ON SEMI-C-REDUCIBILITY OF ($\alpha$, $\beta$)-METRICS

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Every non-Riemannian ($\alpha$, $\beta$)-metric on a manifold with dimension $n \geq 3$ is semi-C-reducible. Thus the study on semi-C-reducible metrics will enhance our understanding on the physical meaning of ($\alpha$, $\beta$)-metrics. In this paper, we study weakly Landsberg semi-C-reducible metrics with some non-Riemannian curvature properties. Then we find some conditions on semi-C-reducible manifolds under which the notions of isotropic Landsberg curvature and of isotropic mean Landsberg curvature are equivalent.

Keywords: Semi-C-reducible metric; Landsberg metric; stretch metric.

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1. Introduction

For a Finsler metric $F = F(x, y)$, its geodesics are characterized by the system of differential equations $\dddot{c}^i + 2G^i_\ jk(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric $F$ is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y \in T_xM$ for any $x \in M$ [1–3]. The Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space [4].

For a Finsler metric $F$ on a manifold $M$, the Minkowski norm $F_x := F|_{T_xM}$ on $T_xM$, for each non-zero tangent vector $y$ at $x$ induces a Riemannian metric $g_x := g_{ij}(x, y)dy^i \otimes dy^j$ on $T_xM_0 = T_xM - \{0\}$. It is well-known that if $F$ is a Berwald metric then all $(T_xM_0, g_x)$ are isometric as Hilbert spaces [5].

Beside the Berwald curvature, there are several important Finslerian curvatures. Let $(M, F)$ be a Finsler manifold. The second derivatives of $\frac{1}{2}F^2_x$ at $y \in T_xM_0$ is an inner product $g_{xy}$ on $T_xM$. The third-order derivatives of $\frac{1}{2}F^2_x$ at $y \in T_xM_0$ is a symmetric trilinear forms $C_{xy}$ on $T_xM$. We call $g_{xy}$ and $C_{xy}$ the fundamental form

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and the Cartan torsion, respectively. The rate of change of $C_y$ along geodesics is the Landsberg curvature $L_y$ on $T_xM$ for any $y \in T_xM_0$. $F$ is said to be Landsbergian if $L = 0$ [6]. It is well-known that on a Landsberg manifold $(M, F)$, all $(T_xM_0, \hat{g}_y)$ are isometric as Banach spaces. It is proved that every Berwald metric is a Landsberg metric [7].

The quotient $L/C$ is regarded as the relative rate of change of $C$ along Finslerian geodesics. $F$ is said to be isotropic Landsberg metric if $L = cFC$, where $c = c(x)$ is a scalar function on $M$. There are many isotropic Landsberg Finsler metrics. For example, the (generalized) Funk metric on the unit ball $B^n \subset \mathbb{R}^n$ is an isotropic Landsberg metric with $c = -1/2$ [8].

Set $I_y := \sum_{i=1}^n C_y(e_i, e_i, \cdot)$ and $J_y := \sum_{i=1}^n I_y(e_i, e_i, \cdot)$, where $\{e_i\}$ is an orthonormal basis for $(T_xM, g_y)$. $I_y$ and $J_y$ is called the mean Cartan torsion and mean Landsberg curvature, respectively. A Finsler metric $F$ is said to be weakly Landsbergian if $J = 0$ [9]. The quotient $J/I$ is regarded as the relative rate of change of $I$ along geodesics. $F$ is said to be isotropic mean Landsberg metric if $J = cFI$, where $c = c(x)$ is a scalar function on $M$.

In order to find explicit examples of weakly Landsberg metrics, we consider $(\alpha, \beta)$-metrics [10]. Let us narrate a brief history of $(\alpha, \beta)$-metrics. This marchen originated in 1941 by a physicist Randers, who first introduced the notion of Randers metrics to consider the unified field theory [11]. A Randers metric $F = \alpha + \beta$ on a manifold $M$ is just a Riemannian metric $\alpha = \sqrt{\alpha_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on $M$ such that $\|\beta\|_\alpha < 1$. In the same time, another event was happened by a geometrician Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by Kropina [12]. Consequently, other match of Randers metric called Kropina metric $F = \alpha^2/\beta$ was born.

