3-connected sT dag), where n and r are the number of the vertices and the number of the sinks of the digraph, respectively [4]. Here we present a new characterization of sphericity of 3-connected sT dags. Then, by this characterization, we develop an $O(n)$ time algorithm for sphericity testing of a 3-connected sT dag with n vertices. The remainder of our work is organized as follows. After some preliminaries in Section 2, some results about assignment graph of an embedded single source digraph are presented in Section 3. In Section 4, we present a new characterization of sphericity of 3-connected sT dag. Then, we develop a linear time algorithm to determine whether

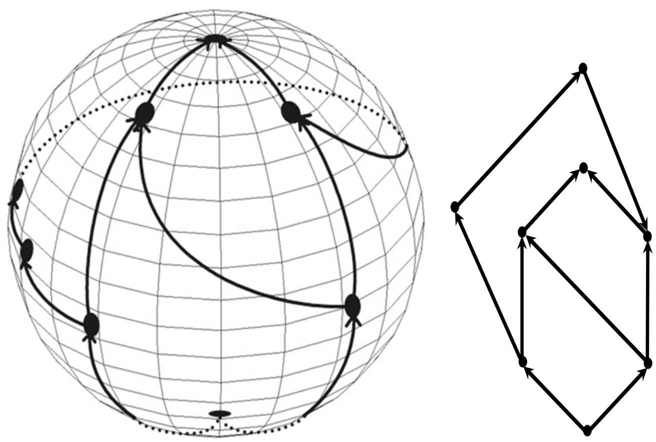

FIGURE 1. A spherical dag and its upward embedding on the round sphere

a 3-connected sT dag is spherical. In Section 5, we present conclusions and some related open problems from our point of view, that is worth to investigate.

2. **Preliminaries**

In this section, we recall some terminologies and basic results which we use throughout the paper. Let D be an embedded digraph on the plane. A *source* of D is a vertex without incoming edges. A *sink* of D is a vertex without outgoing edges. An *internal vertex* of D has both incoming and outgoing edges so that all incoming edges are consecutive. A *saddle vertex* of D has both incoming and outgoing edges so that all incoming edges are not consecutive. An embedded digraph is called *bimodal* if it has no saddle vertex. An *sT dag* is an acyclic digraph with exactly one source. Let f be a face of an embedded digraph D and $b(f)$ denote the subdigraph of D induced by the edges on its boundary. We call a vertex on the boundary of f an *f-local-sink*, if it is a sink in $b(f)$. We call the vertex v a *local sink* of an embedded sT dag D, if v is an f-local-sink for some face f of D. By n_f, we mean the number of f-local-sinks. An *ordinary face* f of an embedded sT dag has exactly one f-local-sink. Let D be an embedded sT dag and F, T, and T' be the set of faces, sinks, and local sinks of D, respectively. Obviously, T

is not empty and $T \subseteq T'$. The number of sinks in terms of local sinks is as follows [4]:

$$|T| = \sum_{f \in F} (n_f - 1) + 1. \tag{2.1}$$

A *sink-assignment* $\mathcal{A}$ is a mapping

$$\mathcal{A} : T \longrightarrow F$$

so that

$$\mathcal{A}^{-1}(f) \subseteq V(b(f)).$$

We say that $\mathcal{A}$ *assigns* v to f, if $v \in \mathcal{A}^{-1}(f)$. If, in addition, there exists a face h of F so that $\mathcal{A}^{-1}(f) = \begin{cases} n_f, & f = h \\ n_f - 1, & f \neq h, \end{cases}$ we call $\mathcal{A}$ a *consistent sink-assignment*. In this case, we call h to be the special face of $\mathcal{A}$.

A *similar embedding* of an embedded graph on a surface is an embedding which preserves the face structure. For example, a similar upward embedding of an embedded dag on the round sphere is depicted in Figure 1. The following theorem characterizes an embedded sT dag that has similar upward embedding on the round sphere.

