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Error analysis and applications of the Fourier–Galerkin Runge–Kutta schemes for high-order stiff PDEs

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ABSTRACT

An integrating factor mixed with Runge-Kutta technique is a time integration method that can be efficiently combined with spatial spectral approximations to provide a very high resolution to the smooth solutions of some linear and nonlinear partial differential equations. In this paper, the novel hybrid Fourier–Galerkin Runge–Kutta scheme, with the aid of an integrating factor, is proposed to solve nonlinear high-order stiff PDEs. Error analysis and properties of the scheme are provided. Application to the approximate solution of the nonlinear stiff Korteweg–de Vries (the 3rd order PDE, dispersive equation), Kuramoto–Sivashinsky (the 4th order PDE, dissipative equation) and Kawahara (the 5th order PDE) equations are presented. Comparisons are made between this proposed scheme and the competing method given by Kassam and Trefethen. It is found that for KdV, KS and Kawahara equations, the proposed method is the best.

1. Introduction

Hybrid schemes based on a combination of the spectral and discrete variable methods have been used in recent years for approximating solutions for a variety of stiff nonlinear PDEs [1–3]. Numerical simulation of the stiff nonlinear PDEs is not an easy task if accurate solutions are to be found efficiently (for example see [4–10]). More recently, Kassam and Trefethen attempt to solve the Kuramoto–Sivashinsky (KS) and Korteweg–de Vries (KdV) equations [3] efficiently by modifying the method introduced with Cox and Matthews [2].

The KdV equation in one dimension can be derived based on nonlinear waves equations in shallow water. It has been found in other physical models such as ion acoustic waves in a plasma [11] and acoustic waves in an anharmonic crystal [12]. For more details one can see the work of Drazin and Johnson [13] and Infeld and Rowlands [11].

The one dimensional KS equation has been studied in the context of inertial manifolds and finite-dimensional attractors, and in the numerical simulations of dynamical behaviors [14]. On the other hand the KS equation models the effect of the particles being knocked out of the interface by the bombarding ions [15].

The Kawahara equation [16,17], is the fifth-order dispersive-type partial differential equation describing one-dimensional propagation of small-amplitude long waves in various problems of fluid dynamics and plasma physics [18,19]. The Kawahara equation is also known as fifth-order KdV or the special version of the Benney–Lin equation [20,21].

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Nomenclature

$\mathcal{L}$	Linear operator (continuous)
$\mathcal{N}$	Nonlinear operator (continuous)
$\mathbb{L}$	Discretized linear operator
$\mathbb{N}$	Discretized nonlinear operator
L	Discretized Fourier–Galerkin linear operator
x	coordinate in horizontal direction
t	time
$f(x,t)$	Real valued source function
$u(x,t)$	Continuous solution
$u^N(x,t)$	Truncated Fourier approximate solution
u_t	Derivative of u with respect to t
u_x	Derivative of u with respect to x
$\hat{u}(t)$	Fourier coefficients of $u(x,t)$
$\hat{u}_k(t)$	kth Fourier coefficients in Fourier–Galerkin method
$\hat{f}_k(t)$	kth Fourier coefficients of $f(x,t)$

Stiff systems of ordinary differential equations arise commonly when solving time dependent partial differential equations by spectral methods, and their numerical solution requires spatial treatment if accurate solutions are to be found efficiently. On the other hand, many time dependent partial differential equations contain low-order nonlinear terms with high-order linear terms. Examples include the third order KdV, the fourth order KS and the fifth order Kawahara equations. In this paper we propose and test the accuracy of a class of numerical methods for integrating stiff systems based on high-order approximations in space and time. The aim of this paper is to bring together three schemes in which the Fourier Galerkin scheme in space and the mixture of integrating factor with Runge–Kutta technique in time (FGIFRK4) is used to discretize the model problems. When we solve a time dependent PDE, it is natural to write the solution as a sum of Fourier modes with time dependent coefficients. The corresponding spectral methods have been shown to be successful for a wide range of applications [22–24]. In problems with periodic boundary conditions, a basis such as Fourier modes are appropriate since the linearized system is diagonal.

The structure of this paper is as follows. In Section 2 we describe the proposed novel scheme. In the third section we discuss the truncation error for the FGIFRK4 scheme. Section 4 provides the stability analysis. In Section 5 we summarize the results of comparison between the Fourier Galerkin exponential time differencing method with fourth order Runge–Kutta time stepping (FGETDRK4), Fourier Collocation exponential time differencing method with fourth order Runge–Kutta time stepping (FCETDRK4) and FGIFRK4 for the KdV, KS and Kawahara equations. Section 6 concludes the paper, and summarizes the advantages of the FGIFRK4 scheme.