Regarding the Cartan tensors of these metrics, Matsumoto introduced the notion of $C$-reducibility and proved that any Randers and Kropina metric are $C$-reducible [13]. Matsumoto–Hōjo proved that the converse is true [14]. Furthermore, by considering Kropina and Randers metrics, Matsumoto introduced the notion of $(\alpha, \beta)$-metrics. An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha$ is a Riemannian metric and $\beta$ is a 1-form on $M$ [15].

In [16], Matsumoto–Shibata introduced the notion of semi-$C$-reducibility by considering the form of Cartan torsion of a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ with dimension $n \geq 3$. A Finsler metric is called semi-$C$-reducible if its Cartan tensor is given by,

$$C_{ijk} = \frac{p}{1+n} \{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{C^2}I_iI_jI_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on $TM$, $h_{ij}$ is the angular metric and $C^2 = F^2$. If $q = 0$, then $F$ is just $C$-reducible Finsler metric and if $p = 0$, then $F$ is called $C^2$-like metric. As a generalization of $C$-reducible metrics, Matsumoto–Shimada introduced the notion of $P$-reducible metrics [12]. A Finsler
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metric is called P-reducible if its Landsberg tensor is given by following,

\[ L_{ijk} = \frac{1}{1+n} \{ h_{ij}J_k + h_{jk}J_i + h_{ki}J_j \} . \]

It is easy to see that every Landsberg metric is a weakly Landsberg metric, but the converse is not true [9]. Every C-reducible or \(C2\)-like weakly Landsberg metric is a Landsberg metric. Thus it is natural to investigate this equivalency for semi-C-reducible metrics. Then, we prove the following.

**Theorem 1.1.** Let \((M, F)\) be a weakly Landsberg semi-C-reducible manifold. Then \(F\) is a Landsberg metric if and only if one of the following holds:

1. \(F\) is C-reducible metric;
2. \(F\) is \(C2\)-like metric;
3. \(q' := q_{|s}y^s = 0\), i.e. \(q\) is constant along any Finslerian geodesics.

More general, it is well-known that on a C-reducible or \(C2\)-like metric the notions of isotropic Landsberg curvature and isotropic mean Landsberg curvature are equivalent [8]. In this paper, we find some conditions on a semi-C-reducible metric under which these notions of curvatures are equivalent. More precisely, we prove the following.

**Theorem 1.2.** Let \((M, F)\) be a compact semi-C-reducible manifold. Suppose that \(F\) is P-reducible. Then the following are equivalent:

1. \(F\) has isotropic Landsberg curvature \(L = cFC\);
2. \(F\) has isotropic mean Landsberg curvature \(J = cFI\);

where \(c\) is a non-zero real constant.

As a generalization of Landsberg curvature, Berwald introduced the notion of stretch curvature. He showed that this tensor vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram [17]. For stretch semi-C-reducible metric, we prove the following.

**Theorem 1.3.** Let \((M, F)\) be a weakly Landsberg semi-C-reducible manifold. Suppose that \(F\) is a stretch metric. Then \(q'' := q_{|[v]}y^{v}y'^{(v)}(x, y) = 0\), i.e. \(q'\) is constant along any Finslerian geodesics.

There are many connections in Finsler geometry [18–21]. In this paper, we use the Berwald connection and the \(h\)- and \(v\)-covariant derivatives of a Finsler tensor field are denoted by “\(|\)” and “\(\:\)”, respectively.

Throughout this paper, we use *Einstein summation convention* for expressions with indices.\(^a\)

\(^a\)That is wherever an index is appeared twice as a subscript as well as a superscript, then that term is assumed to be summed over all values of that index.
2. Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of $M$.

A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0 := TM \setminus \{0\}$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$;
(iii) for each $y \in T_xM$, the following form $g_y$ on $T_xM$ is positive definite,

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]|_{s,t=0}, \quad u, v \in T_yM.$$ 

For a Finsler metric $F = F(x, y)$ on a manifold $M$, the spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a vector field on $TM$, where $G^i = G^i(x, y)$ are defined by,

$$G^i = \frac{\partial}{\partial t} \left( \frac{1}{4} \{ [F^2]_y y^j y^k - [F^2]_x \} \right),$$

where $g_{ij} := \frac{\partial}{\partial y^i} [F^2]_y y^j$ and $(g^{ij}) = (g_{ij})^{-1}$. By definition, $F$ is called a Berwald metric if $G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ for any $x \in M$.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{ds} [g_y + tw(u, v)]|_{s=0}, \quad u, v, w \in T_yM.$$ 

The family $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well-known that $C = 0$ if and only if $F$ is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion $I_y$ by $I_y(u) := I_i(y) u^i$, where $I_i := g^{jk} C_{ijk}$, $C_{ijk} = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} g_{ij}$, and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Deicke's Theorem, $F$ is Riemannian if and only if $I_y = 0$ [5, 22].