Theorem 2.1. [4] *Let D be a bimodal embedded sT dag. Digraph D has a similar upward embedding on the round sphere if and only if there exists a consistent sink-assignment for it.*

A characterization of a general embedded digraph that has a similar upward embedding on the round sphere is as follows.

Theorem 2.2. [7] *An embedded digraph has a similar upward embedding on the round sphere if and only if it has a triangulation with no saddle vertex.*

3. Assignment Graphs of Embedded sT Dags

In this section, we present some results about assignment graph of an embedded sT dag. This graph was introduced by Bertolazzi et al. [1]. Let D be a bimodal embedded sT dag. The *assignment graph* A_D of D is the incidence bipartite graph of the faces and local sinks of D. More precisely, let F and T' be the set of faces and local sinks of D,

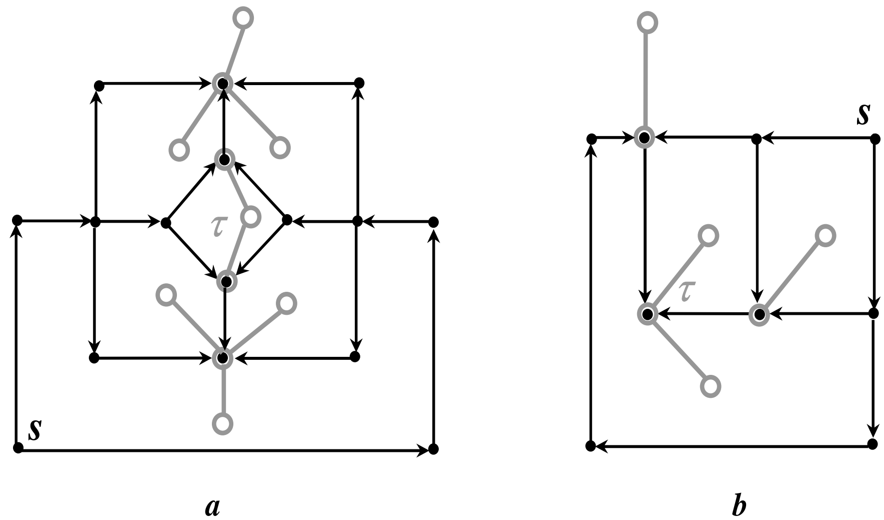

FIGURE 2. Two embedded sT dags and their assignment graphs

respectively. There is a bijection between a part of vertices of A_D and F; the vertices of this part are called *face-nodes* of A_D. There is a bijection between the other part of vertices of A_D and T'; the vertices of this part are called *sink-nodes* of A_D. A face-node f' corresponding to face f is adjacent to a sink-node t' if and only if t' is corresponding to an f-local-sink. Two embedded sT dags and their assignment graphs are depicted in Figure 2. We observe that the assignment graphs of the mentioned embedded sT dags are acyclic. In fact, this is a general property of assignment graphs of the embedded sT dags. It is stated in the following theorem.

Theorem 3.1. *The assignment graph of an embedded sT dag is a forest.*

Proof. Let D be an embedded sT dag. Suppose, for the sake of a contradiction, that there is a cycle C in A_D. Let S be the sink-nodes of C. One can easily see that S is a separation set of D. Therefore, $D \setminus S$ has at least two components. Now, since D is an acyclic digraph, $D \setminus S$ is an acyclic digraph too and each component of $D \setminus S$ has at least one source. Note that the edges incident to each vertex of S point outward from the components of $D \setminus S$, and thus D has at least 1two sources. This is a contradiction. □

Now, we can prove the following result.

Corollary 3.2. *If f and g are two distinct faces of an embedded sT dag D, then the cardinality of the intersection of the sets f-local-sinks and g-local-sinks is at most 1.*

Proof. Let f and g be an arbitrary pair faces of D. Suppose that t_1 and t_2 are two f-local-sinks and also two g-local-sinks. In this case, the subgraph of A_D induced by the face-nodes corresponding to f and g and sink-nodes corresponding to t_1 and t_2 is a cycle of A_D. This is a contradiction. □

The number of sink-nodes of a tree of A_D in terms of the number of local sinks of faces corresponding to its face-nodes is stated in the following lemma.

