New Approximation Method for Structural Optimization

AliReza Habibi[1]

Abstract: The main difficulty with the use of mathematical programming for structural optimization problems in which the structural form is specific is the formulation of constraints, such as displacement and stress limitations, as explicit functions of the design variables. In this research, a new method, called Consistent Approximation (CONAP), was developed to explicitly formulate constraints and objective functions based on an efficient approximation concept. In the proposed method, some important parameters are designed using design sensitivities to increase the method's flexibility and consistency in various optimization problems. It is shown that existing methods based on approximation concepts can be easily derived from CONAP with the definition of special values for the designed parameters. In the presented approach, the primary optimization problem is replaced with a sequence of explicit sub-problems. Each sub-problem is efficiently solved using the Sequential Quadratic Programming (SQP) method. Several examples are given to demonstrate the capability and applicability of the method. It is shown that the proposed method speeds up the convergence of the optimization process. **DOI: 10.1061/(ASCE)CP.1943-5487.0000133.** © *2012 American Society of Civil Engineers.*

CE Database subject headings: Approximation methods; Optimization; Structural design; Computer programming.

Author keywords: Consistent; Approximation; Optimization; Structural; Design; Sensitivity.

Introduction

Structural optimization is currently one of the most important topics in structural engineering and has a wide range of applicability. The objective of structural optimization is to find design variables for a structure that minimize cost and satisfy various design requirements. A large number of optimization techniques have been developed and used in structural optimization. These techniques can be broadly divided into two groups: (i) gradient-based and (ii) direct-search (stochastic, or non-gradient-based). Direct-search techniques explore the design space by generating a number of successive solutions to guide the algorithm to an optimal design. Genetic algorithms (Erbatur et al. 2000; Adeli and Cheng 1994; Togan and Daloglu 2006; Degertekin et al. 2008), simulated annealing algorithms (Kirkpatrick 2002; Lamberti 2008; Hasancebi and Erbatur 2002), evolutionary programming (Kicinger et al. 2005) and evolutionary strategies (Hasancebi 2008) are the most notable direct-search optimization techniques used for the solution of engineering problems. The main characteristic of these algorithms is the imitation of biological and physical events by the evolution of a good-enough or near-optimal solution over a number of successive iterations. These techniques do not require the evaluation of gradients of objective and constraint functions, but they require a significant amount of computing power. In structural optimization problems, design variables are repeatedly perturbed to satisfy nonlinear constraints on displacements, stresses, and critical buckling loads (Haftka and Gurdal 1992; Vanderplaats 1998). For this purpose, gradient-based methods formulate and solve a set of sub-problems in which the original nonlinear functions are replaced by linear, quadratic, or higher-order approximations built using gradient information. The use of these methods can speed up the optimization process. In this regard, most investigators have followed nonlinear programming (NLP) approach and achieved remarkable results (Schmit 1960; Vanderplaats and Moses 1973; Schmit and Farshi 1974; Schmit and Miura 1976; Arora and Haug 1976; Harless 1980; Belegundu and Arora 1985; Adeli and Kamal 1986; Ringertz 1985; Joseph 1987). Some investigators have tried to benefit from the particular physical nature of structural problems, resulting in the development of the optimality criterion (OC) algorithm for finding an optimal design (Razani 1965; Venkayya et al. 1969, Venkayya 1971, and 1978; Fleury 1979; Fleury and Sander 1983; Allwood and Chung 1984; Patnaik et al. 1995). Sequential linear programming (SLP) has been applied to structural problems by several investigators to elicit all the capabilities of this approach. Recent developments in this research strand are attributed to Lamberti and Pappalettere (2000, 2003, and 2004), who have used efficient move-limit definitions incorporating a trust-region method. Sequential quadratic programming (SQP) has also been applied to structural optimization problems (Holzleitner and Mahmoud 1999; Horowitz and Afonso 2002; Mahmoud and Holzleitner 1994; Mahmoud 1997).

Among optimization methods, the mathematical programming method is attractive because of its generality and its rigorous theoretical basis. Because of the dependence of the involved constraints and the variety in the objective function upon each design variable, the achievement of a successful and efficient optimization procedure for different kinds of problem essentially depends on whether appropriate approximation schemes are selected. Valuable research has been carried out on this subject. Schmit (1960) was the first to give a comprehensive statement of the use of mathematical programming techniques to solve the nonlinear-inequality-constrained problem of elastic structures design. Schmit (1981) has provided an excellent historical review of the development of this concept. Schmit and Farshi (1974) have introduced approximation concepts into the structural design process, including design variable linking, temporary constraint deletion, and construction of high-quality explicit approximations of retained constraints using reciprocal variables and first-order Taylor series. These

[1]Assistant Professor, University of Kurdistan, Sanandaj, Iran (corresponding author). E-mail: ar.habibi@uok.ac.ir

Note. This manuscript was submitted on August 6, 2010; approved on May 19, 2011; published online on May 21, 2011. Discussion period open until August 1, 2012; separate discussions must be submitted for individual papers. This paper is part of the *Journal of Computing in Civil Engineering*, Vol. 26, No. 2, March 1, 2012. ©ASCE, ISSN 0887-3801/2012/2-236–247/$25.00.

approximations have led to the emergence of computationally efficient mathematical programming techniques for structural design optimization. A detailed description of approximation concepts with numerous examples has been presented by Schmit and Miura (1976). Definitive research, such as the work of Fleury (1979), has offered fundamental insight into mathematical programming for structural optimization. Schmit and Fleury (1980) have developed an efficient structural synthesis process by combining the Dual method with approximation concepts. These approximation concepts include (1) design variable linking; (2) temporary reduction of the number of inequality constraints by deletion techniques; and (3) construction of high-quality explicit approximations of retained constraints using Taylor series expansions. These approximation concepts convert the optimization problem into a sequence of explicit, primal problems of convex and separable form, which makes the Dual method especially attractive for the solution process. Schmit and Fleury (1980) have also successfully extended the Dual method to deal with pure discrete and mixed continuous-discrete design variable problems. Fleury and Braibant (1986) have introduced conservative first-order approximations by using mixed direct and reciprocal variables at the same time. This property was initially demonstrated by Starnes and Haftka (1979), who employed conservative approximation to handle difficult buckling constraints. Svanberg (1987) has proposed the method of moving asymptotes (MMA) based on a special type of convex approximation. This method can give the user some control of the convergence properties of the overall optimization process. By introducing sequential quadratic programming (SQP) in the dual space, Fleury (1989) has developed the Convex Linearization (CONLIN) optimizer. Some important applications of this method include optimization of trusses (Kuritz and Fleury 1989), optimization of robot path planning (Braibant and Geradin 1985), and optimization of composite structures using NASTRAN (Nagendra and Fleury 1989). Zhang et al. (1998) have applied a mixed approximation method called DQAGMMA to truss configuration optimization. A combination of diagonal quadratic approximation (DQA) and the generalized method of moving asymptotes (GMMA) is used in their method. Habibi (2007) has proposed an efficient optimization algorithm for optimal performance-based design by considering some modifications of the CONLIN method.

Although the convex linearization method can be efficient for problems that have a large feasible design space, one of its disadvantages is that it decreases the feasible domain in such a way that calculation of a feasible solution is difficult in some problems. This disadvantage has been explained by Fleury (1989). In the present study, the elimination of this deficiency was attempted.

In this study, a new method was developed for high-quality nonlinear approximation. This method can generate existing approximation concepts, including linear approximation, concave approximation, and convex approximation. Moreover, the designed parameters of the approximated functions can increase the flexibility and consistency of the proposed method. Accordingly, these parameters can make the method consistent with various structural optimization problems. The efficiency and applicability of the method are illustrated by several numerical examples. It is shown that the proposed method can speed up the convergence of the solution procedure.

Proposed Method

In this section, the proposed method for the approximation of design constraints and objective functions, called the Consistent Approximation method (CONAP), is explained. Because the design sensitivities can affect the approximated function and increase the accuracy of the approximation, a new design variable (x_i') is applied as a function of the design sensitivities with respect to the ordinary design variable (x_i). This new variable is defined as $x_i' = (x_i)^{\alpha_i}$, where the power parameter α_i is a function of the design sensitivity and i denotes the number of variables. In this schema, selecting $\alpha_i = 1$ results in obtaining the ordinary variable x_i. By selecting $\alpha_i = -1$, the reciprocal design variables used by Fleury (1979) are obtained. The mixed design variables (a combination of ordinary and reciprocal variables) used by Fleury and Braibant (1986) can be obtained by selecting $\alpha_i = 1$ and $\alpha_i = -1$ for the positive sensitivities and negative sensitivities, respectively. Accordingly, the new variable introduced can handle the variables applied in the previous research. Moreover, other design variables can be generated to improve the existing approximation methods by proper selection of the power parameter. For this purpose, assuming that the power parameters α_l and α_u correspond to the minimum sensitivity (s_l) and maximum sensitivity (s_u), respectively, of the function under consideration, the following linear equation is proposed to compute each sensitivity parameter:

$$\alpha_i = \alpha_l + (\alpha_u - \alpha_l)\frac{s_i - s_l}{s_u - s_l} \tag{1}$$

