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A New Adaptive Variable Sample Size Approach in EWMA Control Chart

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Adaptive control charts have been developed for improving the capability of control charts in detecting small shifts. In this article, we propose a new exponential weighted moving average control chart with variable sample size, in which the sample size is determined as an integer linear function by EWMA statistic value. The performance of the proposed VSS EWMA control chart is compared with FSS EWMA as well as traditional VSS EWMA control charts. The results show the better performance of the proposed VSS strategy respect to the traditional one and fixed sample size.

Keywords Adaptive control chart; Average run length; EWMA control chart; Statistical process control; Variable sample size

Mathematical Subject Classification

1. Introduction

Control chart is a useful tool in statistical process control. Dr. Shewhart presented the first control chart as $\bar{X}$ control chart in 1924. In this chart, $\mu \pm 3\sigma$ is used as control limits for monitoring process mean. However, the capability of this chart is not well in small shifts, adaptive control charts have been developed to improve the capability of control charts. A control chart is considered adaptive when at least one of its design parameters varies as a function of the process data (Costa and De Magalhães, 2007). Most research about adaptive control charts concentrated on variable sampling intervals (VSI) and variable sample size (VSS) schemes to improve the sensitivity of control charts for detecting small shifts in parameters of process. In this article, we specifically focus on VSS method as an efficient technique in adaptive control charts. As early works in this regard, Prabhu et al. (1993) and Costa (1994) used VSS technique for $\bar{X}$ charts. De Magalhães et al. (2009) presented a statistical design of a hierarchy of two states adaptive parameters $\bar{X}$ charts. In their work, seven techniques in adaptive control charts area (especially VSS) were compared. Zhou and Lian (2011) proposed a VSS-NP chart with adjusting sampling inspection called ASI-NP chart. In their proposed chart, smaller and larger sample sizes are chosen for the next sample when the current $\bar{X}$ value is located in the safe and warning zones, respectively. There are some optimization methods to determine the most efficient sample sizes in VSS

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control charts. See, for example, Wu and Luo (2002) who have developed an algorithm to determine sample sizes for NP control charts.

The exponentially weighted moving average (EWMA) control charts have been introduced by Roberts (1959), and have been widely used in statistical process control (SPC) for monitoring small shifts in process parameters.

To improve the performance of EWMA control chart, adaptive EWMA control charts are also extended. Ji et al. (2006) proposed an EWMA control chart with variable sampling interval for monitoring process mean under non-normality. Chou and Chen (2006) proposed an economic design of VSI EWMA. Traditional variable sample size for the EWMA control chart can be easily developed. In this article, we propose a new variable sample size approach for improving the performance of EWMA control chart and compare it with traditional VSS EWMA control chart as well as fixed sample size (FSS) EWMA control chart. The rest of the article is outlined as follows:

In Section 2, EWMA control chart with fixed sample size (FSS EWMA) and EWMA control chart with variable sample size (VSS EWMA) is introduced. Next, in Section 3, a new EWMA control chart with variable sample size as an integer linear function is developed. Finally, in Section 4, the performances of these charts are compared. The computational results show better performance of the proposed VSS strategy respect to the traditional one and fixed sample size.

2. EWMA Control Chart

2.1. *FSS EWMA*

In this section, we explain about the EWMA control chart with fixed sample size (FSS EWMA) to compare it with our proposed EWMA control chart in Section 4. EWMA control charts were introduced first by Roberts (1959). Assume that a quality characteristic follows a normal distribution with mean μ and standard deviation σ. Hence, the EWMA statistic for ith sample is as follows:

$$Z_i = \lambda \bar{X}_i + (1-\lambda) Z_{i-1} \quad 0 < \lambda \leq 1, \tag{1}$$

where λ is smoothing parameter or exponential weight constant. For $i = 0$, $Z_i = Z_0$ is starting value and is often taken equal to the process target value.

Also, the upper and lower control limits of EWMA control chart is defined as follows:

$$\begin{cases} \text{UCL} = \mu + L\sigma\sqrt{\dfrac{\lambda}{n\,(2-\lambda)}(1-(1-\lambda)^{2i})} \\ \text{CL} = \mu \\ \text{LCL} = \mu - L\sigma\sqrt{\dfrac{\lambda}{n\,(2-\lambda)}(1-(1-\lambda)^{2i})} \end{cases}, \tag{2}$$

where n is the number of samples in each subgroup and L is the coefficient of control limits which is chosen to obtain a desirable in control average run length (ARL_0). The EWMA control chart signals as soon as $Z_i > \text{UCL}$ or $Z_i < \text{LCL}$.