Lemma 3.3. *Let D be an embedded sT dag and A_D be its assignment graph. The number of sink-nodes of a tree τ of A_D is equal to*

$$\sum_{f \in F(\tau)} (n_f - 1) + 1,$$

where $F(\tau)$ is the set of faces corresponding to the face-nodes of τ.

Proof. Suppose that $|F(\tau)| = r$ and s denote the number of sink-nodes of τ. The number of the edges of τ is trivially equal to $\sum_{f \in F(\tau)} n_f$. Therefore,

$$r + s - 1 = \sum_{f \in F(\tau)} n_f.$$

That is,

$$s = \sum_{f \in F(\tau)} (n_f - 1) + 1.$$

□

We observe that the assignment graph of the embedded sT dag depicted in Figure 3, is a tree, i.e., it is connected. After some definitions we have a theorem by which one can characterize the connected assignment graphs. Let f be a face of D. It is called an *extremal face*, if all f-local-sinks are sinks of D. Let τ be a tree of the assignment graph of D. If all sink-nodes of τ are sinks of D, then we call τ an *extremal tree.* If exactly one sink-node of τ is an internal vertex of D, then we call τ an *ordinary tree.* If more than one sink-node of τ are not sinks of D,then we call it a *non-ordinary tree.*

Theorem 3.4. *Let D be an embedded sT dag. A_D is an extremal tree if and only if all faces of D are extremal.*

Proof. If all faces of D are extremal, then all the trees of A_D are trivially extremal. Suppose, for the sake of a contradiction, that A_D has k (for some integer $k \geq 2$) extremal trees $\tau_1, \ldots, \tau_k$. Let s_i, $F(\tau_i)$ $(i = 1, \ldots, k)$, $F(D)$, and T be the number of sink-nodes of τ_i, the set of faces corresponding to the face-nodes of τ_i, the set faces of D, and the set sinks of D, respectively. According to Lemma 3.3, the number of sinks is equal to

$$|T| = \sum_{i=1}^{k} s_i = \sum_{i=1}^{k}(\sum_{f \in F(\tau_i)} (n_f - 1) + 1),$$

or

$$|T| = \sum_{f \in F(D)} (n_f - 1) + k.$$

Since $k \geq 2$, according to (2.1), this is a contradiction.

Conversely, since A_D is an extremal tree, all the local sinks on the boundary of the faces must be sinks of D. This means that all the faces of D are extremal. □

4. **Characterization and Testing**

In this section, we shall introduce a new characterization by which one can characterize whether an embedded sT dag has a similar upward embedding on the round sphere. Then, we shall develop an optimal algorithm for the problem.

In the following theorem, we show that the face-node corresponding to the special face of any consistent assignment is a node of a extremal tree of A_D.

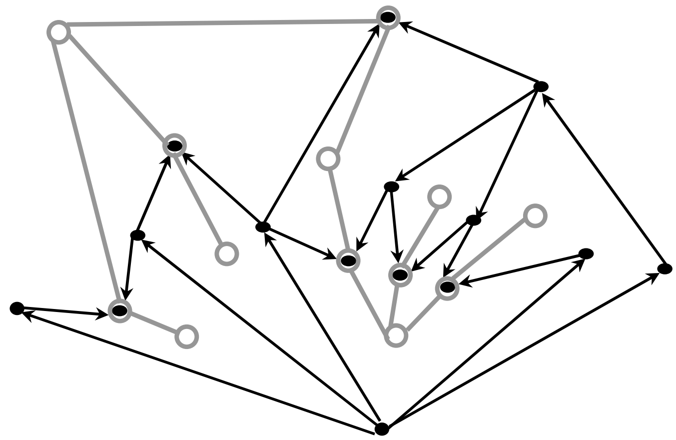

FIGURE 3. An embedded sT dags whose assignment graph is an extremal tree

Theorem 4.1. *Let D be an embedded sT dag that has a similar upward embedding on the round sphere. If $\mathcal{A}$ is a consistent sink-assignment of D and h is the special face of $\mathcal{A}$, then the face-node corresponding to the face h is a node of an extremal tree.*