where s_i is the sensitivity of the function under consideration with respect to the design variable x_i. Eq. (1) shows that the power parameter α_i is a function of its lower and upper limits (α_l and α_u). Moreover, it is dependent on the sign and the magnitude of the function's sensitivity with respect to the design variable under consideration. In conventional approximation methods, variables cannot include the sensitivities. In the convex linearization concept developed by Fleury (1989), variables can only include the signs of the sensitivities of the approximated functions. Accordingly, in the proposed method better approximation is expected to be achieved because of the inclusion of the magnitudes of the sensitivities in addition to their signs. In the proposed equation for the power parameter, ordinary and reciprocal variables can be generated by selecting $\alpha_u = 1$; $s_i = s_u$ and $\alpha_l = -1$; $s_i = s_l$, respectively. The convex linearization method can be achieved by consideration of these two cases simultaneously. Moreover, it is shown by the numerical results that better approximations can be made by selecting other values for the power parameter. Since variation in design variables whose design sensitivity values are high can affect the function value and variation in power parameters whose values are high and can also change the function value, it seems that there exists a relation between the power parameter and the design sensitivity. Accordingly, the assumption of a proportional relation between each minimum sensitivity and its corresponding power parameter ($\alpha_l \propto s_l$), and also between each maximum sensitivity and its corresponding power parameter ($\alpha_u \propto s_u$), can be suggested. It should be noted that such assumptions need to be verified by numerical tests (see Section 4). This assumption leads to the relations $\alpha_l = ks_l$ and $\alpha_u = ks_u$, where k is a non-zero constant. Therefore, the following proportional relation is proposed:

$$\frac{\alpha_u}{\alpha_l} = \frac{s_u}{s_l} \tag{2}$$

From the above assumption, the following simple equation for calculation of the power parameter can be derived:

$$\alpha_i = \frac{s_i}{s_l}\alpha_l = \frac{s_i}{s_u}\alpha_u \tag{3}$$

Eq. (3) can be applied to compute the power parameter given α_l or α_u. Eq. (1) can be generally applied to calculate the power parameter, given α_l and α_u without the assumption in Eq. (2). The last relation shows that the power parameter for each design variable can be obtained by multiplying the ratio of the sensitivity of the function with respect to that variable in the lower limit of the sensitivities by the lower limit of the power parameter. It can also be calculated by multiplying the ratio of the sensitivity of the function with respect to that variable in the upper limit of the sensitivities by the upper limit of the power parameter. Therefore, by selecting a proper value for the lower or upper limit of the power parameter and knowing the values of the sensitivities, the magnitude of the parameter for each design variable can be calculated from Eq. (3).

By defining the new variable by the power parameter, each arbitrary function can be approximated by the first-order Taylor series expansion in terms of new design variables as follows:

$$f(x') \cong f(x'^0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x'_i}(x'_i - x'^0_i) \quad (4)$$

The derivative $\frac{\partial f}{\partial x'_i}$ can be expressed in terms of the main design variable $x_i = (x'_i)^{\frac{1}{\alpha_i}}$

$$\frac{\partial f}{\partial x'_i} = \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial x'_i}\bigg|_{x_i = x_i^0} = \frac{1}{\alpha_i}(x_i'^0)^{\frac{1}{\alpha_i}-1}\frac{\partial f}{\partial x_i} = \frac{1}{\alpha_i}(x_i^0)^{1-\alpha_i}\frac{\partial f}{\partial x_i} \quad (5)$$

By substituting Eq. (5) into Eq. (4) and rearranging Eq. (4) in terms of the main design variable, the following equation can be obtained:

$$f(x) = f(x^0) + \sum_i \frac{1}{\alpha_i}(x_i^0)^{1-\alpha_i} f_i^0[(x_i)^{\alpha_i} - (x_i^0)^{\alpha_i}] \quad (6)$$

In the above equation, f_i is the first derivative of the function $f(x)$ with respect to the design variable x_i. To calculate these derivatives for structural optimization problems in which the constraints or objective function cannot be expressed explicitly in terms of the design variables, the theory of sensitivity analysis must be employed. The symbol $\sum_i$ denotes summation over the terms for which α_i is the power parameter of the ith design variable, whose value depends on the magnitude and sign of the first derivative of the function with respect to the design variable. It can be concluded from Eq. (6) that this equation leads to the Taylor series expansion in terms of the main design variables (x_i) for $\alpha_i = 1$. For $\alpha_i = -1$ in the equation, an approximation based on the reciprocal variables $(1/x_i)$ can be obtained. With $\alpha_i = 1$ for the positive sensitivities and $\alpha_i = -1$ for the negative sensitivities, a convex approximation can be generated from Eq. (6). It is convenient to normalize the design variables so that they are equal to unity at the current point x^0. For this purpose, from the definitions $x_i'' = x_i/x_i^0$ and $f_i'' = f_i^0 x_i^0/\alpha_i$ and from Eq. (6), the following equation can be obtained:

$$f(x'') = f(x^0) + \sum_i f_i''[(x_i'')^{\alpha_i} - 1] = \sum_i f_i''(x_i'')^{\alpha_i} + f_0 - \sum_i f_i'' \quad (7)$$

where f_0 is the value of the function at the current point x^0. Eq. (7) is the basis of the CONAP strategy for approximation of structural optimization problems. In the next section, the proposed optimization algorithm based on the CONAP technique is presented.

Optimization Algorithm

To propose an efficient optimization algorithm based on the CONAP strategy, the structural optimization problem is mathematically considered in the following general form:

$$\min c_0(x) \quad \text{subject to } c_j(x) \le 0, \qquad (j = 1, \cdots) \qquad x_i^l \le x_i \le x_i^u \quad (8)$$

where $c_0(x)$ denotes the objective function, which usually represents a structural characteristic to be minimized (e.g., the weight), and $c_j(x)$ denotes the behavior constraints that impose limitations on the structural response quantities (e.g., the upper bounds on stresses and displacements). The design variables must be bounded by the upper and lower values of x_i^u and x_i^l. The objective function and the constraints can be linear or nonlinear functions of the design variables. Applying the CONAP technique to each function $c_j(x)$ and dropping the superscript //, the optimization problem [Eq. (8)] can be approximated by the following explicit sub-problem:

$$\begin{aligned} &\min \sum_i c_{io}(x_i)^{\alpha_{io}} - \bar{c}_o \\ &\text{subject to}: \sum_i c_{ij}(x_i)^{\alpha_{ij}} \le \bar{c}_j \quad (j = 1, \ldots, m) \qquad x_i^l \le x_i \le x_i^u \end{aligned} \quad (9)$$

where c_{ij} denotes the first derivative of the objective or constraint functions evaluated at the current point x^0 multiplied by the current design variable divided by the power parameter. α_{ij} denotes the power parameters allocated to the design variables. These parameters are not the same for the various constraints and the objective function. The constants $\bar{c}_j$ are calculated as follows:

$$\bar{c}_j = \sum_i c_{ij} - c_j(x^0) \qquad (j = 0, \ldots, m) \quad (10)$$

The approximated sub-problem [Eq. (9)] has a simple algebraic structure and separable design variables. These properties make solution of the sub-problem using mathematical programming attractive. In this research, the Sequential Quadratic Programming method (SQP) was applied to solve each sub-problem. The SQP implementation consists of three main stages, including update of the hessian matrix, solution of the quadratic problem, and determination of the search direction and optimum point. For more details about this method, see Arora (1989).

The CONAP optimization algorithm developed in this study can be carried out in the following steps:

1. Set $k = 1$ and assume proper initial values for the design variables (initial design point x^k) and convergence parameter.
2. Compute the values of the objective and constraint functions at the current design point.
3. Perform a sensitivity analysis and evaluate the gradient vectors of the objective and constraint functions at the current design point.
4. Assume proper values for the upper and lower limits of the power parameters for the objective and constraint functions (it is proposed that positive limits for the positive sensitivities and negative limits for the negative sensitivities be applied).
5. Construct the sub-problem (9) based on the CONAP strategy at the current design point.
6. Solve the sub-problem using the SQP method and find the optimum design variables x^{k+1}.
7. Check the optimality criterion $|\frac{f_{k-1} - f_k}{f_k}| \le \varepsilon$, where f_{k-1} and f_k are the values of the objective function at the optimum point and current point, respectively, and ε is the convergence parameter.
8. If the optimality criterion is satisfied, stop the optimization process; otherwise, set $k = k + 1$ and $x^5 = x^{k+1}$ and then go back to Step 2.

Evaluation of the Efficiency of the Proposed Method

To evaluate the proposed method of structural optimization, six numerical examples are presented in this section. The objective of the first example in the preceding section was to demonstrate the consistency of the proposed method with existing methods, and the capability of the proposed method to vary the degree of convex approximation and increase the quality of approximated design constraints compared with conventional methods. In the other examples, the newly proposed approximation concept was used to perform optimum design for five problems in structural engineering, and the results were compared to those of conventional methods. Based on the algorithm developed in Section 3, a computer program was prepared to design structures using CONAP. For some examples, Finite Element Analysis and Sensitivity Analysis programs were linked with the optimization program. The optimization software was run on a Personal Computer with a Pentium Dual Core 2.66 GHz processor and 3.48 GB of RAM under the Microsoft Windows XP operating system.