2.2. *VSS EWMA*

To design EWMA control chart with variable sample size (VSS EWMA), we use the same statistic and control limits mentioned in Section 2.1 and also two warning limits. Upper warning limit (UWL) and lower warning limit (LWL) can be defined as follows:

$$\begin{cases} \text{UWL} = \mu + L_1\sigma\sqrt{\dfrac{\lambda}{n_i\,(2-\lambda)}(1-(1-\lambda)^{2i})} \\ \text{LWL} = \mu - L_1\sigma\sqrt{\dfrac{\lambda}{n_i\,(2-\lambda)}(1-(1-\lambda)^{2i})} \end{cases}, \tag{3}$$

where, L_1 is the warning coefficient of VSS EWMA.

The area between LWL and UWL is called as safety zone. The area between LCL and LWL, as well as the area between UWL and UCL are called warning zones.

Let n_1, n_2 be the smaller number of samples and larger number of samples, respectively. Hence, the smaller sample size (n_1) is taken in ith sample if Z_{i-1} falls in safety zone and larger sample size (n_2) is taken if Z_{i-1} falls in warning zones.

$$\begin{cases} \text{LWL} < Z_{i-1} < \text{UWL} \rightarrow n_i = n_1 \\ \text{UWL} < Z_{i-1} < \text{UCL} \rightarrow n_i = n_2\,. \\ \text{LCL} < Z_{i-1} < \text{LWL} \rightarrow n_i = n_1 \end{cases} \tag{4}$$

The EWMA statistic with variable sample size is the same as the EWMA statistic in Equation (1) and $Z_0 = \mu$. The control limits of VSS EWMA are modified as below:

$$\begin{cases} \text{UCL}_i = \mu + L_2\sigma\sqrt{\dfrac{\lambda}{n_i\,(2-\lambda)}(1-(1-\lambda)^{2i})} \\ \text{CL} = \mu \\ \text{LCL}_i = \mu - L_2\sigma\sqrt{\dfrac{\lambda}{n_i\,(2-\lambda)}(1-(1-\lambda)^{2i})} \end{cases}. \tag{5}$$

For $i = 0$, $Z_0 = \mu_0$ and $\text{UCL}_0, \text{LCL}_0$ are starting value of control limits and can be determined as follows:

$$\begin{cases} UCL_0 = \mu + L_2\sigma\sqrt{\dfrac{\lambda}{n_1\,(2-\lambda)}} \\ \text{CL} = \mu \\ \text{LCL}_0 = \mu - L_2\sigma\sqrt{\dfrac{\lambda}{n_1\,(2-\lambda)}} \end{cases}. \tag{6}$$

Since $Z_0 = \mu$ is used as a starting point in the traditional VSS control charts and the center-line (μ) is in the safe zone, smaller sample size is used as first number of samples in this method. This is why, n_1 is used as starting sample size in Equation (6). In addition, in some articles in the literature of VSS control charts, such as Flaig (1991), the smaller sample size is chosen as the initial sample size. The sample sizes for next samples (n_i $i = 2, 3, \ldots$) is calculated by Equation (4).

3. Our Proposed Adaptive EWMA Control Chart

In this section, we introduce a new variable sample size approach for improving capability of EWMA control chart. Of course this method can be used for other control charts, but in this article we have used this scheme for EWMA control chart only.

Assume that a process follows normal distributions, in which we can say the process is in-control (IC) if $X_i \sim N\left(0, \sigma^2\right)$ and the process is out-of-control (OC) if $X_i \sim N\left(\delta, \sigma^2\right)$.

In this article, we focus on the mean shifts. Hence, we suppose $\delta \neq 0$.