2. Novel scheme: FGIFRK4

Consider the following stiff nonlinear PDE

$$u_t = \mathcal{L}u + \mathcal{N}(u,t) + f(x,t), \quad x \in [0, 2\pi], t > 0, \tag{1}$$

with periodic boundary conditions and one initial condition, where $f(x,t)$ is a given real valued function of x and t, $\mathcal{L}$ is the linear operator of arbitrary order and $\mathcal{N}$ is the nonlinear operator in the form

$$\mathcal{N}(u,t) = u_x u. \tag{2}$$

Discretization of (1) in space gives rise to a system of ODEs,

$$u_t = \mathbb{L}u + \mathbb{N}(u,t) + f_N(t), \quad t > 0, \tag{3}$$

with initial condition

$$u(x, t = 0) = u_0(x), \tag{4}$$

where f_N contains the prescribed right hand side. In the following theorem we provide the Fourier coefficients of the approximate solution to problem (1) in a fixed spatial mesh.

Theorem 1. *Assume that Eq. (1) is discretized by the Fourier Galerkin technique in space and the integrating factor with fourth order Runge–Kutta in time (FGIFRK4), where the boundary conditions are periodic. Then the Fourier coefficients are:*

$$\hat{u}(t_{n+1}) = E_2\hat{u}(t_n) + \frac{1}{6}(E_2 a_1 + 2E_1(b_1 + c_1) + d), \tag{5}$$

in which

$$E_1 = \mathrm{e}^{\frac{Lh}{2}}, \qquad E_2 = \mathrm{e}^{Lh}, \qquad a_1 = g(\hat{u}(t_n))^2 + h\hat{f}(t_n), \qquad b_1 = gE_1\left(\hat{u}(t_n) + \frac{a_1}{2}\right)^2 + h\hat{f}\left(t_n + \frac{h}{2}\right),$$

$$c_1 = g\left(E_1\hat{u}(t_n) + \frac{b_1}{2}\right)^2 + h\hat{f}\left(t_n + \frac{h}{2}\right), \quad \textit{and} \quad g = -h\frac{\mathrm{i}k}{2}.$$

Proof. We look for a solution that is periodic in space on the interval $[0, 2\pi]$ for Eq. (1). The approximate solution $u^N(x, t)$ is represented as the truncated Fourier series

$$u^N(x, t) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k(t)\mathrm{e}^{\mathrm{i}kx}, \quad t > 0. \tag{6}$$

In this method the fundamental unknowns are the coefficients $\hat{u}_k(t)$ for $k = -N/2, \dots, N/2 - 1$. Substituting the (6) in Eq. (1), multiplying both sides of the resulted equation by test function $\mathrm{e}^{-\mathrm{i}kx}$ and taking the integration over $[0, 2\pi]$, yields the nonlinear system of ordinary differential equations

$$(\hat{u}(t))' = L\hat{u}(t) - \frac{\mathrm{i}k}{2}(\hat{u}(t))^2 + \hat{f}(t), \tag{7}$$

where L is a diagonal matrix. The wavenumber $k = -N/2$ appears unsymmetrically in this approximation. If $\hat{u}_{-N/2}$ has a nonzero imaginary part, then $u^N(t)$ is not a real-valued function. This can lead to a number of difficulties and it is advisable in practice simply to enforce the condition that $\hat{u}_{-N/2}$ is zero (for example see page 123 in [1]). Multiplying both side of the Eq. (7) by integrating factor e^{-Lt}, implying

$$(\mathrm{e}^{-Lt}\hat{u}(t))' = -\frac{\mathrm{i}k}{2}\mathrm{e}^{-Lt}(\hat{u}(t))^2 + \mathrm{e}^{-Lt}\hat{f}(t). \tag{8}$$

Eq. (8) can be expressed as

$$(\hat{v}(t))' = -\frac{\mathrm{i}k}{2}\mathrm{e}^{Lt}(\hat{v}(t))^2 + \mathrm{e}^{-Lt}\hat{f}(t), \tag{9}$$

by using the following change of variable

$$\hat{v}(t) = \mathrm{e}^{-Lt}\hat{u}(t). \tag{10}$$

Applying the Runge-Kutta method to the Eq. (9), yields

$$\begin{aligned}
k_1 &= \alpha h\mathrm{e}^{Lt_n}(\hat{v}(t_n))^2 + h\mathrm{e}^{-Lt_n}\hat{f}(t_n),\\
k_2 &= \alpha h\mathrm{e}^{L\left(t_n+\frac{h}{2}\right)}\left(\hat{v}(t_n) + \frac{k_1}{2}\right)^2 + h\mathrm{e}^{-L\left(t_n+\frac{h}{2}\right)}\hat{f}\left(t_n + \frac{h}{2}\right),\\
k_3 &= \alpha h\mathrm{e}^{L\left(t_n+\frac{h}{2}\right)}\left(\hat{v}(t_n) + \frac{k_2}{2}\right)^2 + h\mathrm{e}^{-L\left(t_n+\frac{h}{2}\right)}\hat{f}\left(t_n + \frac{h}{2}\right),\\
k_4 &= \alpha h\mathrm{e}^{L(t_n+h)}(\hat{v}(t_n) + k_3)^2 + h\mathrm{e}^{-L(t_n+h)}\hat{f}(t_n + h),
\end{aligned}$$

in which $\alpha = -\frac{\mathrm{i}k}{2}$. Thus

$$\hat{v}(t_{n+1}) = \hat{v}(t_n) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{11}$$