Proof. Let τ and $F(\tau)$ be the tree containing the face-node corresponding to h and the set of faces corresponding to its face-nodes, respectively. It is sufficient to show that all members of $F(\tau)$ are extremal. Since $\mathcal{A}$ is a consistent sink assignment of D and h is its special face, thus

$$|\mathcal{A}^{-1}(h)| = n_h, \quad \forall f \in F(\tau) \setminus \{h\}, \ |\mathcal{A}^{-1}(f)| = n_f - 1.$$

which means that at least

$$\sum_{f\in F(\tau)\setminus\{h\}} (n_f - 1) + n_h = \sum_{f\in F(\tau)} (n_f - 1) + 1$$

sinks of D are on the boundary of the faces belonging to $F(\tau)$. In other words, τ has at least $\sum_{f\in F(\tau)}(n_f - 1) + 1$ sink-nodes corresponding to sinks of D. Thus, according to Lemma 3.3, every sink-nodes is a sink of D and τ is an extremal tree of A_D. □

Using the following theorem, we can characterize whether an embedded sT dag has a similar upward embedding on the round sphere. This

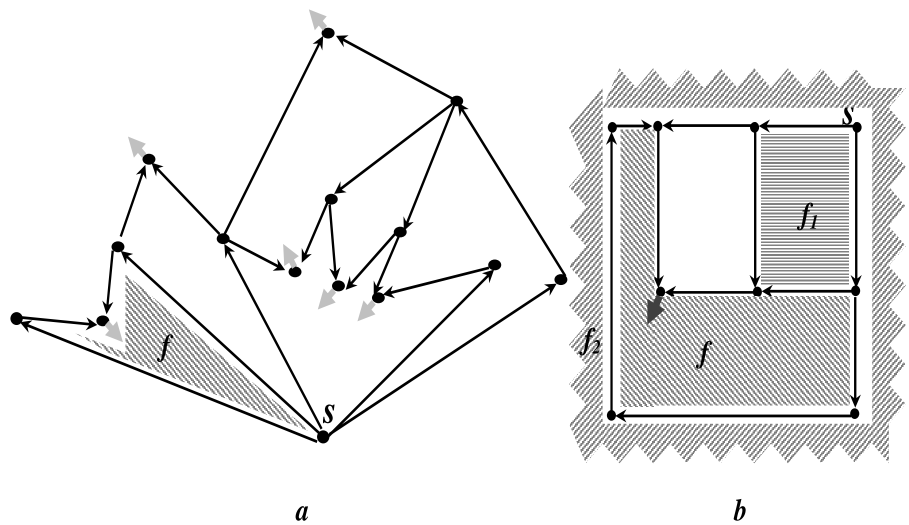

FIGURE 4. Two embedded sT dags and their consistent sink assignment according to their assignment graphs depicted in Figure 2 and Figure 3. The DFS algorithms in the proof of Theorem 4.2 start from the face-nodes corresponding to the shaded faces.

is a new characterization and it can be used to obtain a linear time algorithm to test if an embedded sT dag has a similar upward embedding on the round sphere.

Theorem 4.2. *Let D be a bimodal embedded sT dag and A_D be its assignment graph. D has a similar upward embedding on the round sphere if and only if A_D has exactly one extremal tree and does not have non-ordinary trees.*

Proof. Proof of necessity: Suppose that D has a similar upward embedding on the round sphere. According to Theorem 2.1 there is a consistent sink-assignment for D. Let $\mathcal{A}$ be a consistent sink-assignment for D. Since for every face f of D we have $\mathcal{A}^{-1}(f) \geq n_f - 1$, thus there must be at least $n_f - 1$ sinks on the boundary of f. On the other hand, according to Lemma 3.3, every tree τ of A_D has $\sum_{f\in F(\tau)}(n_f-1)+1$ sink-nodes. Therefore, at most one of these sink-node is an internal vertex of D and consequently τ is an extremal or ordinary tree. By Theorem

