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ON SEMI-C-REDUCIBILITY OF (α, β)-METRICS

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Every non-Riemannian (α, β)-metric on a manifold with dimension $n \geq 3$ is semi-C-reducible. Thus the study on semi-C-reducible metrics will enhance our understanding on the physical meaning of (α, β)-metrics. In this paper, we study weakly Landsberg semi-C-reducible metrics with some non-Riemannian curvature properties. Then we find some conditions on semi-C-reducible manifolds under which the notions of isotropic Landsberg curvature and of isotropic mean Landsberg curvature are equivalent.

Keywords: Semi-C-reducible metric; Landsberg metric; stretch metric.

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1. Introduction

For a Finsler metric $F = F(x, y)$, its geodesics are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric F is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ for any $x \in M$ [1–3]. The Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space [4]. For a Finsler metric F on a manifold M, the Minkwoski norm $F_x := F|_{T_xM}$ on T_xM, for each non-zero tangent vector y at x induces a Riemannian metric $\hat{g}_x := g_{ij}(x, y)dy^i \otimes dy^j$ on $T_xM_0 = T_xM - \{0\}$. It is well-known that if F is a Berwald metric then all $(T_xM_0, \hat{g}_x)$ are isometric as Hilbert spaces [5].

Beside the Berwald curvature, there are several important Finslerian curvatures. Let (M, F) be a Finsler manifold. The second derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third-order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form

and the Cartan torsion, respectively. The rate of change of $\mathbf{C}_y$ along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$ [6]. It is well-known that on a Landsberg manifold (M, F), all $(T_xM_0, \hat{g}_x)$ are isometric as Banach spaces. It is proved that every Berwald metric is a Landsberg metric [7].

The quotient $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along Finslerian geodesics. F is said to be isotropic Landsberg metric if $\mathbf{L} = cF\mathbf{C}$, where $c = c(x)$ is a scalar function on M. There are many isotropic Landsberg Finsler metrics. For example, the (generalized) Funk metric on the unit ball $\mathrm{B}^n \subset \mathbb{R}^n$ is an isotropic Landsberg metric with $c = -1/2$ [8].

Set $\mathbf{I}_y := \sum_{i=1}^n \mathbf{C}_y(e_i, e_i, \cdot)$ and $\mathbf{J}_y := \sum_{i=1}^n \mathbf{L}_y(e_i, e_i, \cdot)$, where $\{e_i\}$ is an orthonormal basis for $(T_xM, \mathbf{g}_y)$. $\mathbf{I}_y$ and $\mathbf{J}_y$ is called the mean Cartan torsion and mean Landsberg curvature, respectively. A Finsler metric F is said to be weakly Landsbergian if $\mathbf{J} = 0$ [9]. The quotient $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. F is said to be isotropic mean Landsberg metric if $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M.

In order to find explicit examples of weakly Landsberg metrics, we consider (α, β)-metrics [10]. Let us narrate a brief history of (α, β)-metrics. This marchen originated in 1941 by a physicist Randers, who first introduced the notion of Randers metrics to consider the unified field theory [11]. A Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$. In the same time, another event was happened by a geometrician Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by Kropina [12]. Consequently, other match of Randers metric called Kropina metric $F = \alpha^2/\beta$ was born.

Regarding the Cartan tensors of these metrics, Matsumoto introduced the notion of C-reducibility and proved that any Randers and Kropina metric are C-reducible [13]. Matsumoto–Hōjō proved that the converse is true [14]. Furthermore, by considering Kropina and Randers metrics, Matsumoto introduced the notion of (α, β)-metrics. An (α, β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β is a 1-form on M [15].

In [16], Matsumoto–Shibata introduced the notion of semi-C-reducibility by considering the form of Cartan torsion of a non-Riemannian (α, β)-metric on a manifold M with dimension $n \geq 3$. A Finsler metric is called semi-C-reducible if its Cartan tensor is given by,

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}J_j\} + \frac{q}{C^2}I_iI_jI_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM, h_{ij} is the angular metric and $C^2 = I^iI_i$. If $q = 0$, then F is just C-reducible Finsler metric and if $p = 0$, then F is called $C2$-like metric. As a generalization of C-reducible metrics, Matsumoto–Shimada introduced the notion of P-reducible metrics [12]. A Finsler

metric is called P-reducible if its Landsberg tensor is given by following,

$$L_{ijk} = \frac{1}{1+n}\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}.$$

It is easy to see that every Landsberg metric is a weakly Landsberg metric, but the converse is not true [9]. Every C-reducible or $C2$-like weakly Landsberg metric is a Landsberg metric. Thus it is natural to investigate this equivalency for semi-C-reducible metrics. Then, we prove the following.