As mentioned before, the sample size for our VSS EWMA control chart should be determined as an integer linear function by EWMA statistic value (VSSILF EWMA). So, we need to define a linear function of EWMA statistic for sample size. Note that smaller sample size is given for the next sample if the current Z is closer to centerline and larger in otherwise. Then, the minimum sample size is given when the current Z value is equal to CL and the maximum sample size is given when the current Z value is equal to UCL or LCL. Let n_1 and n_2 be the minimum and maximum sample size, respectively. The relation between sample size (n) and statistic value (Z) is shown in Figure 1. The sample size function can be defined as follows:

$$n_i = f\left(|Z_{i-1}|\right). \tag{7}$$

The relation between sample size and Z is linear. Also, the value of sample size must be integer. So, an integer linear function can be defined for sample size.

$$n_i = \left[\left(\frac{n_2 - n_1}{\mathrm{UCL}_{i-1} - \mu}\right)|Z_{i-1}| + \left(n_1 - \frac{n_2 - n_1}{\mathrm{UCL}_{i-1} - \mu} \times \mu\right)\right]. \tag{8}$$

Note that the sample sizes (n_i) defined in Equation (8) is positive value between n_1 and n_2. The proof is given in the Appendix.

As mentioned before, n_1 and n_2 are the minimum and maximum sample sizes we want to use, respectively. $\mathrm{UCL}_{i\text{-}1}$ is the upper control limit of our proposed chart, μ is mean of the process (in-control), and $|Z_{i-1}|$ is statistic value for (i-1)th sample.

Since the sample size is variable, the control limits will be variable, too. The upper and lower control limits of this chart can be presented as follows after substituting Equation (8):

$$Z_i = \lambda\left(\frac{\sum_{i=1}^{n_i} x_i}{n_i}\right) + (1-\lambda)\, Z_{i-1} \quad 0 < \lambda \leq 1. \tag{9}$$

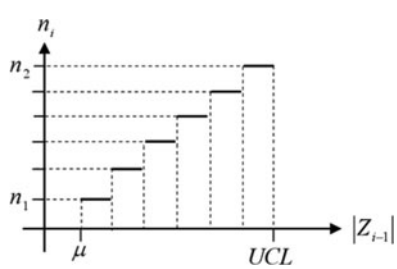

Figure 1. Sample size function plot as statistic value.

$$\begin{cases} \mathrm{UCL}_i = \mu + L_3\sigma\sqrt{\dfrac{\lambda}{\left[\left(\frac{n_2-n_1}{\mathrm{UCL}_{i-1}-\mu}\right)|Z_{i-1}| + \left(n_1 - \frac{n_2-n_1}{\mathrm{UCL}_{i-1}-\mu}\times\mu\right)\right](2-\lambda)}(1-(1-\lambda)^{2i})} \\ \mathrm{CL} = \mu \\ \mathrm{LCL}_i = \mu - L_3\sigma\sqrt{\dfrac{\lambda}{\left[\left(\frac{n_2-n_1}{\mathrm{UCL}_{i-1}-\mu}\right)|Z_{i-1}| + \left(n_1 - \frac{n_2-n_1}{\mathrm{UCL}_{i-1}-\mu}\times\mu\right)\right](2-\lambda)}(1-(1-\lambda)^{2i})} \end{cases}. \tag{10}$$

For $i = 0$, $Z_0 = \mu_0$, $n_0 = n_1$, and UCL_0, LCL_0 are starting value of control limits and can be determined as follows:

$$\begin{cases} \mathrm{UCL}_0 = \mu + L_3\sigma\sqrt{\dfrac{\lambda}{n_1(2-\lambda)}} \\ \mathrm{CL} = \mu \\ \mathrm{LCL}_0 = \mu - L_3\sigma\sqrt{\dfrac{\lambda}{n_1(2-\lambda)}} \end{cases}. \tag{11}$$

For each sample, the sample size (n_i), upper and lower control limit (UCL_i & LCL_i), and statistic value (Z_i) must be calculated. Variable sample size EWMA as integer linear function (VSSILF EWMA) signals as soon as $Z_i > \mathrm{UCL}$ or $Z_i < \mathrm{LCL}$ Note that our