4.1, A_D has at least one extremal tree. Therefore, it is sufficient to show that A_D has no more than one extremal tree. Suppose for the sake of a contradiction that, it has two extremal trees τ_1 and τ_2. In this case, the number of sinks of D

$$|T| \geq \sum_{\tau\in\{\tau_1,\tau_2\}} (\sum_{f\in F(\tau)} (n_f - 1) + 1) + \sum_{\tau \text{ is a tree of } A_D \text{ and } \tau\notin\{\tau_1,\tau_2\}} \sum_{f\in F(\tau)} (n_f - 1).$$

Or equivalently

$$|T| \geq \sum_{f\in F(D)} (n_f - 1) + 2.$$

According to (2.1) it is a contradiction. That means A_D has just one extremal tree and all its other trees (if they exist) are ordinary trees.

Proof of sufficiency: Let A_D consists of an extremal tree τ and k ordinary trees $\tau_1, \ldots, \tau_k$(for some $k \geq 0$). We want to show that D has a similar upward embedding on the round sphere. To this end, by using A_D we obtain a consistent sink-assignment for D. Let f be the corresponding face of an arbitrary face-node of τ. By assumption, the local sinks on its boundary are sinks of D. We assign all of them to f. Other face-nodes of τ are visited by doing a depth-first search (DFS), starting from the face-node corresponding to f. Once a face-node is visited in the DFS, we assign all unassigned sinks on the boundary of the corresponding face to itself. Consequently, we assign n_f sinks to f and $n_h - 1$ sinks to every $h \in F(\tau) \setminus \{f\}$. Now, we consider trees $\tau_1, \ldots, \tau_k$ (if they exist). For $i = 1, \ldots, k$, let t_i, be the sink-node of τ_i that is an internal vertex of D. Let f_i be an arbitrary face for which t_i is an f_i-local-sink. The other f_i-local-sinks must trivially be sinks of D, and we assign them to f_i. Here, again we do a DFS by starting at the face-node corresponding to f_i. Once a face-node is visited in the DFS we assign all unassigned sinks on the boundary of the corresponding face to itself. One can easily see that we assign $n_h - 1$ sinks to every $h \in F(\tau_i)$, for $i = 1, \ldots, k$. This means that by the above argument, we construct a consistent sink-assignment for D. Therefore, D has a similar upward embedding on the round sphere. □

In the following example, we use the proposed algorithm in the proof of Theorem 4.2 for construction of consistent sink-assignments for embedded digraphs in Figure 4.

Example 4.3. *The consistent sink assignment for digraph depicted in Figure 4-a, whose assignment graph is depicted in Figure 3, is obtained by doing a DFS traverse on its assignment graph starting at the face-node corresponding to the shaded face f. The consistent sink assignment for digraph, depicted in Figure 4-b, whose assignment graph is depicted in Figure 2-b is obtained by doing DFS traverses on its assignment graph, starting at the face-node corresponding to the shaded face f for the extremal tree, and the face-nodes corresponding to the shaded faces f_1 and f_2, for the ordinary trees.*

Example 4.4. *The embedded sT dag, depicted in Figure 2-a, has no similar upward embedding on the round sphere, because its assignment graph contains a non-ordinary tree τ. Nevertheless, the digraph depicted in Figure 2-b has an assignment graph consisting of one extremal tree τ and two ordinary trees. Therefore, it has a similar upward embedding on the round sphere. The assignment graph of the embedded sT dag, depicted in Figure 3, consists of only one extremal tree. Therefore, it also has a similar upward embedding on the round sphere.*

Test and complexity. Now, by the characterization in Theorem 4.2, we are ready to test the sphericity of embedded sT dags in an optimal way.