Substituting the Eq. (10) into Eq. (11) gives

$$\hat{u}(t_{n+1}) = \mathrm{e}^{Lh}\hat{u}(t_n) + \frac{\mathrm{e}^{L(t_n+h)}}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{12}$$

By setting

$$\begin{aligned}
a &= \mathrm{e}^{L(t_n+h)}k_1 = \alpha h\mathrm{e}^{Lh}(\hat{u}(t_n))^2 + h\mathrm{e}^{Lh}\hat{f}(t_n),\\
b &= \mathrm{e}^{L(t_n+h)}k_2 = \alpha h\mathrm{e}^{L\frac{h}{2}}\left(\mathrm{e}^{L\frac{h}{2}}\left(\hat{u}(t_n) + \frac{a_1}{2}\right)\right)^2 + h\mathrm{e}^{L\frac{h}{2}}\hat{f}\left(t_n + \frac{h}{2}\right)\\
&= E_1\left(\alpha hE_1\left(\hat{u}(t_n) + \frac{a_1}{2}\right) + h\hat{f}\left(t_n + \frac{h}{2}\right)\right),
\end{aligned}$$

$$c = \mathrm{e}^{L(t_n+h)}k_3 = E_1\left(\alpha h\left(E_1\hat{u}(t_n) + \frac{b_1}{2}\right)^2 + h\hat{f}\left(t_n + \frac{h}{2}\right)\right),$$

$$d = \mathrm{e}^{L(t_n+h)}k_4 = \alpha h(E_2\hat{u}(t_n) + E_1c_1)^2 + h\hat{f}(t_n + h),$$

Fourier coefficients (5) are found and this proves the theorem. □

3. Truncation error

In the novel scheme FGIFRK4 the spatial discretization is spectral but the temporal discretization uses discrete variable method [25]. For spectral method, convergence at the rate $O(N^{-m})$ for every constant m is achieved, provided the solution is infinitely differentiable, and an even faster convergence at a rate $O(c^N)(0 < c < 1)$ is achieved if the solution is suitably analytic [26]. If the spatial discretization is presumed fixed, then we use the term "order of the scheme" in its ODE context. In this section we show the truncation error in the technique (FGIFRK4) that is proposed in Theorem 1 for Eq. (1).

Theorem 2. *Suppose that* $F(t, \hat{v}(t)) = -\frac{\mathrm{i}k}{2}\mathrm{e}^{Lt}(\hat{v}(t))^2 + \mathrm{e}^{-Lt}\hat{f}(t)$*, for* $t \in [0, a]$ *and* $\hat{v}(t) \in \mathbb{R}$*. Under the assumption of Theorem 1, if* $|F(t, \hat{v})| < Q$ *and* $\left|\frac{\partial^{i+j}F}{\partial t^i \partial \hat{v}^j}\right| < \frac{P^{i+j}}{Q^{j-1}}$*,* $i + j < 5$*, in which P and Q are some positive constants, then the temporal local truncation error of the novel scheme FGIFRK4 is* $O(h^5)$*.*

Proof. Consider the numerical solution of

$$(\hat{v}(t))' = F(t, \hat{v}(t)), \quad t \in [0, a], \hat{v}(t_0) = \hat{v}_0, \tag{13}$$

using the fourth order Runge–Kutta

$$\hat{v}_{n+1} = \hat{v}_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{14}$$

where

$$k_1 = hF(t_n, \hat{v}_n), \qquad k_2 = hF\left(t_n + \frac{h}{2}, \hat{v}_n + \frac{k_1}{2}\right), \qquad k_3 = hF\left(t_n + \frac{h}{2}, \hat{v}_n + \frac{k_2}{2}\right),$$
$$\text{and} \quad k_4 = hF(t_n + h, \hat{v}_n + k_3).$$

The temporal local truncation error $T_{n+1}(\hat{v})$ at time $n + 1$ is defined by

$$T_{n+1} = \hat{v}(t_{n+1}) - \left(\hat{v}_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\right). \tag{15}$$