Table 1
The average run length of different EWMA control chart types

	FSS EWMA		VSS EWMA		VSSILF EWMA	
	$n = 5$		$n_1 = 5\, n_2 = 10$		$n_1 = 5\, n_2 = 10$	
	$\lambda = 0.2$		$\lambda = 0.2$		$\lambda = 0.2$	
	$L = 2.8$		$L_2 = 3.33$		$L_3 = 3.15$	
	$\sigma = 2$		$\sigma = 2$		$\sigma = 2$	
δ	ARL	ASS	ARL	ASS	ARL	ASS
−3	2.30	5	2.03	6.37	2.0298	6.41
−2.5	2.70	5	2.39	6.34	2.2712	6.40
−2	3.29	5	3.01	6.26	2.7452	6.47
−1.5	4.30	5	4.30	6.12	3.6233	6.58
−1	7.63	5	6.93	5.97	5.9632	6.73
−0.5	25.42	5	21.54	5.68	19.0077	6.58
−0.25	84.22	5	74.58	5.33	68.5428	6.73
0	**280**	**5**	**280**	**5.14**	**280**	**5.89**
0.25	85.68	5	73.42	5.33	67.9153	6.71
0.5	25.74	5	21.39	5.68	18.9802	6.64
1	7.64	5	6.90	5.97	5.9821	6.73
1.5	4.28	5	4.28	6.12	3.6325	6.58
2	3.29	5	3.02	6.26	2.7265	6.47
2.5	2.38	5	2.71	6.34	2.2726	6.40
3	2.32	5	2.32	6.37	2.0241	6.41

Table 2
The ARL of EWMA control charts with roughly equal in-control ASS with $\lambda = 0.2$

	FSS EWMA		VSS EWMA		VSSILF EWMA	
	$n = 6$		$n_1 = 6\, n_2 = 10$		$n_1 = 5\, n_2 = 11$	
	$\lambda = 0.2$		$\lambda = 0.2$		$\lambda = 0.2$	
	$L = 2.76$		$L_2 = 3.43$		$L_3 = 3.05$	
	$\sigma = 2$		$\sigma = 2$		$\sigma = 2$	
δ	ARL	ASS	ARL	ASS	ARL	ASS
−3	1.8852	6	2.1047	7.49	2.0262	6.72
−2.5	2.2120	6	2.4319	7.36	2.2438	6.71
−2	2.7384	6	2.9609	7.31	2.6920	6.78
−1.5	3.8060	6	3.8361	7.19	3.5434	6.93
−1	6.6482	6	6.0863	7.08	5.6716	7.10
−0.5	21.5042	6	18.6979	6.82	17.0892	7.04
−0.25	73.1768	6	65.0164	6.41	63.2374	6.56
0	**280**	**6**	**280**	**6.16**	**280**	**6.15**
0.25	73.5858	6	65.0232	6.41	63.6286	6.56
0.5	21.8130	6	18.6982	6.82	17.7828	7.04
1	6.7240	6	6.0851	7.08	5.7060	7.10
1.5	3.8654	6	3.8363	7.19	3.5158	6.93
2	2.7292	6	2.9703	7.31	2.6772	6.78
2.5	2.2010	6	2.4401	7.36	2.2470	6.71
3	1.8678	6	2.1063	7.49	2.0202	6.72

proposed method is the extension of the traditional VSS method. So, we used $Z_0 = \mu$ as the starting point of the EWMA statistic and n_1 (smaller sample size) as the first sample size.

4. Computational Results

In this section, we compare the performance of three EWMA control charts, explained in the previous sections including FSS EWMA, VSS EWMA, and VSSILF EWMA control charts.

We assume that the process follows normal distribution with $\mu = 0$ and $\sigma = 2$. As mentioned before, we just focus on mean shifts in this article and assume that the standard deviation is fixed over time. So, the process is considered in-control if X_i follows normal distribution with mean zero and variance 4, and out-of-control if X_i follows normal distribution with mean δ and variance 4 ($\delta \neq 0$). Process mean shifts might be increasing or decreasing and δ can be positive or negative. Since the sampling intervals are fixed, we use average run length (ARL) as a criterion for our comparison. The coefficients of control limits in FSS EWMA, VSS EWMA, and VSSILF EWMA control charts are set through 10,000 simulation runs to obtain the in-control average run length (ARL_0) equal to 280. Different positive and negative shifts from in-control mean ($\mu = 0$) are considered. Out-of-control average run length (ARL_1) and average number of samples (ASS) is computed

Table 3
A comparison between FSS and VSSILF EWMA

	FSS EWMA		VSSILF EWMA	
	$n = 6$, $\lambda = 0.2$, $L = 2.76$, $\sigma = 2$		$n_1 = 5\, n_2 = 10$, $\lambda = 0.2$, $L_3 = 3.15$, $\sigma = 2$	
δ	ARL	ASS	ARL	ASS
−3	1.8852	6	2.0298	6.41
−2.5	2.2120	6	2.2712	6.40
−2	2.7384	6	2.7452	6.47
−1.5	3.8060	6	3.6233	6.58
−1	6.6482	6	5.9632	6.73
−0.5	21.5042	6	19.0077	6.58
−0.25	73.1768	6	68.5428	6.73
0	**280**	**6**	**280**	**5.89**
0.25	73.5858	6	67.9153	6.71
0.5	21.8130	6	18.9802	6.64
1	6.7240	6	5.9821	6.73
1.5	3.8654	6	3.6325	6.58
2	2.7292	6	2.7265	6.47
2.5	2.2010	6	2.2726	6.40
3	1.8678	6	2.0298	6.41

via 10,000 simulation runs. ASS is average number of samples in each subgroup, i.e., the average sample size.

