The lattice structure of Signed chip firing games and related models

A. Dolati *
Shahed University
Department of mathematics
dolati@shahed.ac.ir

S. Taromi
Shahed University
Department of Mathematics
taroomi@shahed.ac.ir

B. Bakhshayesh
Shahed University
Department of Mathematics
Bakhshayesh@shahed.ac.ir

Abstract: In this paper the lattice structure of Signed Chip Firing Games are studied and the class of lattices induced by Signed Chip Firing Games with the class of ULD lattices and the class of lattices induced by Mutating Chip Firing Games and Ablian sandpile model are compared.

Keywords: lattice, Signed Chip Firing Game ($SCFG$), Ablian sandpile model (ASM), Mutating Chip Firing Game ($MCFG$), ULD lattices.

1 Introduction

The Signed Chip Firing Game ($SCFG$) is a discrete dynamical system that is introduced by R. Cori and T.T.T. Huong [6]. It is defined over an undirected graph $G = (V, E)$, which is called the support graph. An Integer-valued weight is considered for each vertex $v \in V$, which is the number of the chips stored at v and the sum of weights of all vertices is zero. The game proceeds with an evolution rule, called the firing rule. It may be stopped after a limited time or may be continued forever.

The chip ring game (CFG), Signed Chip Firing Game (SFG), the Abelian sandpile model (ASM), and the mutating chip ring game ($MCFG$) are discrete dynamical models which are used in physics [1, 8] and computer science [3, 4, 11]. Lots of researches have been accomplished on the models of discrete dynamical m (such as CFG, ASM etc.) to name a few: the study of the set of all configurations reached from an initial configuration ([12, 14]) and the configurations that reach the initial configuration after passing some stages of the game which will be called hereafter the recurrent configurations ([2, 5, 8]). Also, the study of whether a given game will be terminated or continued forever is of high interest among researchers of this field([3, 4, 9]). In this paper we want to study these subjects more broadly. The set of all configurations reached from an initial configuration, called the configuration space, of the $SCFG$ is studied. So far, the configuration space of the mentioned models have been characterized and some conclusions have been made. For instance the class of lattices induced by CFG is placed between the distributive and the ULD lattices ([14]). It has been proved that the class of lattices induced by ASM includes the distributive lattices and the class of lattices induced by CFG includes the class of lattices induced by ASM. Also, it has been shown that the class of lattices induced by CFG and $MCFG$ are the same [13].For more detailed information see [10].
In this paper the class of lattices induced by $SCFG$, ASM and $MCFG$ are compared. It will be proven that the $SCFG$ induces some new lattices which are not induced by the other models. Hence this game is important in terms of inducing new lattices.

2 Definitions

In this section, a definition of posets and lattices will be given and the models will be defined.

*Corresponding Author, Tehran, PO Box: 18151-159, Iran.

2.1 Posets and Lattices

A partially ordered set (poset) is a set with a relation that has three properties: transitive, reflexive and antisymmetric. For each two elements x and y of a poset if $x < y, x \leq z < y$ implies $z = x$ then it is said that x is covered by y or x is a lower cover of y (or y is an upper cover of x) and we write $x \prec y$. Also the interval $[x, y]$ is the set containing all elements of the poset between x and y, including both x and y. The Hasse diagram is used to represent a poset P [13]. Two posets P and P' are called isomorphism if there exists a bijective function $\varphi : P \to P'$ such that for all the inequality $x \leq y$ implies $\varphi(x) \leq \varphi(y)$.

The least upper bound of two elements x and y of a poset is called the join of x and y and is denoted by $y \vee x$. The greatest lower bound of any two elements x and y of a poset is called the meet of x and y and is denoted by $y \wedge x$. A poset $L$is called a lattice if any two elements of the poset have a least upper bound and a greatest lower bound. As all lattices studied here are finite, each lattice has a unique maximal as well as minimal element.
A lattice is called a hypercube of dimension n if it is isomorphic to the power set of a set of n elements, ordered by inclusion. A lattice is called lower locally distributive (LLD) if the interval between any element and the meet of all its lower covers is a hypercube. Upper locally distributive (ULD) lattices are defined dually [15]. A lattice is ranked if all the paths in the covering relation from the minimal to the maximal element have the same length. The reader is referred to the references provided at the end of paper for more detailed study of posets and lattices [7].