Approximation of a Design Constraint

Approximation of the following design constraint with two design variables at the design point $x^0 = (2, 2)$ was considered:

$$g(x) = 5x_2 - x_1^2 \leq 10 \tag{11}$$

This constraint had previously been evaluated using four different methods, including linear approximation, approximation using reciprocal variables, concave approximation, and convex approximation (Fleury and Braibant 1986). The results are given as follows:

1. Linear: $5x_2 - 4x_1 \leq 6$
2. Reciprocal: $-20/x_2 + 16/x_1 \leq 2$
3. Concave: $-20/x_2 - 4x_1 \leq -14$
4. Convex: $5x_2 + 16/x_1 \leq 22$

As can be observed, in Case 1 (linear approximation), ordinary design variables had been used; in Case 2 (reciprocal approximation), reciprocal design variables had been used; in Case 3 (concave approximation), an ordinary variable and a reciprocal variable had been used for Design Variables 1 and 2, respectively; and in case 4 (convex approximation), an ordinary variable and a reciprocal variable had been used for Design Variables 2 and 1, respectively. To prove that each of these approximations can be derived from the proposed approximation concept [Eq. (6)], it was necessary to use $\alpha_1 = \alpha_2 = 1$, $\alpha_1 = \alpha_2 = -1$, $\alpha_1 = -\alpha_2 = 1$ and $\alpha_1 = -\alpha_2 = -1$ for cases 1, 2, 3 and 4, respectively, as follows:

1. $$6 + \frac{1}{1}(2)^{1-1}(-4)[(x_1)^1 - 2^1] + \frac{1}{1}(2)^{1-1}(5)[(x_2)^1 - 2^1] \leq 10 \Rightarrow 5x_2 - 4x_1 \leq 6 \tag{12}$$

2. $$6+\frac{1}{-1}(2)^{1-1}(-4)[(x_1)^{-1}-2^{-1}]+\frac{1}{-1}(2)^{1+1}(5)[(x_2)^{-1}-2^{-1}] \leq 10 \Rightarrow -20/x_2 + 16/x_1 \leq 2 \tag{13}$$

3. $$6 + \frac{1}{1}(2)^{1-1}(-4)[(x_1)^1 - 2^1] + \frac{1}{-1}(2)^{1+1}(5)[(x_2)^{-1} - 2^{-1}] \leq 10 \Rightarrow -20/x_2 - 4x_1 \leq -14 \tag{14}$$

4. $$6 + \frac{1}{-1}(2)^{1+1}(-4)[(x_1)^{-1} - 2^{-1}] + \frac{1}{1}(2)^{1-1}(5)[(x_2)^1 - 2^1] \leq 10 \Rightarrow 5x_2 + 16/x_1 \leq 22 \tag{15}$$

The above inequalities show that the existing approximation methods can be simply derived from the CONAP method. Accordingly, it can be concluded that the existing methods are special cases of the proposed method by assuming special values of the power parameter. To show the applicability and efficiency of CONAP in increasing the quality of approximation and controlling the degree of convex approximation, two additional cases were considered. Relation 3 was applied to calculate the power parameters for these two cases. The lower limits of the power parameter were assumed to be $\alpha_l = -0.5$ and $\alpha_l = -2$ for Case 5 and Case 6, respectively. Using these assumptions and taking the sensitivities of the function with respect to Design Variables 1 and 2 at point x^0 to be -4 and 5, respectively, the power parameters were obtained from Relation 3 (-0.5 and -0.625 for Design Variables 1 and 2, respectively, for Case 5 and -2 and 2.5 for Design Variables 1 and 2, respectively, for Case 6). Accordingly, the design constraint was approximated using Relation 6, as follows:

1. CONSISTENT 1

$$6 + \frac{1}{-0.5}(2)^{1+0.5}(-4)[(x_1)^{-0.5} - 2^{-0.5}] + \frac{1}{0.625}(2)^{1-0.625}(5)[(x_2)^{0.625} - 2^{0.625}] \leq 10 \Rightarrow 10.3747(x_2)^{0.625} + 22.6274/(x_1)^{0.5} \leq 36 \tag{16}$$

2. CONSISTENT 2

$$6 + \frac{1}{-2}(2)^{1-2}(-4)[(x_1)^{-2} - 2^{-2}] + \frac{1}{2.5}(2)^{1-2.5}(5)[(x_2)^{2.5} - 2^{2.5}] \leq 10 \Rightarrow 0.7071(x_2)^{2.5} + 16/(x_1)^2 \leq 12 \tag{17}$$

To better explain the core steps of the proposed procedure for approximation of the constraint, Case 6 ($\alpha_l = -2$) was considered. The following steps were needed to approximate the constraint $g(x) = 5x_2 - x_1^2$ at the point $x^0 = (2, 2)$:

Step 1. Calculate the sensitivity (s_i) of the function with respect to the design variables, using Eq. (18).

$$s_1 = \left.\frac{\partial \mathbf{g}}{\partial x_1}\right|_{x_1=x_2=2} = -\mathbf{2}x_1 = -2 \times 2 = -4; \qquad s_2 = \left.\frac{\partial \mathbf{g}}{\partial x_2}\right|_{x_1=x_2=2} = \mathbf{5} \tag{18}$$

Step 2. Calculate the minimum and maximum sensitivities (s_l; s_u) using Eq. (19).

$$s_l = \min(s_1, s_2) = \min(-4, 5) = -4; \qquad s_u = \max(s_1, s_2) = \max(-4, 5) = 5 \tag{19}$$

Step 3. Calculate α_u using Eq. (2), as follows:

$$\frac{\alpha_u}{\alpha_l} = \frac{s_u}{s_l} \Rightarrow \alpha_u = \frac{s_u}{s_l}\alpha_l = \frac{5}{-4} \times -2 = 2.5 \tag{20}$$

Step 4. Calculate the power parameters (α_i) for all of the design variables using Eq. (1) or Eq. (3), as follows:

$$\alpha_1 = \frac{s_1}{s_l}\alpha_l = \frac{-4}{-4} \times -2 = -2; \qquad \alpha_2 = \frac{s_2}{s_l}\alpha_l = \frac{5}{-4} \times -2 = 2.5 \tag{21}$$

Step 5. Calculate the values of the function and its derivative with respect to the new variables using Eq. (5), as follows:

$$g(x'_0) = g(x_0) = 5 \times 2 - 2^2 = 6;$$
$$\frac{\partial g}{\partial x'_1} = \frac{1}{\alpha_1}(x_1^0)^{1-\alpha_1}\frac{\partial g}{\partial x_1} = \frac{1}{-2}(2)^{1-(-2)} \times -4 = 16;$$
$$\frac{\partial g}{\partial x'_2} = \frac{1}{\alpha_2}(x_2^0)^{1-\alpha_2}\frac{\partial g}{\partial x_2} = \frac{1}{2.5}(2)^{1-2.5} \times 5 = 0.7071 \quad (22)$$

Step 6. Determine the approximated function as a function of the new variables using Eq. (4), as follows:

$$g(x') \cong g(x'^0) + \sum_{i=1}^{2}\frac{\partial g}{\partial x'_i}(x'_i - x'^0_i)$$
$$= 6 + 16 \times (x'_1 - 2^{-2}) + 0.7071 \times (x'_2 - 2^{2.5})$$
$$= 16x'_1 + 0.7071x'_2 - 2 \quad (23)$$

Step 7. Determine the approximated function as a function of the main variables by substituting $x'_i = x_i^{\alpha_i}$ into $g(x')$ or using Eq. (6), as follows:

$$g(x) \cong 16x'_1 + 0.7071x'_2 - 2 = 16x_1^{\alpha_1} + 0.7071x_2^{\alpha_2} - 2$$
$$= 16x_1^{-2} + 0.7071x_2^{2.5} - 2 = \frac{16}{x_1^2} + 0.7071x_2^{2.5} - 2 \quad (24)$$

Step 8. Determine the approximated function as a function of the normalized variables ($x''_i = x_i/x_i^0$) using Eq. (7), as follows:

$$g''_1 = g_1^0 x_1^0/\alpha_1 = -4 \times 2/-2 = 4;$$
$$g''_2 = g_2^0 x_2^0/\alpha_2 = 5 \times 2/2.5 = 4 \quad (25)$$

$$\therefore g(x'') \cong \sum_i g''_i (x''_i)^{\alpha_i} + g_0 - \sum_i g''_i$$
$$= 4(x''_1)^{-2} + 4(x''_2)^{2.5} + 6 - (4 + 4)$$
$$= \frac{4}{(x''_1)^2} + 4(x''_2)^{2.5} - 2$$

Six approximated constraint surfaces (Cases 1 to 6) and the real constraint surface are plotted in Fig. 1. This figure shows that the concave approximation (Case 3) had a maximum feasible sub-domain and that the second consistent approximation (Case 6) had a minimum feasible sub-domain. The feasible sub-domains of the concave and reciprocal approximations (Cases 2 and 3) were larger than the feasible sub-domain of the real constraint, and the feasible sub-domains of the linear, convex and consistent approximations (Cases 1, 4, 5 and 6) were smaller than the feasible sub-domain of the real constraint. Comparison of the various approximation methods for this example shows that CONAP can generate the various approximations (Cases 1 to 4). Moreover, it can be intuitively verified that CONAP can yield the most conservative approximation (Case 6).

As observed from Fig. 1, the feasible sub-domain of the second consistent approximation (Case 6) was smaller than that of the convex approximation (Case 3), while the feasible sub-domain of the first consistent approximation (Case 5) was larger. Therefore, the proposed approximation can enlarge the feasible sub-domain (see Case 5) and consequently does not have the lack-of-feasible-solution problem that the convex approximation method has. The proposed method can also be efficiently applied to approximate complicated nonlinear design constraints because of its conservative results (see Case 6).