By Taylor's expansion at time n for each term in the right hand side of (15), we have

$$\begin{aligned}
T_{n+1} = \frac{h^5}{5!}\Bigg[&\frac{1}{24}\frac{\partial^4 F}{\partial t^4} + \frac{1}{2}\frac{\partial F}{\partial \hat{v}}\frac{\partial^2 F}{\partial t \partial \hat{v}}\frac{\partial F}{\partial t} + \frac{1}{4}\frac{\partial^4 F}{\partial t^2 \partial \hat{v}^2}F^2 \\
&+ \frac{1}{6}\frac{\partial^4 F}{\partial \hat{v}^3 \partial t}F^3 + \frac{1}{6}\frac{\partial^4 F}{\partial t^3 \partial \hat{v}}F + \frac{1}{24}\frac{\partial^4 F}{\partial \hat{v}^4}F^4 + \frac{1}{2}F\frac{\partial^3 F}{\partial \hat{v}^2 \partial t}\frac{\partial F}{\partial t} \\
&- \frac{1}{4}\frac{\partial^2 F}{\partial \hat{v}^2}\frac{\partial^2 F}{\partial t^2}F - \frac{1}{4}F^3\left(\frac{\partial^2 F}{\partial \hat{v}^2}\right)^2 + 2\frac{\partial^2 F}{\partial \hat{v}^2}\frac{\partial F}{\partial \hat{v}}\frac{\partial F}{\partial t}F \\
&- \frac{1}{4}\frac{\partial^3 F}{\partial \hat{v}\partial t^2}\frac{\partial F}{\partial \hat{v}}F + \frac{1}{12}\frac{\partial^3 F}{\partial \hat{v}^3}\frac{\partial F}{\partial \hat{v}}F^3 - \frac{1}{6}\frac{\partial^3 F}{\partial t^3}\frac{\partial F}{\partial \hat{v}} + \frac{1}{4}\frac{\partial^3 F}{\partial t^2 \partial \hat{v}}\frac{\partial F}{\partial t} \\
&- \frac{1}{4}\frac{\partial^2 F}{\partial \hat{v}\partial t}\frac{\partial^2 F}{\partial t^2} - \frac{1}{2}\left(\frac{\partial^2 F}{\partial \hat{v}\partial t}\right)^2 F - \frac{3}{4}\frac{\partial^2 F}{\partial \hat{v}\partial t}\frac{\partial^2 F}{\partial \hat{v}^2}F^2 \\
&+ \frac{3}{4}\frac{\partial^2 F}{\partial \hat{v}^2}\left(\frac{\partial F}{\partial t}\right)^2 + \frac{3}{2}\frac{\partial^2 F}{\partial \hat{v}^2}\left(\frac{\partial F}{\partial \hat{v}}\right)^2 F^2 - \left(\frac{\partial F}{\partial \hat{v}}\right)^3\frac{\partial F}{\partial t} - \left(\frac{\partial F}{\partial \hat{v}}\right)^4 F \\
&+ \frac{1}{4}F^2\frac{\partial^3 F}{\partial \hat{v}^3}\frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial \hat{v}\partial t}\left(\frac{\partial F}{\partial \hat{v}}\right)^2 F + \frac{1}{4}\frac{\partial^2 F}{\partial t^2}\left(\frac{\partial F}{\partial \hat{v}}\right)^2\Bigg]_{(t_n, \hat{v}_n)} + O(h^6).
\end{aligned}$$

Since by assumptions F and its mixed partial derivatives are bounded, therefore the principal part of the absolute truncation error is $\frac{1}{36}h^5QP^4$. Hence

$$|T_{n+1}| = O(h^5). \quad \square$$

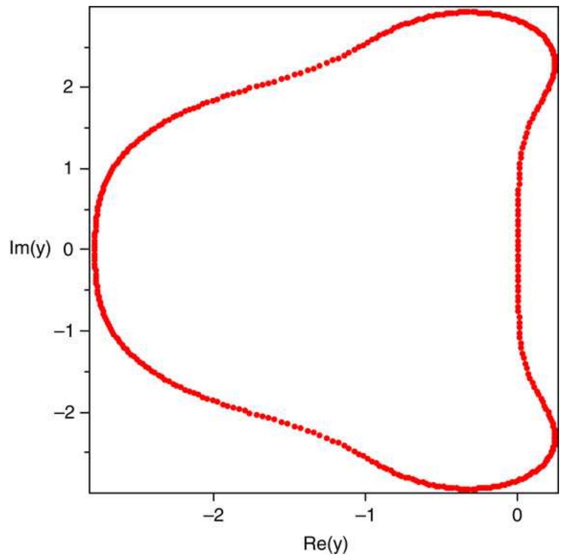

Fig. 1. Stability region for FGIFRK4 scheme in the complex plane for $y = \lambda t$.

The results of this theorem were announced without proofs in [27]. The important point to note here is that we obtained another leading factor for local truncation error. The leading factor $\frac{1}{36}$ in the course of proof is correct. Numerical scheme FGIFRK4 reduces the error in space spectrally and monitors the local truncation error in time to be $O(h^5)$. The scheme has two main goals. One is to get an accurate solution and the other is to accomplish the integration as inexpensively as possible.