Theorem 1.1. *Let (M, F) be a weakly Landaberg semi-C-reducible manifold. Then F is a Landsberg metric if and only if one of the following holds*:

(1) *F is C-reducible metric*;
(2) *F is $C2$-like metric*;
(3) *$q' := q_{|s}y^s = 0$, i.e. q is constant along any Finslerian geodesics.*

More general, it is well-known that on a C-reducible or $C2$-like metric the notions being of isotropic Landsberg curvature and isotropic mean Landsberg curvature are equivalent [8]. In this paper, we find some conditions on a semi-C-reducible metric under which these notions of curvatures are equivalent. More precisely, we prove the following.

Theorem 1.2. *Let (M, F) be a compact semi-C-reducible manifold. Suppose that F is P-reducible. Then the following are equivalent*:

(1) *F has isotropic Landsberg curvature $\mathbf{L} = cF\mathbf{C}$*;
(2) *F has isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$*;

where c is a non-zero real constant.

As a generalization of Landsberg curvature, Berwald introduced the notion of stretch curvature. He showed that this tensor vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram [17]. For stretch semi-C-reducible metric, we prove the following.

Theorem 1.3. *Let (M, F) be a weakly Landsberg semi-C-reducible manifold. Suppose that F is a stretch metric. Then $q'' := q_{|s|r}y^s y^r(x, y) = 0$, i.e. q' is constant along any Finslerian geodesics.*

There are many connections in Finsler geometry [18–21]. In this paper, we use the Berwald connection and the h- and v-covariant derivatives of a Finsler tensor field are denoted by "$|$" and "$,$", respectively.

Throughout this paper, we use *Einstein summation convention* for expressions with indices.[a]

[a]That is wherever an index is appeared twice as a subscript as well as a superscript, then that term is assumed to be summed over all values of that index.

2. Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M.

A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) F is C^∞ on $TM_0 := TM\backslash\{0\}$;
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM;
(iii) for each $y \in T_xM_0$, the following form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y + su + tv)]|_{s,t=0}, \quad u, v \in T_xM.$$

For a Finsler metric $F = F(x, y)$ on a manifold M, the spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a vector field on TM, where $G^i = G^i(x, y)$ are defined by,

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\},$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ and $(g^{ij}) = (g_{ij})^{-1}$. By definition, F is called a *Berwald metric* if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ for any $x \in M$.

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}[\mathbf{g}_{y+tw}(u, v)]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well-known that $\mathbf{C} = 0$ if and only if F is Riemannian. For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$, $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Deicke's Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ [5, 22].

Let (M, F) be a Finsler manifold. For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^i v^j w^k$ where,

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\},$$

$h_{ij} := FF_{y^i y^j}$ is the angular metric.

Lemma 2.1 ([14]). *A Finsler metric F on a manifold M of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall\, y \in TM_0$.*

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by:

$$C_{ijk} = \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{C^2}I_iI_jI_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM and $C^2 = I^iI_i$. Contracting the last relation by g^{jk} shows that p and q satisfy $p + q = 1$. In [16], Matsumoto and Shibata proved that every (α, β)-metric is semi-C-reducible.

Theorem 2.2 ([12, 16]). *Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.*

The horizontal covariant derivatives of the Cartan torsion $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$, where,

$$L_{ijk} := C_{ijk|s} y^s.$$

The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$. The quantity $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. A Finsler metric F is said to be isotropic Landsberg metric if,

$$\mathbf{L} = cF\mathbf{C},$$

where $c = c(x)$ is a scalar function on M [8, 23].

The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature $\mathbf{J}_y(u) := J_i(y)u^i$, where $J_i := g^{jk}L_{ijk}$. A Finsler metric is called a weakly Landsberg metric if $\mathbf{J} = 0$ [7]. The quantity $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. Then F is said to be isotropic mean Landsberg metric if $\mathbf{J} = cF\mathbf{I}$, for some scalar function $c = c(x)$ on M [8]. It is easy to see that every isotropic Landsberg metric is an isotropic mean Landsberg metric.

Define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}(y)u^i v^j w^k z^l$ where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}).$$

A Finsler metric F is said to be stretch metric if $\mathbf{\Sigma} = 0$. According to Berwald's theorem, stretch curvature vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram [17]. By definition, every Landsberg metric is a stretch metric.

3. Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1. First, we consider a general case, i.e. semi-C-reducible metrics with isotropic Landsberg curvature.

Proposition 3.1. *Let (M, F) be a semi-C-reducible manifold. Suppose that F has isotropic Landsberg curvature. Then q is constant along any Finslerian geodesic.*

Proof. F is semi-C-reducible metric

$$C_{ijk} = \frac{q}{C^2} I_i I_j I_k + \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}. \tag{1}$$

Taking horizontal covariant derivation of (1) yields

$$\begin{aligned} C_{ijk|s} = {} & \frac{q_{|s}}{C^2} I_i I_j I_k - \frac{q(I_{m|s}I^m + I^m_{|s} I_m)}{C^4} I_i I_j I_k \\ & + \frac{q}{C^2}\{I_{i|s}I_jI_k + I_iI_{j|s}I_k + I_iI_jI_{k|s}\} \\ & + \frac{p_{|s}}{n+1}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{p}{n+1}\{h_{ij}I_{k|s} + h_{jk}I_{i|s} + h_{ki}I_{j|s}\}. \end{aligned} \tag{2}$$

Contracting the above equation with y^s implies that

$$L_{ijk} = \frac{q'}{C^2} I_i I_j I_k - q\frac{J_m I^m + J^m I_m}{C^4} I_i I_j I_k + \frac{q}{C^2}\{J_i I_j I_k + I_i J_j I_k + I_i I_j J_k\}$$
$$+ \frac{p'}{n+1}\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\} + \frac{p}{n+1}\{h_{ij} J_k + h_{jk} J_i + h_{ki} J_j\}, \quad (3)$$

where $p' = p_{|s} y^s$ and $q' = q_{|s} y^s$. By assumption we have $\mathbf{L} = cF\mathbf{C}$, where $c = c(x)$ is a scalar function on M. Then we have

$$cFC_{ijk} = \frac{q' + cFq}{C^2} I_i I_j I_k + \frac{cFp + p'}{n+1}\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\}. \quad (4)$$

If $q = 0$, then the proof is done. Suppose that $q \neq 0$, then by (1) we get

$$\frac{1}{C^2} I_i I_j I_k = \frac{1}{q} C_{ijk} - \frac{p}{q(n+1)}\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\}. \quad (5)$$

Plugging (5) into (4) implies that,

$$cFC_{ijk} = \frac{q' + cFq}{q} C_{ijk} + \frac{p'q - q'p}{q(n+1)}\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\}. \quad (6)$$

Since $p = 1 - q$, then we have,

$$p' = -q'.$$

Therefore, (6) reduces to following:

$$\frac{q'}{q}\left\{C_{ijk} - \frac{1}{n+1}\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\}\right\} = 0. \quad (7)$$

If $q' \neq 0$, then F is a C-reducible metric, which is a contradiction. Thus $q' = 0$, i.e. q is constant along any Finslerian geodesic. □

Proof of Theorem 1.1. Let (M, F) be a weakly Landsberg semi-C-reducible manifold. Then by (3), we have,

$$L_{ijk} = q'\left\{\frac{1}{C^2} I_i I_j I_k - \frac{1}{n+1}(h_{ij} I_k + h_{jk} I_i + h_{ki} I_j)\right\}. \quad (8)$$

By (8) it results that F is a Landsberg metric if and only if $q' = 0$ or the following holds:

$$\frac{1}{C^2} I_i I_j I_k = \frac{1}{n+1}\{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\}. \quad (9)$$

Putting (9) in (1) implies that $q = 0$ or $p = 0$, which implies that F is a C-reducible metric or $C2$-like metric. □

4. Proof of Theorem 1.2

In this section, we are going to prove the Theorem 1.2. First, we consider compact semi-C-reducible Finsler manifold with condition $q' + qcF = 0$, where c is a non-zero real constant.

Lemma 4.1. *Let (M, F) be a compact semi-C-reducible manifold. Suppose that F satisfies $q' + qcF = 0$, where c is a non-zero real constant. Then F is a C-reducible metric.*

Proof. Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c(t)$ be the geodesic with $\dot{c}(0) = y$ and $V(t)$ be the parallel vector field along c with $V(0) = v$. Define $q(t)$ and $q'(t)$ as follows,

$$q(t) := q_{\dot{c}}(V(t)), \tag{10}$$

$$q'(t) := q'_{\dot{c}}(V(t)). \tag{11}$$

By (10), (11) and $q' + qcF = 0$ we get,

$$q(t) = e^{-ct} q(0). \tag{12}$$

By assumption M is a compact manifold, thus q is bounded. By letting $t \to +\infty$ or $t \to -\infty$, we have $q = 0$. This means that F is C-reducible. □

In the rest of this section, we consider the case $q' + qcF \neq 0$.