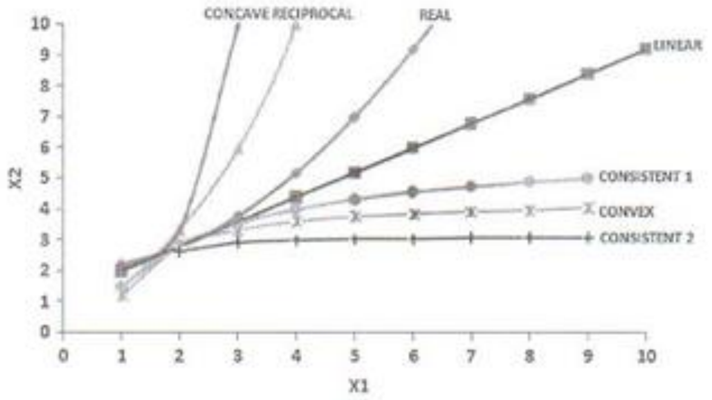

Fig. 1. Comparison of different approximation methods for the constraint $g(x) = 5x_2 - x_1^2 \leq 10$

Optimum Design of a Beam with a Rectangular Cross-Section

As a numerical test of the proposed method, CONAP was applied to the optimum design of a beam with a rectangular cross-section. This problem had previously been solved using the analytical method, the graph method and the Constraint Steepest Descent (CSD) method (Arora 1989). For solving such simple problems, in which the objective and constraint functions are explicit functions of the design variables, it is not necessary to apply approximation methods. This problem was selected because it can be used to compare the proposed and existing methods. The objective function to be minimized was the area of the cross-section. The design variables were the width (b) and the height (d) of the section. The design constraints were limitations (g_1), the shear stress (g_2), the ratio of height to width (g_3), and the lower and upper limits of the design variables (g_4, g_5, g_6 and g_7). Using these definitions, the optimization problem was formulated as follows:

$$\min f(b,d) = bd \quad \text{subject to}: \; g_1 = \frac{6M}{\sigma_{all}bd^2} - 1 \leq 0$$
$$g_2 = \frac{1.5V}{\tau_{all}bd} - 1 \leq 0 \qquad g_3 = d - rb \leq 0 \qquad g_4 = b_l - b \leq 0$$
$$g_5 = b - b_u \leq 0 \qquad g_6 = d_l - d \leq 0 \qquad g_7 = d - d_u \leq 0 \quad (26)$$

where M is the maximum bending moment of the beam; V is the maximum shear force of the beam; σ_{all} is the allowable bending stress; τ_{all} is the allowable shear stress; r is the maximum ratio of height to width; b_l and b_u are the lower and upper limits of the width, respectively; and d_l and d_u are the lower and upper limits of the height, respectively. For this test example, it was assumed that $b_l = d_l = 10$ mm; $b_u = d_u = 1{,}000$ mm; $r = 2$; $\frac{6M}{\sigma_{all}} = 24 \times 10^6$ mm^2 and $\frac{1.5V}{\tau_{all}} = 1{,}12{,}500$ mm^2. The optimal analytical solution of this problem is a curve. The minimum value of the objective function that had been obtained was 112,500 mm^2; the active constraint was g_2, and its Lagrange multiplier had been obtained as 112,500 (Arora 1989). This problem was solved with CONAP assuming (50, 200) mm as an initial point, which violated some constraints. This initial point had been considered by Arora (1989). The convergence criterion for the objective function was assumed to be 0.001. The following two states were considered for the solution:

a. Use of $\alpha_i = 1$ for the positive sensitivities and $\alpha_i = -1$ for the negative sensitivities; and
b. Determination of the power parameter (α_i) from Eq. (3) assuming $\alpha_l = -1$ for the design constraints and $\alpha_l = 1$ for the objective function.

Case (a) led to the convex approximation (CONLIN) method, developed by Fleury (1989),that mixes ordinary and reciprocal variables. This is a special case of the method developed in this paper. In this state, the optimization algorithm converged in five iterations.

Case (b) illustrated the capability of the proposed method compared with the other methods. The lower limit of the power parameter was assumed to be positive for the positive sensitivities and negative for the negative sensitivities. It had been shown in previous work that this assumption can lead to convex approximation. Considering this fact and the fact that the sensitivities of the objective function are usually positive while those of the design constraints are negative, a positive value for the lower limit of the power parameter of the objective function and a negative value for lower limit of the power parameter of the design constraints were selected. In this state, the optimization algorithm also converged in five iterations.

The results of the optimization for the four methods, including CONAP (Case b), CONLIN (Case a), CSD (Arora 1989) and the exact solution (Arora 1989), are compared in Fig. 2 and Table 1. It can be observed that the minimum value of the objective function resulting from the four methods was 112,500 mm^2. The optimum values of the design variables resulting from CONAP, CONLIN and CSD were different, while the minimum values of the objective function from the three methods were the same. This observation is related to existing solutions to this problem. The exact results validate this observation; that is, the optimum values obtained from the three methods can be obtained from the exact solution. The iteration history, shown in Fig. 2, shows that CONAP led to conservative results compared with CONLIN and CSD in all iterations. This property can be efficient for the design optimization of structures with complicated nonlinear design constraints. Fig. 2 shows that although a jump existed in the CONAP iteration history in Iteration 1 because of the violation of some of the constraints, CONAP rapidly converged in the following iterations. This illustrates the capability of the proposed method to approximate design constraints. To compare the computational efforts of CONLIN and CONAP for this problem, the execution times were measured. The execution time of CONLIN was 0.32 s and that of CONAP was 0.30 s. This shows that the computational effort of CONAP was less than that of CONLIN.

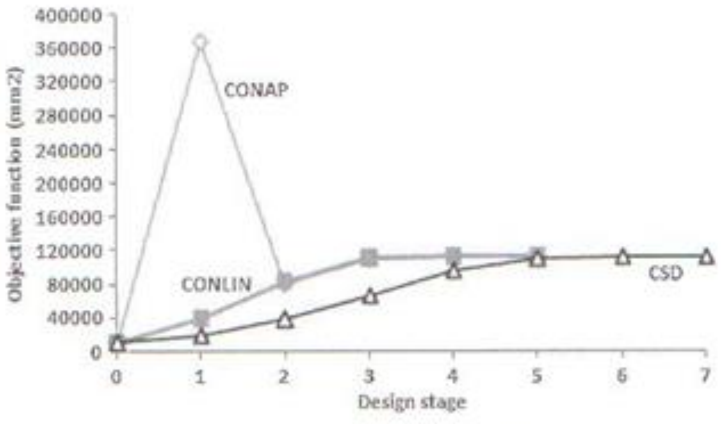

Fig. 2. The iteration history of the beam with rectangular cross-section

Table 1. Optimization Results for the Beam with a Rectangular Cross-Section

Design variable	CSD	CONLIN	Exact solution	CONAP
b (mm)	315.17	289.5465	$b \geq 237.1708$	256.5504
d (mm)	356.9	388.5387	$112{,}500/b$	438.5103
Objective function (mm^2)	112,500	112,500	112,500	112,500

Optimum Design of a Cantilever Beam

A cantilever beam, built from five beam elements as shown in Fig. 3, was considered. Each beam element had a quadratic cross-section as shown in Fig. 3. The beam was rigidly supported at Node 1, and subjected to an external vertical force at Node 6 (see Fig. 3). This problem was a good and important design example because its analytical solution exists (Svanberg 1987). This problem had previously been solved using the MMA method (Svanberg 1987). Therefore, it was possible to verify the results of the proposed method. The design variables were the dimensions x_j of the different beam elements, and the thicknesses were held fixed. The objective function to be minimized was the weight of the beam. There was only one behavior constraint, which was a limit on the vertical displacement of Node 6, where the given load acted. The lower bounds on the design variables were so small, and the upper bounds were so large, that they never became active in this problem. Using classical beam theory, after some manipulations and assumptions (Svanberg 1987), the optimization problem was formulated as follows:

$$\begin{aligned} &\text{Minimize } C_1(x_1 + x_2 + x_3 + x_4 + x_5) \\ &\text{Subject to } 61/x_1^3 + 37/x_2^3 + 19/x_3^3 + 7/x_4^3 + 1/x_5^3 \leq C_2 \end{aligned} \tag{27}$$

where

$$C_1 = 4Lt\gamma \quad \text{and} \quad C_2 = \frac{2Et\Delta_{all}}{PL^3} \tag{28}$$

where L is the length of the beam elements; t is the thickness of the elements; γ is the unit weight; E is the modulus of elasticity; P is the magnitude of the given load and Δ_{all} is the allowable displacement. In this test problem, it was assumed that $C_1 = 0.0624$ kN/cm and $C_2 = 1/\text{cm}^3$.

As a starting point for the numerical test of CONAP, the feasible solution $x_j = 5$ cm was used for all j. This solution had also been considered as a starting point for the MMA method (Svanberg 1987). At this point, the displacement constraint was satisfied with equality, and the total weight of the beam was 1.560 kN.

To illustrate the effect of the power parameter on the efficiency and convergence of the proposed method and to compare the CONAP results with the CONLIN results, the following five cases of power parameter lower limit were considered:

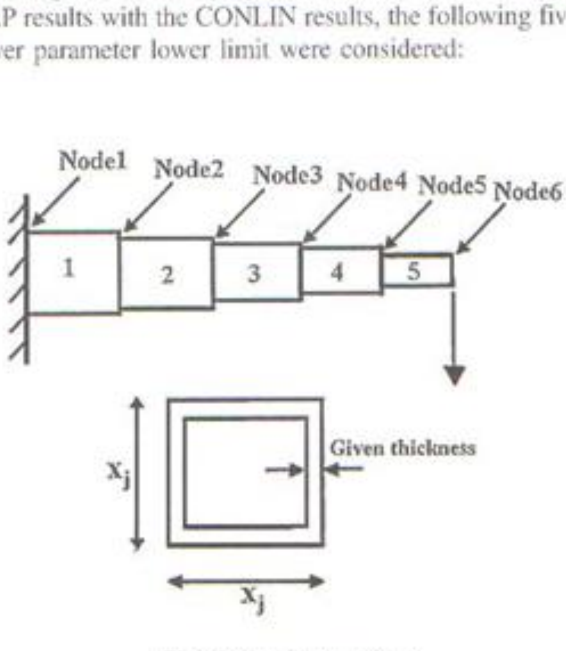

Fig. 3. Cantilever beam

1. $\alpha_l = 1$ for the objective function and $\alpha_l = -1$ for the design constraint (CONAP 1).
2. $\alpha_l = 2$ for the objective function and $\alpha_l = -1$ for the design constraint (CONAP 2).
3. $\alpha_l = 3$ for the objective function and $\alpha_l = -1$ for the design constraint (CONAP 3).
4. $\alpha_l = 1$ for the objective function and $\alpha_l = -2$ for the design constraint (CONAP 4).
5. $\alpha_i = 1$ for the positive sensitivities and $\alpha_i = -1$ for the negative sensitivities (CONLIN).