4. Stability analysis

In this section we investigate the temporal stability analysis of FGIFRK4 scheme. Applying Eq. (10) to Eq. (7) we conclude that

$$(\hat{v}(t))' = F(t, \hat{v}(t)), \quad t \in [0, a], \hat{v}(t_0) = \hat{v}_0, \tag{16}$$

where $F(t, \hat{v}(t)) = -\frac{ik}{2} e^{Lt} (\hat{v}(t))^2 + e^{-Lt} \hat{f}(t)$. We denote by

$$(\hat{v}(t))' = \lambda(\hat{v}(t)), \tag{17}$$

the linear form of Eq. (16), where λ is in general a complex constant.

Theorem 3. *Under the hypotheses of Theorem 1, if moreover λ is a real and negative scaler, then for $h \leq \frac{-2.785294}{\lambda}$ the FGIFRK4 scheme is stable.*

Proof. Applying the fourth order Runge–Kutta method to Eq. (17) yields

$$\hat{v}_{n+1} = \hat{v}_n + \frac{1}{6}\left(6h\lambda + 3(h\lambda)^2 + (h\lambda)^3 + \frac{(h\lambda)^4}{4}\right)\hat{v}_n. \tag{18}$$

Setting $y = h\lambda$, we can write the Eq. (18) in the form

$$\hat{v}_{n+1} = g_1(y)\hat{v}_n, \tag{19}$$

where $g_1(y) = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}$. The stability condition $|g_1(y)| \leq 1$ holds for $y \in [-2.785294, 0.0]$. Hence $h \leq \frac{-2.785294}{\lambda}$ for negative λ, and the proof is complete. □

In the case that λ is complex-valued scaler, the region of stability for FGIFRK4 scheme is shown in Fig. 1.

5. Numerical results

To elucidate the efficiency of the proposed scheme FGIFRK4 we present some model problems that are solved by our written code in Matlab7.3, numerically.

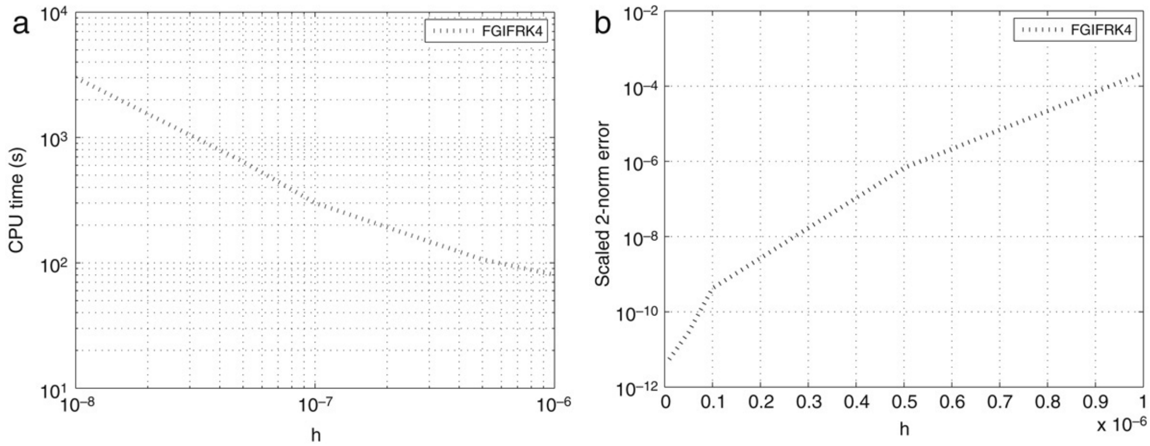

Fig. 2. (a) CPU time(s) for different values of h and (b) Scaled 2-norm magnitude of the relative error for the Model Problem 1 when $N = 2048$, using FGIFRK4 scheme.

5.1. Model Problem 1

Consider the Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \quad x \in [-10, 10], t > 0, \tag{20}$$

which is becoming a standard test for spectral solvers [28,29,26]. The stiffness results from the term u_{xxx} and manifests itself in rapid linear oscillation of the high-wavenumber modes [2].

The computations are spatially 2π-periodic and follow a soliton solution, $u = g(x - ct)$, where $g(x) = 3c\, sech^2\left(\frac{\sqrt{c}x}{2}\right)$ for one period, i.e., up to $t = \frac{2\pi}{c}$, with $c = 625$. The scaled 2-norm of the relative error i.e. $\sqrt{\left(\sum(u_j - g_j)^2/\sum g_j^2\right)}$ is plotted in Fig. 2(b). Fig. 2(a) represents the CPU time calculated for different values of time steps when $N = 2048$. We succeeded in solving Eq. (20) whereas the methods presented by Kassam and Trefethen [3] were not successful. A comparison between Fig. 2(b) and Fig. 9. in [2] shows that FGIFRK4 scheme yields more accurate solutions than the ETD4RK scheme of Cox and Matthews [2].