2.2 Definitions of the models

Each of the following models are defined over a multi graph $G = (V, E)$, called the support graph of the model. A configuration is a mapping that assigns a weight to each vertex $v \in V$ that is the number of chips stored at v. This weight can be positive, negative or even zero. If σ is a configuration of the game then the configuration of each vertex v is the number of chips stored at v, denoted by $\sigma(v)$. The games start with an initial configuration and continue by an evolution rule, called the firing rule.

The Chip Firing Game (CFG) [13] is played over a directed graph $G = (V, E)$ and the configuration of each vertex is positive. The firing rule is as follows: A vertex $v \in V$ whose number of chips is greater than or equal to its outgoing degree, can be fired and it sends a chip along each of its arcs. If there exists no such vertex then the game will stop and the last configuration is called the fixed point of the game. In other words a configuration σ is called a fixed point if the following statement holds:

$$\forall v \in V : \sigma(v) < deg_G(v).$$

which $deg_G(v)$ is the outgoing degree of vertex v in G.

The Abelian Sandpile Model (ASM) [13] is played over an undirected graph $G = (V, E)$ with a special vertex, called sink, which never fires and is denoted by $\perp$ and the configuration of each vertex is positive. Note that a directed graph will be achieved if two arcs (i, j) and (j, i) are replaced by each edge i, j of the given undirected graph. The firing rule and the fixed point of this game are similar to the preceding game, CFG.

The Mutating Chip Firing Game ($MCFG$) [13] is played over a directed graph $G = (V, E)$ and the configuration of each vertex is positive. An infinite sequence $M_v = (S_v^1, S_v^2, ...)$ of multisets of nodes in V is fixed for each node $v \in V$, which is called the mutation sequence of v. If the number of chips stored at a node $v \in V$ is greater than or equal to its current outgoing degree then $v$fires according to the firing rule of the CFG. After firing a vertex the mutation of node v occurs in a way that the outgoing arcs of v are removed and a new arc (v, w) is added for each $ww \in S_v^1$. Then S_v^1 is removed from M_v so that $M_v = (S_v^2, S_v^3, ...)$. The support graph of the initial state is called the initial support graph of a $MCFG$.

The Signed Chip Firing Game ($SCFG$) [6] is played over an undirected graph $G = (V, E)$ where the configuration of each vertex is an integer such that the sum of the configuration of all vertices equals zero. For any positive vertex (i.e. a vertex whose number of chips is positive) the firing rule is similar to the CFG. If the configuration of a negative vertex v (i.e. a vertex whose number of chips is negative) is less than or equal to $-deg_v$ (where $-deg_v$ is the degree of a node v), then v can be fired and receives a chip from each of its neighbors. The game stops when no vertex is able to fire.
If a vertex v of an initial configuration σ can be fired fired and a configuration σ' is obtained, then σ is called the predecessor of σ' and we write $\sigma \longrightarrow^v \sigma'$. This relation is called the predecessor relation. Such a game may last forever or may converge to a unique fixed point. The sequence of all vertices that have been fired through the game in order to reach the final configuration from the initial configuration is called an execution of the game. The set of all configurations reachable from the initial configuration C, ordered by the

predecessor relation, is called the configuration space of C and is denoted by $L(C)$. Two convergent games C, C' are equivalent if $L(C)$ is isomorphic to $L(C)'$. From now on we will denote the configuration space of CFG, ASM and $MCFG$ by $L(CFG)$, $L(ASM)$ and $L(MCFG)$, respectively.

3 Main results

In this section we study a special class of $L(SCFG)$ and show the $SCFG$s related to this class are equivalent to the ASM, $MCFG$ and ULD. First we state some definitions: A $SCFG$ is simple if each of its vertices can be fired at most once during an execution.