For the convergence criterion of 0.001 (the relative difference between the objective functions at the last two iterations had to be less than 0.001), the results of the iteration history for Cases 1 to 4 (CONAP) are plotted in Fig. 4 and compared with those for Case 5 (CONLIN) and MMA (with the parameter $t = 1/16$). For this problem, the execution times for Cases 1 to 4 were 3.146, 0.98, 0.6, and 2.664 seconds, respectively. In Case 5, the method diverged. That is, its execution time was extreme. The optimal design variable values are summarized in Table 2 for Cases 1 to 4 and compared with those of the exact solution. The optimal results show that the minimum objective value resulting from Cases 1 to 4 was 1.34, the same result obtained from the exact solution and MMA. This proves that the results of the proposed method were correct. The iteration history shows that CONAP converged in 22, 7, 4, and 18 iterations for Cases 1, 2, 3, and 4, respectively. The MMA method converged in 12 iterations. By increasing the lower limit of the power parameter of the objective function from 1 to 2 and 3, considerable improvement was made in the convergence and speed of CONAP. As Fig 4 and Table 2 show, selection of proper and consistent values of the power parameter can speed up the convergence of the proposed method. In Case 4, the convergence of CONAP was further improved by decreasing the lower limit of the power parameter of the design constraint from −1 to −2, although the convergence rate compared with Cases 2 and 3 was very low. For this problem, it can be concluded that the rate of convergence of CONAP was considerably affected by a change in the power parameter of the objective function, while the effect of a change in the power parameter of the constraint was not as significant.

For Case 5 (CONLIN), it can be observed that the iteration history never converged. The divergence of another traditional optimization method has also been shown for this problem (Svanberg 1987).

It is remarkable that for this problem, CONLIN did not converge at all, whereas CONAP converged to the optimal solution obtained from the analytical method for all of the tested values of α_l ($1 \le \alpha_l \le 3$ for the objective function and $-2 \le \alpha_l \le -1$ for the design constraint). The best values of α_l for this specific problem were 3 and −1 for the objective function and the design constraint, respectively. With these values of α_l, the convergence was remarkably fast; only 3 iterations were needed.

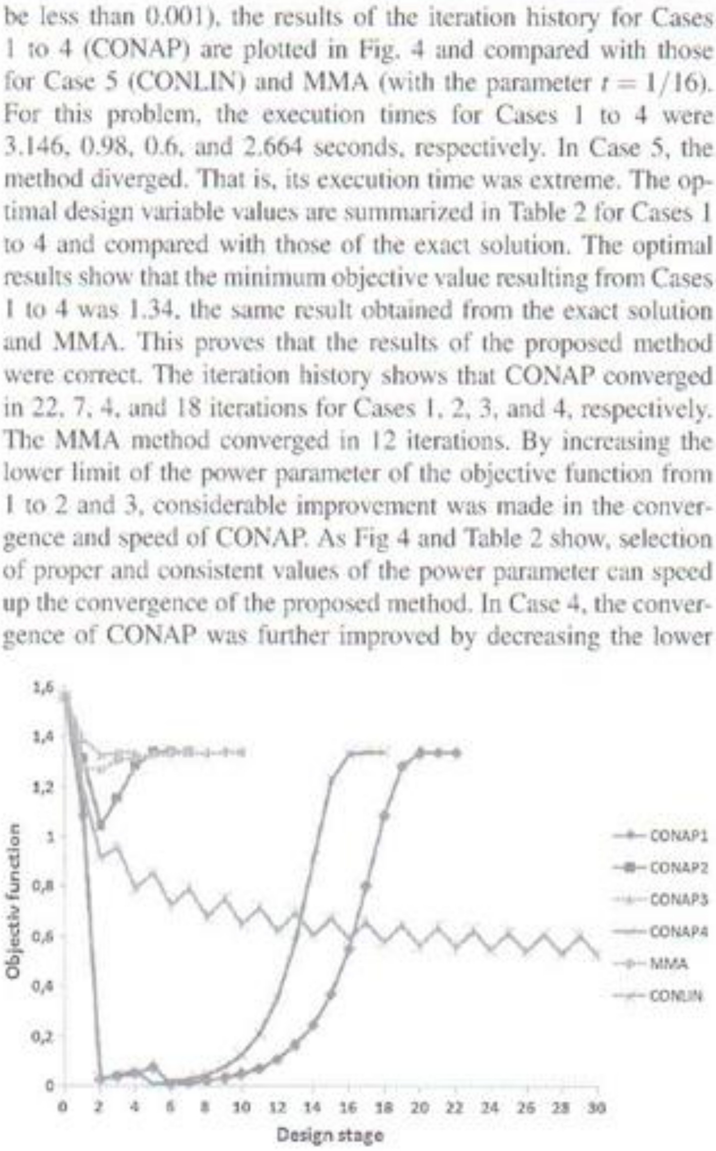

Fig. 4. Iteration history for the cantilever beam

Table 2. Optimal Solution of the Design Variables for the Cantilever Beam

Design variable	Method				
	CONAP 1	CONAP 2	CONAP 3	CONAP 4	Exact solution
X_1	6.0163	6.0193	6.0126	6.0133	6.016
X_2	5.3081	5.3134	5.3095	5.3092	5.309
X_3	4.4895	4.4990	4.4971	4.4958	4.494
X_4	3.5042	3.4932	3.5007	3.5036	3.502
X_5	2.1555	2.1483	2.1538	2.1517	2.153
Weight (kN)	1.340	1.339	1.340	1.340	1.340

Optimum Design of a Two-Bar Truss

This problem, shown in Fig. 5, was selected because it contained both a configuration and a sizing variable. It was also a good problem because the results of CONAP could be compared to those of SLP, CONLIN and MMA (Svanberg 1987) as well as the exact solution. The two design variables were the cross-sectional area of the members (x_1) and half of the distance between Nodes 1 and 2 (x_2). The truss was subjected only to stress constraints. The truss was subjected to an external force $F = (F_x, F_y)$ at the unsupported node (Node 3), where $F_x = 24.8$ kN, $F_y = 198.4$ kN, and $|F| = 200$ kN. The height of the truss (h) was assumed to be 1 m. The lower bounds on the variables were $x_1^l = 0.2$ cm^2 and $x_2^l = 0.1$ m, respectively. The upper bounds on the variables were $x_1^u = 4$ cm^2 and $x_2^u = 1.6$ m, respectively. None of these four bounds became active at the optimal solution.

The objective function to be minimized was the weight of the truss. The tensile stress had to be less than 100 MPa ($\sigma_{all} = 100$) in each of the two members. This simple problem was formulated analytically (this was not necessary for the proposed method to work), as follows:

$$\text{Minimize } C_1 x_1 \sqrt{h^2 + x_2^2}$$

$$\text{Subject to } \frac{\sigma_1}{\sigma_{all}} = \frac{\sqrt{h^2 + x_2^2}}{2\sigma_{all}} \left(\frac{F_y}{x_1 h} + \frac{F_x}{x_1 x_2} \right) \le 1 \quad \text{(Bar 1)} \quad \text{and}$$

$$\frac{\sigma_2}{\sigma_{all}} = \frac{\sqrt{h^2 + x_2^2}}{2\sigma_{all}} \left(\frac{F_y}{x_1 h} - \frac{F_x}{x_1 x_2} \right) \le 1 \quad \text{(Bar 1)}$$

$$x_1^l \le x_1 \le x_1^u, \qquad x_2^l \le x_2 \le x_2^u \tag{29}$$

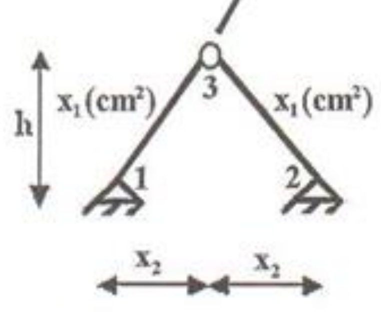

Fig. 5. Two-bar truss

where $C_1 = 1\ \mathrm{g/cm^3}$. It is obvious from this formulation that the second constraint (the stress constraint on Bar 2) could never become active, because the stress in Bar 1 was always strictly greater than the stress in Bar 2. Therefore, the first constraint (the stress constraint in Bar 1) was active. To find the exact solution for this problem, first, the variable x_1 was determined by solving the active constraint equation, as follows:

$$\frac{\sqrt{h^2+x_2^2}}{2\sigma_{\mathrm{all}}}\left(\frac{F_y}{x_1 h}+\frac{F_x}{x_1 x_2}\right)=1 \Rightarrow x_1=\frac{\sqrt{h^2+x_2^2}}{2\sigma_{all}}\left(\frac{F_y}{h}+\frac{F_x}{x_2}\right) \tag{30}$$

By substituting the above equation into the objective function, the following unconstrained problem was then generated:

$$\begin{aligned}\text{Minimize } f(x) &= C_1 x_1\sqrt{h^2+x_2^2}\\ &= C_1\frac{\sqrt{h^2+x_2^2}}{2\sigma_{\mathrm{all}}}\left(\frac{F_y}{h}+\frac{F_x}{x_2}\right)\sqrt{h^2+x_2^2}\\ &= \frac{C_1}{2\sigma_{\mathrm{all}}}\left(\frac{F_y}{h}+\frac{F_x}{x_2}\right)(h^2+x_2^2)\end{aligned} \tag{31}$$

To minimize $f(x)$, its first derivative had to vanish, as follows:

$$\frac{df}{dx_2}=0 \Rightarrow 2\frac{F_y}{h}x_2^3+(2F_x-1)x_2^2-h^2=0 \tag{32}$$

By solving Eq. (32), the exact value of x_2 was obtained, and then by substituting its value in the x_1 equation, the exact value of x_1 was computed.