5.2. Model Problem 2

Consider the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \quad x \in [-20, 20], t > 0.$$

The initial condition at $t = 0$ is assumed to be $u(x, 0) = 2\,\mathrm{sech}^2(x)$. This equation with periodic boundary conditions represents the one dimensional nonlinear wave in shallow water. The exact solution is $u(x, t) = 2\,\mathrm{sech}^2(x - 4t)$. Here we compare the FGETDRK4 and FCETDRK4 in [3] against the FGIFRK4 scheme. The CPU time and the scaled 2-norm relative errors in estimating the quantity u^N, that calculated in different values of time steps h, are shown in Fig. 3.

Fig. 4 demonstrates the rate of CPU time changes for FCETDRK4 and FGETDRK4 with respect to the FGIFRK4. As it is shown, the proposed method is one hundred times faster than FCETDRK4, while it is as fast as FGETDRK4 for coarse meshes. Moreover the proposed method in Theorem 1 can achieve more accurate results than the other methods, as is shown in Fig. 3(b).

5.3. Model Problem 3: A dissipative partial differential equation

In this model problem we apply the FGIFRK4 to the well-known Kuramoto–Sivashinsky equation [30]

$$u_t = -2u_{xx} - u_{xxxx} - uu_x. \tag{21}$$

The boundary conditions are periodic, with spatial period 2π and the initial condition is $u(x, 0) = 0.03\sin(x)$. The fourth-derivative term makes the linear part extremely stiff, with rapid linear decay of the high-wavenumber modes. Thus standard explicit methods are impractical. This problem has the dissipative terms to remove the aliasing errors. Kassam and Terfethen [3] introduced the FCETDRK4 and FGETDRK4 to calculate approximation solution to (21). Tables 1–3 represent the approximate solutions to the problem with three different schemes FGETDRK4, FCETDRK4 and FGIFRK4 for various nodes (x_i, t_j), respectively. As one can see, the results of the FGIFRK4 have a good agreement with the work of Kassam and Terfethen.

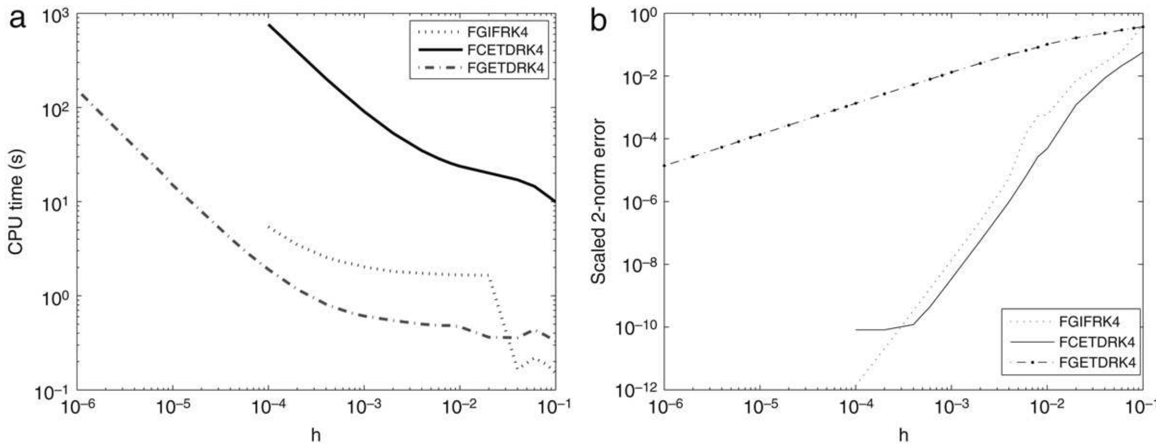

Fig. 3. (a) CPU time(s) and (b) Scaled 2-norm magnitude of the relative error for the Model Problem 2 using three methods for different values of time steps h when $N = 256$.

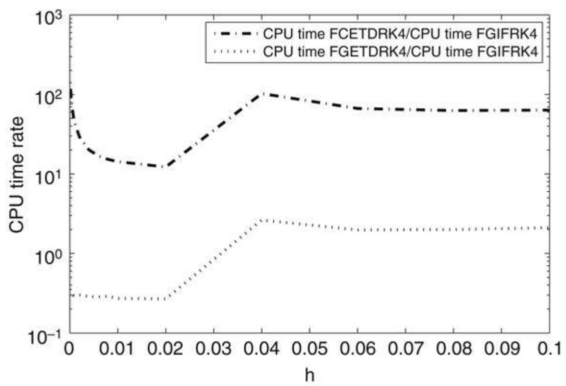

Fig. 4. Comparison of the CPU time rates between three different methods for Model Problem 2.