Let $(M, F)$ be a Finsler manifold. For $y \in T_xM_0$, define the Matsumoto torsion $M_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $M_y(u, v, w) := M_{ijk}(y) u^i v^j w^k$ where,

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \}.$$ 

$h_{ij} := FF y^i y^j$ is the angular metric.

**Lemma 2.1** ([14]). A Finsler metric $F$ on a manifold $M$ of dimension $n \geq 3$ is a Randers metric if and only if $M_y = 0$, $\forall y \in TM_0$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by:

$$C_{ijk} = \frac{p}{1+n} \{ h_{ij} I_k + h_{ik} I_j + h_{jk} I_i \} + \frac{q}{C^2} I_i I_j I_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on $TM$ and $C^2 = I^2 I_i$. Contracting the last relation by $g^{ik}$ shows that $p$ and $q$ satisfy $p + q = 1$. In [16], Matsumoto and Shibata proved that every $(\alpha, \beta)$-metric is semi-C-reducible.

**Theorem 2.2** ([12, 16]). Let $F = \alpha \phi(\beta)$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Then $F$ is semi-C-reducible.
The horizontal covariant derivatives of the Cartan torsion $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $L_y: T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $L_y(u,v,w) := L_{ijk}(y)u^iv^jw^k$, where

$$L_{ijk} := C_{ijk|s}y^s.$$ 

The family $\mathbf{L} := \{L_y\}_{y \in TM}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$. The quantity $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. A Finsler metric $F$ is said to be isotropic Landsberg metric if,

$$\mathbf{L} = c \mathbf{FC},$$

where $c = c(x)$ is a scalar function on $M$ [8, 23].

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $I_y(u) := J_i(y)u^i$, where $J_i := g^{ij}L_{ijk}$. A Finsler metric is called a weakly Landsberg metric if $\mathbf{J} = 0$ [7]. The quantity $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. Then $F$ is said to be isotropic mean Landsberg metric if $\mathbf{J} = c \mathbf{FI}$, for some scalar function $c = c(x)$ on $M$ [8]. It is easy to see that every isotropic Landsberg metric is an isotropic mean Landsberg metric.

Define the stretch curvature $\Sigma_y: T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\Sigma_y(u,v,w,z) := \Sigma_{ijk}(y)u^iv^jw^kz^l$ where

$$\Sigma_{ijk} := 2(L_{ijk|l} - L_{ijl|k}).$$

A Finsler metric $F$ is said to be stretch metric if $\Sigma = 0$. According to Berwald’s theorem, stretch curvature vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram [17]. By definition, every Landsberg metric is a stretch metric.

3. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1. First, we consider a general case, i.e. semi-C-reducible metrics with isotropic Landsberg curvature.

**Proposition 3.1.** Let $(M, F)$ be a semi-C-reducible manifold. Suppose that $F$ has isotropic Landsberg curvature. Then $q$ is constant along any Finslerian geodesic.

**Proof.** $F$ is semi-C-reducible metric

$$C_{ijk} = \frac{q}{C^2}I_iI_jI_k + \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}. \quad (1)$$

Taking horizontal covariant derivation of (1) yields

$$C_{ijk|s} = \frac{q}{C^2}I_iI_jI_k - \frac{q(I_{ij|s}I^m + I_{ij}I_{|s}I^m)}{C^4}I_iI_jI_k$$

$$+ \frac{q}{C^2}\{I_{ij|s}I_kI_k + I_iI_{ij|s}I_k + I_iI_jI_{|s}\}$$

$$+ \frac{p}{n+1}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{p}{n+1}\{h_{ij}I_{|s}I_k + h_{jk}I_{ij|s} + h_{ki}I_{ij|s}\}. \quad (2)$$
Contracting the above equation with $y^s$ implies that

$$L_{ijk} = \frac{q}{C^2} I_{i} I_{j} I_{k} - q \frac{J_{m} I_{m}}{C^{4}} I_{i} I_{j} I_{k} + \frac{q}{C^2} \{J_{i} I_{j} I_{k} + I_{i} J_{j} I_{k} + I_{i} I_{j} J_{k}\}$$