A feasible starting point, in which the design variables were $x_1 = 1.5\ \mathrm{cm^2}$ and $x_2 = 0.5$ m (the same initial point assumed

Table 3. Optimal Results for the Two-Bar Truss

Iteration number	Variable 1, variable 2, stress ratio and weight	Exact solution	CONAP	MMA	CONLIN	SLP
0	x_1 (cm^2)	1.4116	1.50	1.50	1.50	1.50
	x_2 (m)	0.3771	0.50	0.50	0.50	0.50
	$\sigma_1/\sigma_{\mathrm{all}}$	1	0.92	0.92	0.92	0.92
	$W\times10^{-2}$ (g)	1.5086	1.68	1.68	1.68	1.68
1	x_1 (cm^2)		1.3864	1.39	1.39	1.38
	x_2 (m)		0.1000	0.10	0.25	0.25
	$\sigma_1/\sigma_{\mathrm{all}}$		1.6180	1.62	1.11	1.11
	$W\times10^{-2}$ (g)		1.3933	1.40	1.43	1.42
2	x_1 (cm^2)		1.4114	0.63	1.33	1.14
	x_2 (m)		0.3006	0.62	0.50	0.50
	$\sigma_1/\sigma_{\mathrm{all}}$		1.0392	2.23	1.04	1.22
	$W\times10^{-2}$ (g)		1.4737	0.74	1.49	1.27
3	x_1 (cm^2)		1.4048	1.45	1.39	1.34
	x_2 (m)		0.3760	0.10	0.25	0.25
	$\sigma_1/\sigma_{\mathrm{all}}$		1.0052	1.54	1.11	1.14
	$W\times10^{-2}$ (g)		1.5008	1.46	1.43	1.38
4	x_1 (cm^2)		1.4100	1.04	1.33	1.15
	x_2 (m)		0.3806	0.34	0.50	0.50
	$\sigma_1/\sigma_{\mathrm{all}}$		1.0000	1.38	1.04	1.21
	$W\times10^{-2}$ (g)		1.5086	1.10	1.49	1.28
5	x_1 (cm^2)		The method	1.42	1.39	1.34
	x_2 (m)		is converged	0.40	0.25	0.25
	$\sigma_1/\sigma_{\mathrm{all}}$			0.99	1.11	1.14
	$W\times10^{-2}$ (g)			1.53	1.43	1.38
6	x_1 (cm^2)			1.41	1.33	1.15
	x_2 (m)			0.38	0.50	0.5
	$\sigma_1/\sigma_{\mathrm{all}}$			1.00	1.04	1.21
	$W\times10^{-2}$ (g)			1.51	1.49	1.28
7	x_1 (cm^2)			The method	1.39	1.34
	x_2 (m)			is converged	0.25	0.25
	$\sigma_1/\sigma_{\mathrm{all}}$				1.11	1.14
	$W\times10^{-2}$ (g)				1.43	1.38
8	x_1 (cm^2)				The method	The method
	x_2 (m)				is diverged	is diverged
	$\sigma_1/\sigma_{\mathrm{all}}$					
	$W\times10^{-2}$ (g)					

for SLP, CONLIN and MMA), was chosen. At this point, the weight of the truss was 1.677×10^2 g and the stress ratio in Bar 1 was 0.925. Optimum design of the truss was performed using CONAP with $\alpha_l = 3$ for the objective function and $\alpha_l = -1$ for the design constraints. For this problem, the computation time of CONAP was 0.264 s and that of CONLIN was extreme (the CONLIN method diverged). The results of optimizations using the five methods (CONAP, SLP, CONLIN, MMA and the exact solution) are shown in Table 3. It can be observed that CONAP converged to the exact solution: that is, the CONAP results were in good agreement with the exact results. Neither SLP nor CONLIN converged, while MMA and CONAP converged to the optimal solution in 6 and 4 iterations, respectively. Therefore, CONAP can guarantee good convergence in problems in which size and shape variables are simultaneously considered. (Of course, this claim needs more case studies to support it.)

Optimum Design of a Ten-Bar Truss

In this problem, optimum design of a practical example of a structural system was performed. This example was concerned with the famous ten-bar truss, shown in Fig. 6, which had been studied previously (Arora 1989; Fleury and Braibant 1986). This truss had been specially devised to make the problem difficult to solve with conventional methods (Fleury and Braibant 1986). The objective was to minimize the weight of the truss. The cross-sectional area of each member was considered to be a design variable, and there were ten design variables. The displacements at Nodes 4 and 5 were limited to 2 in (5.08 cm) and 1 in (2.54 cm), respectively. Instead of assigning a maximum allowable stress limit to the critical Member 6, the stress flow (i.e., the force) in Member 6 was limited to 2,500 lb (1,1124 N). The lower bound and upper bound on the variables were assumed to be 0.25 in^2 (1.613 cm^2) and 100 in^2 (645.16 cm^2), respectively. The difference between this example and three aforementioned examples was that the design constraints were not explicit functions of the variables in this problem. In general, this presents a serious difficulty for structural optimization in practical problems. This difficulty causes conventional optimization algorithms to be inapplicable.

The CONAP method was used to solve this problem, in which there were no explicit constraints. The finite element method was used to perform the structural analysis. To evaluate the design sensitivities, a sensitivity analysis of the truss was performed using the Finite Difference Method (FDM). In the initial design, which was used to start the optimization process, all of the design variables were considered to be 20 in^2 (129.03 cm^2). This initial point was seriously infeasible and violated all of the constraints. For values of 1 and -1

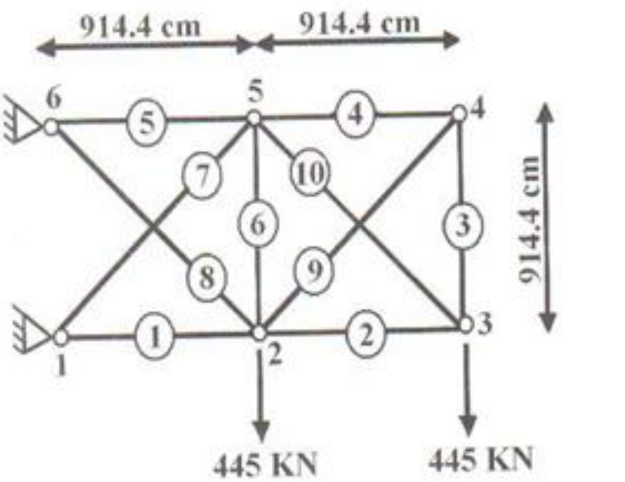

Fig. 6. Ten-bar truss

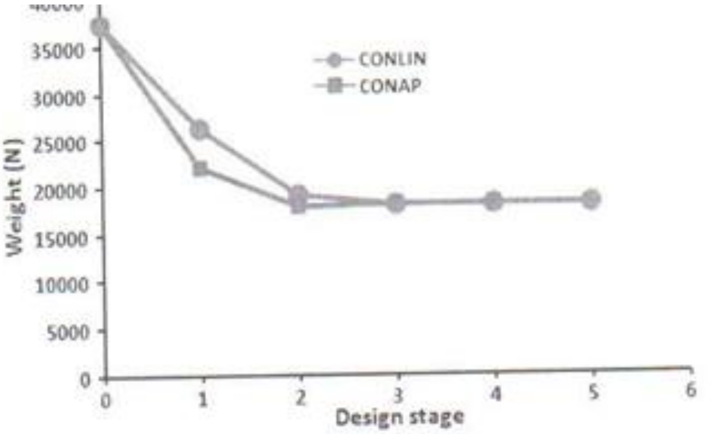

Fig. 7. Iteration history for ten-bar truss example

for the lower bounds of the power parameters of the objective function and the constraints, respectively, the iteration history of CONAP is shown in Fig. 7 and compared with that of CONLIN. The optimal solutions for Design Variables 1 to 10 were, respectively,

23.9759, 4.2366, 0.25, 3.9182, 21.1317, 0.25, 13.0401, 16.6808, 5.5397, and 6.0475.