Table 1
Approximate solution to the Model Problem 3 using the FGETDRK4.

t_j	x_i			
	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$
0.6	0.038503177641	0.038801235566	−0.038801235566	−0.038503177641
1.8	0.126640459560	0.129931929100	−0.129931929100	−0.126640459560
3.0	0.406330023400	0.442342332931	−0.442342332931	−0.406330023400
4.2	1.170629181546	1.537368937438	−1.537368937438	−1.170629181546
6.0	2.354793901227	5.559638780860	−5.559638780860	−2.354793901227

Table 2
Approximate solution to the Model Problem 3 using the FCETDRK4.

t_j	x_i			
	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$
0.6	0.038503177639	0.038801235566	−0.038801235566	−0.038503177643
1.8	0.126640459585	0.129931929150	−0.129931929142	−0.126640459620
3.0	0.406330038666	0.442342350372	−0.442342350324	−0.406330038814
4.2	1.170633427246	1.537375375192	−1.537375375163	−1.170633427611
6.0	2.354906644952	5.560995166315	−5.560995170212	−2.354906642431

5.4. Model Problem 4

Consider an inhomogeneous periodic boundary value problem for the Kuramoto–Sivashinsky (KS) equation [31]

$$u_t + u_{xx} + \nu u_{xxxx} + uu_x = f(x, t), \quad x \in [0, 2\pi], t \in [0, 1], \tag{22}$$

Table 3
Approximate solution to the Model Problem 3 using the FGIFRK4.

t_j	x_i			
	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$
0.6	0.038503177644	0.038801235579	−0.038801235579	−0.038503177644
1.8	0.126640459850	0.129931929462	−0.129931929462	−0.126640459850
3.0	0.406330048589	0.442342355792	−0.442342355792	−0.406330048589
4.2	1.170633510822	1.537374062325	−1.537374062325	−1.170633510822
6.0	2.355020930540	5.560484954840	−5.560484954840	−2.355020930540

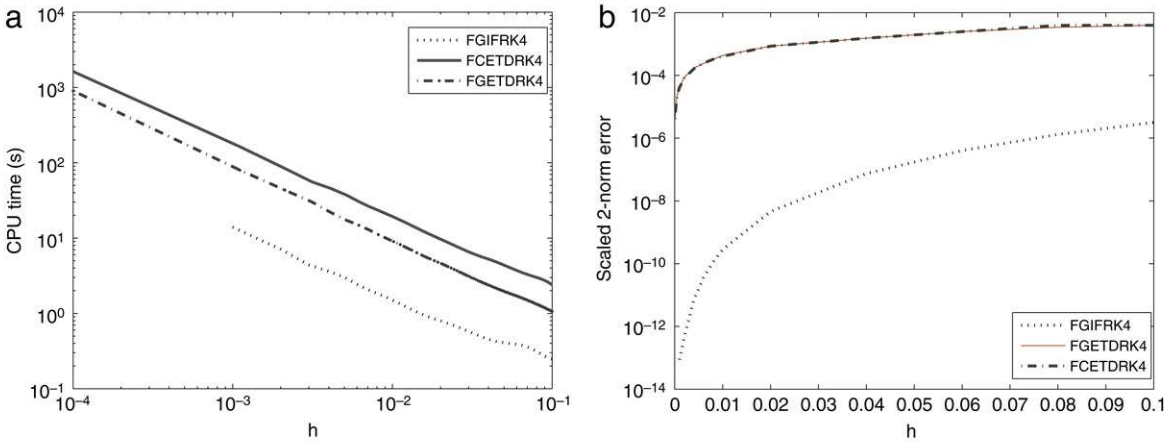

Fig. 5. (a) CPU time(s) and (b) Scaled 2-norm magnitude of the relative error for the Model Problem 4 using three methods for different values of time steps h when $N = 64$.

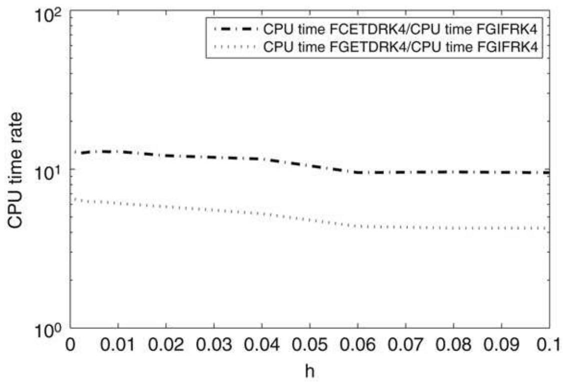

Fig. 6. Comparison of the CPU time rates between three different methods for Model Problem 4.

with initial condition chosen as $u(x, 0) = \sin(x)$. $u(x, 0)$ is a 2π-periodic function and ν is a positive parameter playing the role of viscosity. The exact solution is given by $u_{exact} = \sin(x + t)$. An attempt has been made here to compare the schemes FGETDRK4 and FCETDRK4 [3] against FGIFRK4. Fig. 5 represents the CPU time and the scaled 2-norm relative errors for different values of time steps h. Fig. 5(a) illustrates that FGIFRK4 is almost eight and sixteen times faster than FGETDRK4 and FCETDRK4, respectively (see Fig. 6). Fig. 5(b) shows that the proposed method is more accurate than the two other schemes.