$$+ \frac{p'}{n+1}\{h_{ij} I_{k} + h_{jk} I_{i} + h_{ki} I_{j}\} + \frac{p}{n+1}\{h_{ij} J_{k} + h_{jk} J_{i} + h_{ki} J_{j}\}, \quad (3)$$

where $p' = p_{i} y^{i}$ and $q' = q_{i} y^{i}$. By assumption we have $L = c F C$, where $c = c(x)$ is a scalar function on $M$. Then we have

$$c F C_{ijk} = \frac{q'}{q} + \frac{c F q}{C^2} I_{i} I_{j} I_{k} + \frac{c F p}{n+1} + \frac{p'}{n+1}\{h_{ij} I_{k} + h_{jk} I_{i} + h_{ki} I_{j}\}. \quad (4)$$

If $q = 0$, then the proof is done. Suppose that $q \neq 0$, then by (1) we get

$$\frac{1}{C^2} I_{i} I_{j} I_{k} = \frac{1}{q} C_{ijk} - \frac{p}{q(n+1)}\{h_{ij} I_{k} + h_{jk} I_{i} + h_{ki} I_{j}\}. \quad (5)$$

Plugging (5) into (4) implies that,

$$c F C_{ijk} = \frac{q'}{q} + \frac{c F q}{q} C_{ijk} + \frac{p' q - q' p}{q(n+1)}\{h_{ij} I_{k} + h_{jk} I_{i} + h_{ki} I_{j}\}. \quad (6)$$

Since $p = 1 - q$, then we have,

$$p' = -q'.$$

Therefore, (6) reduces to following:

$$\frac{q'}{q} \left\{ C_{ijk} - \frac{1}{n+1}\{h_{ij} I_{k} + h_{jk} I_{i} + h_{ki} I_{j}\} \right\} = 0. \quad (7)$$

If $q' \neq 0$, then $F$ is a C-reducible metric, which is a contradiction. Thus $q' = 0$, i.e. $q$ is constant along any Finslerian geodesic.

**Proof of Theorem 1.1.** Let $(M, F)$ be a weakly Landsberg semi-C-reducible manifold. Then by (3), we have,

$$L_{ijk} = q' \left\{ \frac{1}{C^2} I_{i} I_{j} I_{k} - \frac{1}{n+1}\{h_{ij} I_{k} + h_{jk} I_{i} + h_{ki} I_{j}\} \right\}. \quad (8)$$

By (8) it results that $F$ is a Landsberg metric if and only if $q' = 0$ or the following holds:

$$\frac{1}{C^2} I_{i} I_{j} I_{k} = \frac{1}{n+1}\{h_{ij} I_{k} + h_{jk} I_{i} + h_{ki} I_{j}\}. \quad (9)$$

Putting (9) in (1) implies that $q = 0$ or $p = 0$, which implies that $F$ is a C-reducible metric or C2-like metric.
4. Proof of Theorem 1.2
In this section, we are going to prove the Theorem 1.2. First, we consider compact semi-C-reducible Finsler manifold with condition $q' + qcF = 0$, where $c$ is a non-zero real constant.

**Lemma 4.1.** Let $(M, F)$ be a compact semi-C-reducible manifold. Suppose that $F$ satisfies $q' + qcF = 0$, where $c$ is a non-zero real constant. Then $F$ is a C-reducible metric.

**Proof.** Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c(t)$ be the geodesic with $c(0) = y$ and $V(t)$ be the parallel vector field along $c$ with $V(0) = v$. Define $q(t)$ and $q'(t)$ as follows,

$$q(t) := q_c(V(t)), \quad (10)$$
$$q'(t) := q'_c(V(t)). \quad (11)$$

By (10), (11) and $q' + qcF = 0$ we get,

$$q(t) = e^{-ct}q(0). \quad (12)$$

By assumption $M$ is a compact manifold, thus $q$ is bounded. By letting $t \to +\infty$ or $t \to -\infty$, we have $q = 0$. This means that $F$ is C-reducible. 

In the rest of this section, we consider the case $q' + qcF \neq 0$.

**Lemma 4.2.** Let $(M, F)$ be a semi-C-reducible manifold. Suppose that $F$ is a P-reducible metric satisfies $q' + qcF \neq 0$. Then $F$ is isotropic mean Landsberg metric if and only if it is an isotropic Landsberg metric.