Algorithm 1. *Sphericity testing.*
Input: *An embedded sT dag D.*
Output: **Yes** *or* **No** *depending on whether D has a similar upward embedding on the round sphere.*
Step 1: *Test if D is bimodal. If not return* **No**.
Step 2: *Construct assignment graph A_D.*
Step 3: *Check conditions of Theorem 4.2. If these conditions are not met, then return* **No** *else return* **Yes**.

By Algorithm Sphericity testing, one can test whether an embedded sT dag has a similar upward embedding on the round sphere. The complexity of each step of the algorithm is obviously, $O(n)$. Therefore, the complexity of the algorithm is, $O(n)$. Note that a 3-connected planar graph has exactly one embedding on the plane [12]. On the other hand, there are linear time algorithms for testing planarity of a graph and constructing a planar embedding if one exists [2, 3, 9, 10, 13]. Therefore, the unique embedding, of a planar 3-connected graph can be constructed in a linear time. By these facts and Algorithm 1, we have the following theorem.

Theorem 4.5. *Given a three connected* sT *dag* D*, we can decide whether it is spherical in* $O(n)$ *time, where* n *is the number of the vertices of* D*.*

5. Conclusion and Some Open Problems

We focused on the sphericity of embedded sT dags. We presented some results about the assignment graph of an embedded sT dag. Then, presented a new characterization for embedded sT dag having a similar upward embedding on the round sphere. Finally, by this characterization, we developed a linear algorithm for testing whether an embedded sT dag had a similar upward embedding on the round sphere. The following are some open problems:

(1) Characterize all 3-connected digraphs which admit upward embedding on round sphere.
(2) Characterize all sT dag which admit upward embedding on round sphere.
(3) Is it possible to find a polynomial time algorithm for upward embedding testing of a given 3-connected digraph on round sphere?
(4) Is it possible to find a polynomial time algorithm for upward embedding testing of a given sT dag on round sphere?

References

[1] P. Bertolazzi, G. Di Battista, C. Mannino and R. Tamasia, Optimal upward planarity testing of single-source digraphs, *SIAM J. Comput.* **27** (1998) 132-169.
[2] K. S. Booth and G.S. Lueker, Testing for the consecutive ones property, interval graphs and graph planarity using PQ-tree algorithms, *J. Comput. System Sci.* **13** (1976) 335-379.
[3] N. Chiba, T. Nishizeki, S. Abe and T. Ozawa, A linear algorithm for embedding planar graphs using PQ-trees, *J. Comput. System Sci.* **30** (1985), 54-76.
[4] A. Dolati and S. M. Hashemi, On the sphericity testing of single source digraphs, *Discrete Math.* **308** (2008) 2175-2181.
[5] A. Dolati, S. M. Hashemi and M. Khosravani, On the upward embedding on the torus, *Rocky Mountain J. Math.* **38** (2008) 107-121.
[6] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1987.

[7] S. M. Hashemi, Digraph embedding, graph theory (Prague, 1998), *Discrete Math.* **233** (2001) 321-328.
[8] S. M. Hashemi, I. Rival and A. Kisielewicz, The Complexity of upward drawings on spheres, *Order* **14** (1997/98) 327-363.
[9] J. E. Hopcroft and R. Tarjan, Efficient planarity testing, *J. Assoc. Comput. Mach.* **21** (1974), 549-568.
[10] B. Mohar, A linear time algorithm for embedding graphs in an arbitrary surface, *SIAM J. Discrete Math.* **12** (1999) 6-26.
[11] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press, Baltimore 2001.
[12] H. Whitney, A set of topological invariants for graphs, *Amer. J. Math.* **55** (1933) 231-235.
[13] S. G. Williamson, Depth-first search and Kuratowski subgraphs, *J. Assoc. Comput. Mach.* **31** (1984), 681-693.

Ardeshir Dolati
Department of Mathematics and Computer Science, Shahed University, P.O. Box 18151-159, Tehran, Iran
Email: `dolati@shahed.ac.ir`