5.5. Model Problem 5: The Kawahara equation

Consider the following nonlinear partial differential equation

$$u_t + u_{3x} - u_{5x} + uu_x = 0, \quad x \in [-100, 100], t \in [0, 0.5], \tag{23}$$

which is called Kawahara equation. Eq. (23) subject to initial condition

$$u(x, 0) = \frac{-72}{169} + \frac{105}{169\cosh^4(kx)}, \tag{24}$$

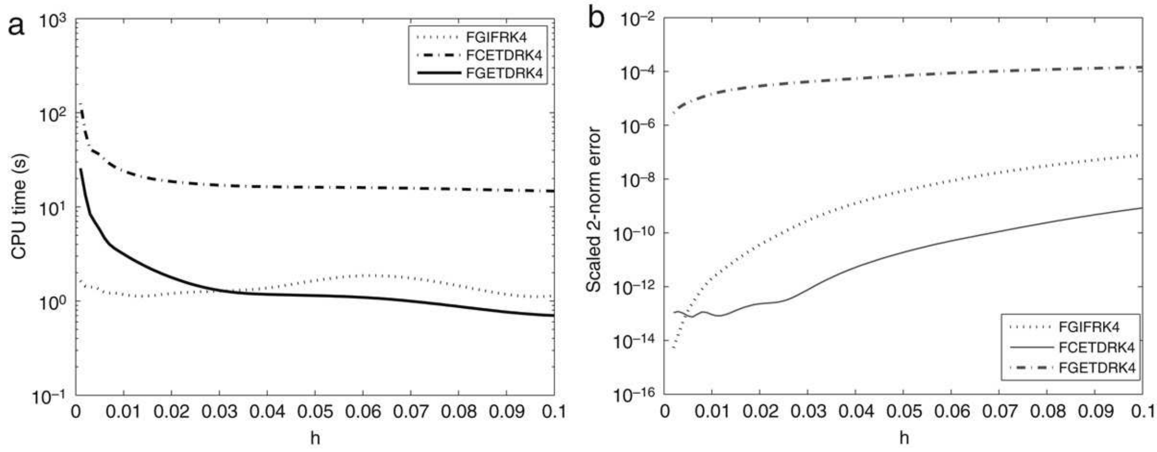

Fig. 7. (a) CPU time(s) and (b) Scaled 2-norm magnitude of the relative error for the Model Problem 5 using three methods for different values of time steps h when $N = 256$.

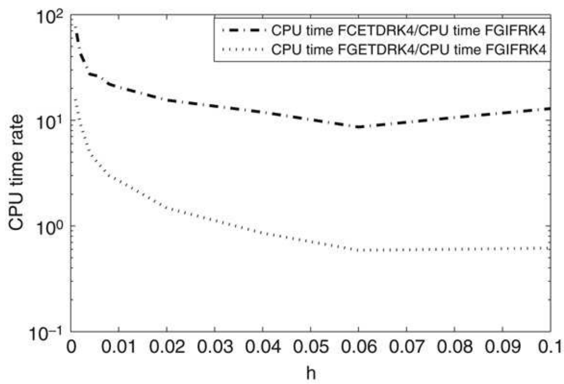

Fig. 8. Comparison of the CPU time rates between three different methods for Model Problem 5.

is solved where $k = \frac{1}{2\sqrt{13}}$. The exact solution is

$$u(x,t) = \frac{-72}{169} + \frac{105}{169\cosh^4(k(x+ct))}, \tag{25}$$

in which $c = \frac{36}{169}$. The Kawahara equation occurs in the theory of magneto-acoustic waves in a [17] and in the theory of shallow water waves with surface tension [32]. Fig. 7 demonstrate the comparison between three different schemes for (a) the CPU time and (b) scaled 2-norm of the relative error. Indeed, as shown in Fig. 8 for finer meshes, FGIFRK4 is almost ten and a hundred times faster than FGETDRK4 and FCETDRK4, respectively. For the large time steps h, the nature of the CPU time FGIFRK4 is the same as the FGETDRK4 scheme, while it is ten times faster than the FCETDRK4 scheme.

6. Conclusions

The present work considers the Fourier Galerkin spectral method combined with a fourth-order Runge–Kutta time-stepping technique for stiff high-order PDEs. The truncation error, stability analysis and stability region for the novel scheme are presented. Numerical tests have been carried out to examine the efficiency of the proposed scheme against the similar methods in the literature. The present numerical results have demonstrated the advantage of the novel scheme for the three following reasons. First, the novel scheme enables one to obtain the approximate solution for hard stiff problems as presented in Model Problems 1 and 4, while the similar methods either cannot solve the problem or they yield less accurate solutions. Second, in contrast with other schemes, when the novel scheme is used a highly accurate solution with less computational effort is yielded. Third, the proposed scheme in this paper succeeded in solving the inhomogeneous stiff PDEs in the presence of the source function (Model Problem 4). Additionally, the numerical results have confirmed the general behavior of the proposed scheme which was discussed in this paper.

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