The solution values show that the lower bounds on Design Variables 3 and 6 were active in the optimal solution. That is, to minimize the weight of the structure, Members 3 and 6 had to be selected for minimum possible strength or deleted from the truss, as long as it remained stable. It seems that this was because of the limit of the force of Member 6 that had been used as a constraint according to the assumptions of Fleury and Braibant (1986). This problem could be eliminated by using the stress constraint instead of the force constraint. The results show that two displacement constraints were

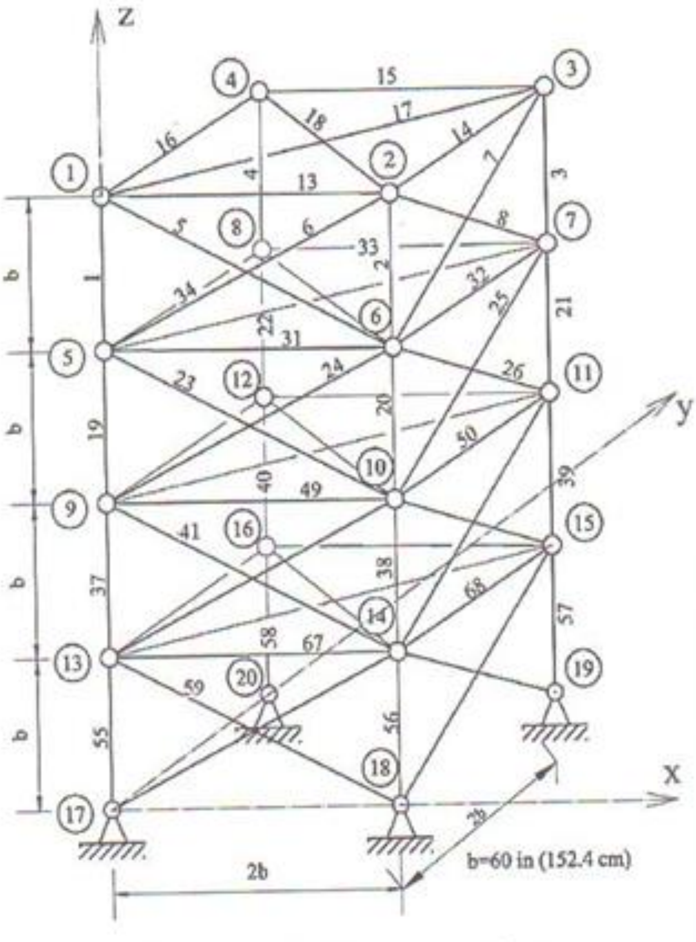

Fig. 8. Geometry of a seventy-two-bar truss

Table 4. Element Grouping for the Seventy-Two-Bar Truss Structure

Group	Elements	Group	Elements
1	1, 2, 3,4	9	37, 38, 39, 40
2	5, 6, 7, 8, 9, 10, 11, 12	10	41, 42, 43, 44, 45, 46, 47, 48
3	13, 14, 15, 16	11	49, 50, 51, 52
4	17, 18	12	53, 54
5	19, 20, 21, 22	13	55, 56, 57, 58
6	23, 24, 25, 26, 27, 28, 29, 30	14	59, 60, 61, 62, 63, 64, 65, 66
7	31, 32, 33, 34	15	67, 68, 69, 70
8	35, 36	16	71, 72

active in the optimal design. That is, in the optimal design, the displacements of Nodes 4 and 5 were 2 in (5.08 cm) and 1 in (2.54 cm), respectively. Comparison of the CONAP results with those of CONLIN (Fig. 7) shows that the total number of iterations required for CONAP was less than that for CONLIN. It can also be seen that the execution time of CONAP was 6.88 s, while that of CONLIN was 8.43 s. This shows that the computational effort of CONAP was less than that of the CONLIN. Therefore, with the proper choice of power parameters in the proposed method, the convergence rate can increase in comparison with other methods such as CONLIN. The results show that the optimal design satisfied all of the constraints despite its reduced cost, but that the initial design, with an almost 50% greater cost, violated the constraints.

Optimum Design of a Seventy-Two-Bar Truss Structure

To test the functionality and reliability of the proposed method with a more complex example problem, the 72-bar four-level skeletal tower shown in Fig. 8, which had been previously treated by other investigators (Schmit and Miura 1976; Arora and Haug 1976; Chao et al. 1984; Sedaghati 2005; Lamberti 2008; Farshi and Alinia-ziazi 2010), was considered. The Young's modulus of the material was 68.971 GPa, and the mass density was 2,767.991 kg/m^3. Because of the symmetry of the structure, variable linking was adopted by grouping the cross-sectional areas of the truss members into

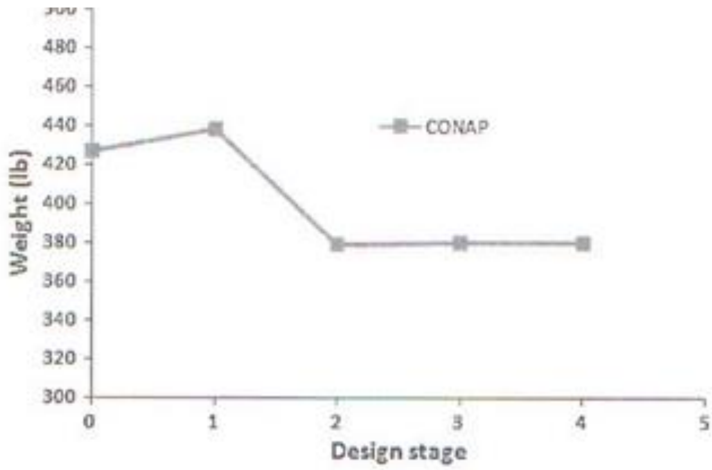

Fig. 9. The iteration history for the seventy-two-bar truss structure

16 groups, and the cross-sectional area of the bars of each group was taken as a design variable, yielding 16 sizing variables. The member numbers and the corresponding group numbers are given in Table 4. The structure was subjected to two loading conditions:

a. 22,248.706 N (i.e., 5 kips) in the positive x- and y-directions and the negative z-direction at Node 1.
b. 22,248.706 N (i.e., 5 kips) in the negative z-direction at Nodes 1, 2, 3 and 4.

The objective was to minimize the weight of the truss. The optimization was performed with 320 non-linear constraints on the nodal displacements and member stresses. The displacements at the uppermost nodes, Nodes 1, 2, 3, and 4, in the x- and y-directions had to be less than ± 6.35 mm (i.e., ± 0.25 in). The allowable stress (tensile and compressive) for all of the members was 172.4 MPa (i.e., 25 ksi). The lower bound of the cross-sectional areas was 64.516 mm^2 (i.e., 0.1 in^2). The optimization was started from an infeasible design. In the initial design, all of the cross-sectional areas were set to 322.58 mm^2 (i.e., 0.5 in^2). This initial point violated all of the displacement constraints.

Table 5. Optimization Results for the Seventy-Two-Bar Truss Structure

Group Number	Members	Venkayya (1971)	Schmit and Farshi (1974)	Schmit and Miura (1976)	Arora and Haug (1976)	Chao et al. (1984)	Lee and Geem (2004)	Sedaghati (2005)	Farshi and Alinia-ziazi (2010)	GA	CONAP
1	A1:A4	0.1610	0.1580	0.1570	0.1564	0.1570	0.1560	0.1565	0.1565	0.2874	0.1567
2	A5:A12	0.5570	0.5940	0.5460	0.5464	0.5490	0.5470	0.5456	0.5457	0.7907	0.5371
3	A13:A16	0.3770	0.3410	0.4110	0.4110	0.4060	0.4420	0.4104	0.4106	0.2967	0.4155
4	A17:A18	0.5060	0.6080	0.5700	0.5712	0.5550	0.5900	0.5697	0.5697	0.3300	0.5707
5	A19:A22	0.6110	0.2640	0.5230	0.5263	0.5130	0.5170	0.5237	0.5237	0.3296	0.5289
6	A23:A30	0.5320	0.5480	0.5170	0.5178	0.5290	0.5040	0.5171	0.5171	0.6411	0.5125
7	A31:A34	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1158	0.1000
8	A31:A34	0.1000	0.1510	0.1000	0.1000	0.1000	0.1010	0.1000	0.1000	0.1043	0.1000
9	A37:A40	1.2460	1.1070	1.2670	1.2702	1.2520	1.2290	1.2684	1.2685	0.8205	1.2681
10	A41:A48	0.5240	0.5790	0.5120	0.5124	0.5240	0.5220	0.5117	0.5118	0.6931	0.5112
11	A49:A52	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1033	0.1000
12	A53:A54	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1049	0.1000
13	A55:A58	1.8180	2.0780	1.8850	1.8656	1.8320	1.7900	1.8862	1.8864	1.5597	1.9266
14	A59:A66	0.5240	0.5030	0.5130	0.5131	0.5120	0.5210	0.5123	0.5122	0.7375	0.5134
15	A67:A70	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1418	0.1000
16	A71:A72	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1014	0.1000
Weight (N)		1,695.58	1,728.63	1,688.64	1,688.55	1,688.55	1,686.99a[a]	1,688.55	1,688.68	1,924.4	1,688.73

[a]Some of the constraints were slightly violated.

The iteration history of the 72-bar truss, as shown in Fig. 9, shows that CONAP converged in four iterations. The CONAP algorithm found the optimal weight to be 379.6575 lbs, and there were no constraint violations after 132 structural analyses (one analysis for evaluation of the constraints and 32 analyses for calculation of the sensitivities at each iteration), which took 200 s. The initial design, with a 12.4% greater cost, violated all of the displacement constraints. Optimal designs reported in the literature, in which the initial design variables have been assumed to be 0.1 in^2, are listed in Table 5 together with the optimal design from CONAP. It can be observed that the optimum design weights obtained from the algorithms developed by Schmit and Miura (1976), Arora and Haug (1976), Chao et al. (1984), Sedaghati (2005), and Farshi and Alinia-ziazi (2010) are almost the same as that obtained from CONAP. The problem was also solved using a GA algorithm. The GA program was run eight times to achieve the global minimum, on the same computer which had been used for the run of the CONAP program. Total execution time of these runs was 116.5 s. Among them, the best solution, for which the objective function value was less than that of other solutions, is been reported in Table 5. For this solution, execution time is 12.9 s and the objective function value is equal to 1,924.4 N; that is about 14% heavier than for the CONAP design. The results show that the computational effort of GA is less than that of the CONAP; however, the minimum weight resulting from CONAP is less than the minimum weight resulting from GA.