DEFINITION 1: A vertex that stores a positive (negative) number of chips is called a positive (negative) vertex.

DEFINITION 2: The firing of a positive (negative) node is called a positive (negative) firing.

Now we will present two properties for the SCFG in order to reach our result.

DEFINITION 3: For a given $SCFG$ if the number of chips at all vertices of the initial configuration of a $SCFG$ is nonzero, then the $SCFG$ is said to have p_1 property.

DEFINITION 4: For a given $SCFG$ if no negative node prevents a positive node from being fired as well as no positive node prevents a negative node from being fired, then the $SCFG$ is said to have p_2 property.

Now we can state the main results:

THEOREM 1: The configuration space of simple $SCFG$ with p_2 property is a lattice.

THEOREM 2: Any simple $SCFG$ with p_1 and p_2 properties is equivalent to a convergent ASM.

THEOREM 3: The configuration space of a simple $SCFG$with properties p_1 and p_2 is an ULD lattice.

THEOREM 4: The configuration space of a simple $SCFG$ with properties p_1 and p_2 is equivalent to a $MCFG$.

4 Discussion and Future Works

In this paper, we have introduced a class of SCFG with p_1 and p_2 properties, which are equivalent to other known models. We intend to study those class of SCFG, not necessarily have p_1 and p_2 properties, that are equivalent to other known models. Therefore we would be able to find the lattices induced by SCFG which does not overlap the lattices induced by other known models and hence introduce a new class of lattices.

Refrences

[1] P. Bak, C. Tang, and K. Wiesenfeld, *Self-organized criticality: An explanation of the 1/f noise*, Phys. Rev. Lett. (1987).

[2] N. Biggs, *Chip firing and the critical group of a graph*, Journal of Algebraic Combinatorics **9** (1999), 2545.

[3] A. Bjorner and L. Lovasz, *Chip-firing games on directed graphs*, Journal of Algebraic Combinatorics **1** (1992), 304-328.

[4] A. Bjorner, L. Lovasz, and W. Shor, *On computing fixed points for generalized sandpiles*, European Journal of Combinatorics **12** (1991), 283-291.

[5] R. Cori and D. Rossin, *On the sandpile group of a graph*, European Journal of Combinatorics **21** (2000), 447459.

[6] R. Cori and T.T.T.Huong, *Signed chip firing games on some particular casesand its applications*, LIX, Lecole Polytechnique, France Institute of Mathematics, Hanoi, Vietnam, October 26 (2009).

[7] B. Davey and H. Priestley, *Introduction to Lattices and Orders*, 1990.

[8] D. Dhar, P. Ruelle, S. Sen, and D. Verma, *Algebraic aspects of sandpile models*, Vol. 28, 1995.

[9] K. Eriksson, *Chip firing games on mutating graphs*, SIAM Journal of Discrete Mathematics **9** (1996), 118128.

[10] E. Goles, M. Latapy, C. Magnien, and M. Movan, *Sandpile Models and Lattices: A Comprehensive Survey*, Theoretical Computer Science **322** (2004), 383 407.

[11] E. Goles, M. Morvan, and H. Phan, *Lattice structure and convergence of a game of cards*, Annals of Combinatorics (1998).

[12] M. Latapy and H. D. Phan, *The lattice structure of chip firing games and related models*, Physica D 155 (2001), 69-82.

[13] C. Magnien, *Classes of lattices induced by chip firing (and sandpile) dynamics*, European Journal of Combinatorics (2003), 665 - 683.

[14] H.D. Phan, L. Vuillon, and C. Magnien, *Characterization of lattices induced by (extended) chip firing games*, Discrete Mathematics and Theoretical Computer Science, Proc. 1st Internat. Conf. Discrete Models: Combinatorics, Computation, and Geometry (DM-CCG01), MIMD (July 2001), 229-244.

[15] B. Monjardet, K.P. Bogart, R. Freese, and J. Kung, *The consequences of Dilworths work on lattices with unique irreductible decompositions*, The Dilworth Theorems Selected Papers of Robert P. Dilworth, Birkhauser, Boston (1990), 192-201.