Table 1 shows the results when the smoothing parameter (λ) is equal to 0.2 for all EWMA control charts. The sample size n_1 and n_2 in VSS and VSSILF EWMA control charts are assumed to be 5 and 10, respectively.

The results in Table 1 show that the VSSILF EWMA control chart outperforms both FSS and VSS EWMA control charts in terms of ARL. When the process is in-control, the ASS for VSSILF is more than VSS. In addition, when the process mean shifts out-of-control the ASS for VSSILF EWMA is larger than for VSS EWMA. However, as the magnitude of the shifts increases, the ASS for both VSS and VSSILF get close to each other.

It is clear that there is a tradeoff between costs corresponding to ARL and ASS criteria under out-of-control conditions. If the cost of being the process out-of-control is more than the fixed and variable sampling costs, we propose using VSSILF EWMA control chart.

Moreover, when the process is in control, the ASS for VSSILF is more than ASS for VSS chart. We can change n_1 and n_2 to reduce this difference (see Table 2). In this table, we set n_1, n_2, and L for all EWMA control charts to have in-control ARL equal to 280 and ASS roughly equal to 6. It is clear that the VSSILF EWMA control charts outperform both FSS and VSS EWMA control charts in terms of ARL, except in very large shifts, in which FSS EWMA chart is the best method.

In addition, in medium to large shifts, ASS in VSSILF is also less than ASS in VSS EWMA control chart. Note that the ARL and ASS criteria are symmetric in positive and negative shifts.

Table 3 presents a comparison between FSS EWMA and VSS EWMA, in which the in-control ASS for FSS chart is more than in-control ASS for VSSILF chart.

In other words, in this table we set n_1, n_2, and L for VSSILF EWMA chart to obtain in-control ASS less than in-control ASS for FSS EWMA control chart. Because we believe that a process is usually in-control condition and when the process is in-control, less sample size is required. This leads to designing an economic control chart as well. In this situation, again, the performance of VSSILF EWMA chart is better than FSS EWMA chart in terms of ARL.

5. Conclusions

In this article, we proposed a new VSS approach for EWMA control charts in which the sample size was varied as an integer linear function by EWMA statistics. For this purpose, a function was defined which relates the current sample size to the value of previous EWMA statistics. The performance of the proposed EWMA statistics (VSSILF EWMA) was compared with the traditional FSS and VSS control charts through simulation studies. The results showed that VSSILF EWMA chart outperforms both FSS and VSS EWMA control charts in small and medium shifts in terms of ARL. In addition, when the in-control ASS for VSS EWMA and VSSILF are equal, the out-of-control ASS for VSSILF EWMA chart in large shifts is also less than the ASS for VSS EWMA control chart.

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Appendix: The Proof That n_i Is a Positive Value

It is obvious that $\mu \leq |Z_{i-1}| \leq \mathrm{UCL}_{i-1}$. By substituting $|Z_{i-1}|$ with its lower bound and upper bound in Equation (8) we will have:

$$n_i \leq \left[\left(\frac{n_2 - n_1}{\mathrm{UCL}_{i-1} - \mu}\right)|\mathrm{UCL}_{i-1}| + \left(n_1 - \frac{n_2 - n_1}{\mathrm{UCL}_{i-1} - \mu} \times \mu\right)\right] \rightarrow n_i \leq n_2$$

and

$$n_i \geq \left[\left(\frac{n_2 - n_1}{\mathrm{UCL}_{i-1} - \mu}\right)|\mu| + \left(n_1 - \frac{n_2 - n_1}{\mathrm{UCL}_{i-1} - \mu} \times \mu\right)\right] \rightarrow n_i \geq n_1.$$

As a result

$$n_1 \leq n_i \leq n_2.$$

Since n_1 and n_2 are both positive, it can be concluded that n_i in Equation (8) is positive.

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