**Proof.** $F$ is P-reducible metric,

$$L_{ijk} = \frac{1}{1 + n}\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}. \quad (13)$$

Since $J = cFI$, we get,

$$L_{ijk} = \frac{cF}{1 + n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}, \quad (14)$$

or equivalently

$$\frac{1}{1 + n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} = \frac{1}{cF}L_{ijk}. \quad (15)$$

$F$ is semi-C-reducible metric,

$$C_{ijk} = \frac{q}{C^2}I_iI_jI_k + \frac{p}{1 + n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}. \quad (16)$$
Taking horizontal covariant derivation of (16) yields

\[ L_{ijk} = \frac{q'}{C^2} I_i I_j I_k - \frac{q J_m I^m}{C^4} I_i I_j I_k + \frac{q}{C^2} (J_i I_j I_k + I_i J_j I_k + I_i I_j J_k) \]

\[ + \frac{p'}{n+1} \{ h_{ij} I_k + h_{jk} I_i + h_{ki} I_j \} + \frac{p}{n+1} \{ h_{ij} J_k + h_{jk} J_i + h_{ki} J_j \} \]

(17)

where \( p' = p_i y^i \) and \( q' = q_i y^i \). Taking \( J = cFI \) in (17) and using (13)–(15), one can obtain the following:

\[ L_{ijk} = \left( \frac{p'}{cF} + p \right) L_{ijk} + (q' + qcF) \frac{1}{C^2} I_i I_j I_k. \]

(18)

Thus we get,

\[ (q' + qcF) L_{ijk} = (q' + qcF) cF \frac{1}{C^2} I_i I_j I_k. \]

(19)

By assumption \( q' + qcF \neq 0 \), we get,

\[ L_{ijk} = cF \frac{1}{C^2} I_i I_j I_k. \]

(20)

By (16) we have,

\[ \frac{1}{C^2} I_i I_j I_k = \frac{1}{q} C_{ijk} - \frac{p}{q(1+n)} \{ h_{ij} I_k + h_{jk} I_i + h_{ki} I_j \}, \]

(21)

which, considering (14), reduces to following:

\[ \frac{1}{C^2} I_i I_j I_k = \frac{1}{q} C_{ijk} - \frac{p}{qcF} L_{ijk}. \]

(22)

By (20) and (22), we conclude that,

\[ L_{ijk} = cFC_{ijk}, \]

(23)

which mean that \( F \) is an isotropic Landsberg metric.

**Proof of Theorem 1.2.** By Lemmas 4.1 and 4.2, we get the proof.

5. **Proof of Theorem 1.3**

In this section, we are going to prove the Theorem 1.3.

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Proof of Theorem 1.3. By assumption we have,
\[ L_{ijk;k} - L_{ijl;j} = 0. \]  \(24\)
Contracting (24) with \(y^i\) implies that,
\[ L_{ijk} y^l = 0. \]  \(25\)
On the other hand, by (5) and (8) we have,
\[ L_{ijk} = p' q \left\{ \frac{1}{n+1} \left( h_{ij} I_k + h_{jk} I_i + h_{ki} I_j \right) - C_{ijk} \right\} = q' M_{ijk}. \]  \(26\)
We have,
\[ M_{ijk} y^l = L_{ijk} - \frac{1}{n+1} \left\{ h_{ij} J_k + h_{jk} J_i + h_{ki} J_j \right\}. \]  \(27\)
Since \(F\) is weakly Landsbergian, (27) reduces to following:
\[ M_{ijk} y^l = L_{ijk}. \]  \(28\)
Taking a horizontal derivation of (26) and using (28) yields,
\[ L_{ijk} y^l = \left[ \left( \frac{q'}{q} \right)' + \left( \frac{q'}{q} \right)^2 \right] M_{ijk} = \frac{q''}{q} M_{ijk}, \]  \(29\)
where \(q'' = q_{ijkl} y^i y^j y^k\). By (29), we conclude that,
\[ q'' = 0, \quad \text{or} \quad M_{ijk} = 0. \]
In the second case, \(F\) is a C-reducible metric and then we get \(q = 0\), which is a contradiction. Thus \(q'' = 0\), which means that \(q'\) is constant function along any Finslerian geodesics. \(\square\)

References

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