Conclusions

In this paper, a new method for structural optimization is developed. In this method, named Consistent Approximation (CONAP), the design sensitivities with respect to the design variables are used to explicitly express the objective function and the design constraints. It is shown that the power parameters used in the approximated functions can expand (or constrict) the design space. This property makes the proposed method consistent with various structural problems. It is also shown that existing approximation methods can be generated as specific cases of the proposed method by selection of specific values of the power parameter used for the approximated functions.

It is not claimed that the results obtained for the test problems are typical. Additional aspects, including discrete design variables and non-linear structural behavior, need to be investigated to generalize the results presented in this paper. For many problems, CONLIN can be very efficient. However, it is believed that the results obtained demonstrate the flexibility and consistency of CONAP, which gives the user some control of the convergence of the overall optimization process. From the results of the evaluated numerical examples, it is shown that the computational effort of CONAP is less than that of CONLIN.

References

Adeli, H., and Cheng, N. T. (1994). "Concurrent genetic algorithms for optimization of large structures." *J. Aerosp. Eng.*, 7(3), 276–296.

Adeli, H., and Kamal, O. (1986). "Efficient optimization of space trusses." *Comput. Struct.*, 24(3), 501–511.

Allwood, R. J., and Chung, Y. S. (1984). "Minimum-weight design of trusses by an optimality criteria method." *Int. J. Numer. Methods Eng.*, 20(4), 697–713.

Arora, J. S. (1989). *Introduction to optimum design*, McGraw-Hill.

Arora, J. S., and Haug, J. R. (1976). "Efficient optimal design of structures by generalized steepest descent programming." *Int. J. Numer. Methods Eng.*, 10(4), 747–766.

Belegundu, A. D., and Arora, J. S. (1985). "A study of mathematical programming methods for structural optimization, Part I: Theory; Part II: Numerical results." *Int. J. Numer. Methods Eng.*, 21(9), 1583–1623.

Braibant, V., and Geradin, M. (1985). "Optimum path planning of robot arms." *LTAS Report RA-2*, Univ. of Liege, Belgium.

Chao, N., Fenves, S., and Westerberg, A. (1984). "Application of reduced quadratic programming technique to optimal structural design." *New Directions in Optimum Structural Design*, Atrex, E., Gallagher, R. H., Radgsdell, K. M. and Zienkiewicz, O. C., Eds., Wiley, NY.

Degertekin, S. O., Saka, M. P., and Hayaloglu, H. S. (2008). "Optimal load and resistance factor design of geometrically nonlinear steel space frames via tabu search and genetic algorithm." *Eng. Struct.*, 30(1), 197–205.

Erbatur, F., Hasancebi, O., Tutuncil, I., and Kilci, H. (2000). "Optimum design of planar and space structures with genetic algorithms." *Comput. Struct.*, 75(2), 209–224.

Farshi, B., and Alinia-ziazi, A. (2010). "Sizing optimization of truss structures by method of centers and force formulation." *Int. J. Solids Struct.*, 47(18–19), 2508–2524.

Fleury, C. (1979). "A unified approach to structural weight minimization." *Comput. Methods Appl. Mech. Eng.*, 20(1), 17–38.

Fleury, C. (1989). "CONLIN: An efficient dual optimizer based on convex approximation concepts." *Struct. Optim.*, 1(2), 81–89.

Fleury, C., and Braibant, V. (1986). "Structural optimization: A new dual method using mixed variables." *Int. J. Numer. Methods Eng.*, 23(3), 409–428.

Fleury, C., and Sander, G. (1983). "Dual methods for optimizing finite element flexural systems." *Comput. Methods Appl. Mech. Eng.*, 37(3), 249–275.

Habibi, A. R. (2007). "Optimal performance-based seismic design of reinforced concrete frameworks." Ph.D. thesis, Tarbiat Modares Univ., Tehran, Iran.

Haftka, R. T., and Gurdal, Z. (1992). *Elements of structural optimization*, 3rd Ed., Kluwer, Dordrecht.

Harless, R. I. (1980). "A method for synthesis of optimal weight." *Comput. Struct.*, 12(6), 791–804.

Hasancebi, O. (2008). "Adaptive evolution strategies in structural optimization: Enhancing their computational performance with applications to large-scale structures." *Comput. Struct.*, 86(1–2), 119–132.

Hasancebi, O., and Erbatur, F. (2002). "On efficient use of simulated annealing in complex structural optimization problems." *Acta Mechanica*, 157, 27–50.

Holzleitner, L., and Mahmoud, K. G. (1999). "Structural shape optimization using MSC/NASTRAN and sequential quadratic programming." *Comput. Struct.*, 70(5), 487–514.

Horowitz, B., and Afonso, S. M. B. (2002). "Quadratic programming solver for structural optimisation using SQP algorithm." *Adv. Eng. Softw.*, 33(7–10), 669–674.

Joseph, K. T. (1987). "A basis change strategy for the reduced gradient method and the optimum design of large structures." *Int. J. Numer. Methods Eng.*, 24(7), 1269–1281.

Kicinger, R., Arciszewki, T., and Jong, K. D. (2005). "Evolutionary computation and structural design: a survey of the state-of-the-art." *Comput. Struct.*, 83(23–24), 1943–1978.

Kirkpatrick, S., Gelatt, C. D., and Vecchi, M. P. (1983). "Optimization by simulated annealing." *Science*, 220(4598), 671–680.

Kuritz, S. P., and Fleury, C. (1989). "Mixed variable structural optimization using convex linearization techniques." *Eng. Optim.*, 15(1), 27–41.

Lamberti, L. (2008). "An efficient simulated annealing algorithm for design optimization of truss structures." *Comput. Struct.*, 86(19–20), 1936–1953.

Lamberti, L., and Pappalettere, C. (2000). "Comparison of the numerical efficiency of different sequential linear programming based algorithms for structural optimization problems." *Comput. Struct.*, 76(6), 713–728.

Lamberti, L., and Pappalettere, C. (2003). "Move limits definition in structural optimization with sequential linear programming. Parts I & II." *Comput. Struct.*, 81(4), 197–238.

Lamberti, L., and Pappalettere, C. (2004). "Improved sequential linear programming formulation for structural weight minimization." *Comput. Methods Appl. Mech. Eng.*, 193(33–35), 3493–3521.

Lee, K. S., and Geem, Z. W. (2004). "A new structural optimization method based on the harmony search algorithm." *Comput. Struct.*, 82(9–10), 781–798.
Mahmoud, K. G. (1997). "An efficient approach to structural optimization." *Comput. Struct.*, 64(1–4), 97–112.
Mahmoud, K. G., Engl, H. W., and Holzleitner, L. (1994). "Optimum structural design using MSC/NASTRAN and sequential quadratic programming." *Comput. Struct.*, 52(3), 437–447.
Nagendra, G. K., and Fleury, C. (1989). "Sensitivity and optimization of large scale composite structures." *Finite Elem. Anal. Des.*, 5(3), 223–235.
Patnaik, S. N., Guptill, J. D., and Berke, L. (1995). "Merits and limitations of optimality criteria method for structural optimization." *Int. J. Numer. Methods Eng.*, 38(18), 3087–3120.
Razani, R. (1965). "The behaviour of the fully stressed design of structures and its relationship to minimum weight design." *AIAA J.*, 3(12), 2262–2268.
Ringertz, U. T. (1985). "On topology optimization of trusses." *Eng. Optim.*, 9(3), 209–218.
Schmit, L. A. (1960). "Structural design by systematic synthesis." *Proc., The 2nd Conference on Electronic Computation*, ASCE, Reston, VA, 105–122.
Schmit, L. A. (1981). "Structural synthesis: Its genesis and development." *AIAA J.*, 19(10), 1249–1263.
Schmit, L. A., and Fleury, C. (1980). "Structural synthesis by combining approximation concepts and dual methods." *AIAA J.*, 18(10), 1252–1260.
Schmit, L. A., and Farshi, B. (1974). "Some approximation concepts for structural synthesis." *AIAA J.*, 12(5), 692–699.
Schmit, L. A., and Miura, H. (1976). "Approximation concepts for efficient structural synthesis." *Rep. NASA-CR-2552*, NASA, Washington, DC.
Sedaghati, R. (2005). "Benchmark case studies in structural design optimization using the force method." *Int. J. Solids Struct.*, 42(21-22), 5848–5871.
Starnes, J. H., and Haftka, R. T. (1979). "Preliminary design of composite wings for buckling, stress and displacement constraints." *J. Aircr.*, 16(8), 564–570.
Svanberg, K. (1987). "The method of moving asymptotes: A new method for structural optimization." *Int. J. Numer. Methods Eng.*, 24(2), 359–373.
Togan, V., and Daloglu, A. T. (2006). "Optimization of 3D trusses with adaptive approaches in genetic algorithms." *Eng. Struct.*, 28(7), 1019–1027.
Vanderplaats, G. N. (1998). *Numerical optimization techniques for engineering design*, Vanderplaats Research and Development (VR&D), Inc., Colorado Springs, CO.
Vanderplaats, G. N., and Moses, F. (1973). "Structural optimization by methods of feasible directions." *Comput. Struct.*, 3(4), 739–755.
Venkayya, V. B. (1971). "Design of optimum structures." *Comput. Struct.*, 1(1–2), 265–309.
Venkayya, V. B. (1978). "Structural optimization: A review and some recommendations." *Int. J. Numer. Methods Eng.*, 13(2), 203–228.
Venkayya, V. B., Khot, N. S., and Reddy, V. S. (1969). "Energy distribution in an optimum structural design." *Rep. AFFDL-TR-68-156*, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Dayton, OH.
Zhang, W. H., Domaszewski, M., and Fleury, C. (1998). "A new mixed convex approximation method with applications for truss configuration optimization." *Struct. Optim.*, 15(3-4), 237–241.