Lemma 4.2. *Let (M, F) be a semi-C-reducible manifold. Suppose that F is a P-reducible metric satisfies $q' + qcF \neq 0$. Then F is isotropic mean Landsberg metric if and only if it is an isotropic Landsberg metric.*

Proof. F is P-reducible metric,

$$L_{ijk} = \frac{1}{1+n}\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}. \tag{13}$$

Since $J = cFI$, we get,

$$L_{ijk} = \frac{cF}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}, \tag{14}$$

or equivalently

$$\frac{1}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} = \frac{1}{cF}L_{ijk}. \tag{15}$$

F is semi-C-reducible metric,

$$C_{ijk} = \frac{q}{C^2}I_iI_jI_k + \frac{p}{1+n}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\}. \tag{16}$$

Taking horizontal covariant derivation of (16) yields

$$L_{ijk} = \frac{q'}{C^2} I_i I_j I_k - q \frac{J_m I^m + J^m I_m}{C^4} I_i I_j I_k + \frac{q}{C^2} \{J_i I_j I_k + I_i J_j I_k + I_i I_j J_k\}$$
$$+ \frac{p'}{n+1} \{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\} + \frac{p}{n+1} \{h_{ij} J_k + h_{jk} J_i + h_{ki} J_j\}, \quad (17)$$

where $p' = p_{|s} y^s$ and $q' = q_{|s} y^s$. Taking $J = cFI$ in (17) and using (13)–(15), one can obtains the following:

$$L_{ijk} = \left(\frac{p'}{cF} + p \right) L_{ijk} + (q' + qcF) \frac{1}{C^2} I_i I_j I_k. \quad (18)$$

Thus we get,

$$(q' + qcF) L_{ijk} = (q' + qcF) \frac{cF}{C^2} I_i I_j I_k. \quad (19)$$

By assumption $q' + qcF \neq 0$, we get,

$$L_{ijk} = \frac{cF}{C^2} I_i I_j I_k. \quad (20)$$

By (16) we have,

$$\frac{1}{C^2} I_i I_j I_k = \frac{1}{q} C_{ijk} - \frac{p}{q(1+n)} \{h_{ij} I_k + h_{jk} I_i + h_{ki} I_j\}, \quad (21)$$

which, considering (14), reduces to following:

$$\frac{1}{C^2} I_i I_j I_k = \frac{1}{q} C_{ijk} - \frac{p}{qcF} L_{ijk}. \quad (22)$$

By (20) and (22), we conclude that,

$$L_{ijk} = cFC_{ijk}, \quad (23)$$

which mean that F is an isotropic Landsberg metric. □

Proof of Theorem 1.2. By Lemmas 4.1 and 4.2, we get the proof. □

5. Proof of Theorem 1.3

In this section, we are going to prove the Theorem 1.3.

Proof of Theorem 1.3. By assumption we have,

$$L_{ijk|l} - L_{ijl|k} = 0. \tag{24}$$

Contracting (24) with y^l implies that,

$$L_{ijk|l} y^l = 0. \tag{25}$$

On the other hand, by (5) and (8) we have,

$$L_{ijk} = \frac{p'}{q}\left\{\frac{1}{n+1}\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} - C_{ijk}\right\} = \frac{q'}{q}M_{ijk}. \tag{26}$$

We have,

$$M_{ijk|l}y^l = L_{ijk} - \frac{1}{n+1}\{h_{ij}J_k + h_{jk}J_i + h_{ki}J_j\}. \tag{27}$$

Since F is weakly Landsbergian, (27) reduces to following:

$$M_{ijk|l}y^l = L_{ijk}. \tag{28}$$

Taking a horizontal derivation of (26) and using (28) yields,

$$L_{ijk|l}y^l = \left[\left(\frac{q'}{q}\right)' + \left(\frac{q'}{q}\right)^2\right] M_{ijk} = \frac{q''}{q}M_{ijk}, \tag{29}$$

where $q'' = q_{|s|l}y^l y^s$. By (29), we conclude that,

$$q'' = 0, \quad \text{or} \quad M_{ijk} = 0.$$

In the second case, F is a C-reducible metric and then we get $q = 0$, which is a contradiction. Thus $q'' = 0$, which means that q' is constant function along any Finslerian geodesics. □

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