Identifying the Time of a Step Change in the Mean of a Two-Stage Process

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Abstract – **when a control chart signals is not necessarily the real time of a change known as the change point in the literature. Estimating the real time of a change after a control chart signals is crucial and leads to saving time and cost in finding special causes. In this paper an MLE approach is used to estimate step change point in the mean of multistage process in Phase II. We specially focus on a two stage process each one including a variable quality characteristic with normal distribution. The quality characteristics in these two stages are linked to each other through a linear regression model. The performance of the proposed method is evaluated through a numerical example. The results show that the estimator performs well.**

Keywords - **Change point estimation, multistage, linear regression model, maximum likelihood estimator**

I. INTRODUCTION

In most manufacturing industries, a product is usually manufactured in more than one stage that called as multistage process. Multistage processes have a property that distinguishes them from single step processes. In these processes, quality of the product in precise stages affects its quality in the next stages. This effect is referred to as cascade property. So the quality of final products not only depends on the final stage, but also on quality characteristic of precise stages.

Control charts are typically used to detect changes in a process. The aim is improving the performance of process by distinguishing between special causes and common causes of variation. Different approaches are proposed for monitoring multistage processes in the literature of statistical process control. Monitoring the quality characteristics in each stage with a separate control chart, monitoring all quality characteristics of all stages simultaneously with one multivariate control chart and finally using cause selecting control charts are the most important approaches frequently used in the literature of multistage processes. Using the first approach, the effect of the previous stages on quality of product in the final stage is neglected. In the second approach, one cannot determine the quality characteristic and the stage responsible for out-of-control signal. To overcome these problems, the cause selecting control charts are proposed as the third approach. This type of control charts eliminates the effect of previous stages from current stage.

When a change is detected, search for finding the special causes should be initiated. Control chart signal is not exactly the real time of the change. Knowing the real time of the change would simplify the search and save the time to find the source of process change. Consequently, the effects of delay in finding the assignable causes such as the number of nonconformance items are decreased.

In the literature of change point estimation, different types of change and estimating approaches have been considered. Change types are classified as single step change, multiple step change, drift and monotonic change. The main proposed approaches is including maximum likelihood estimate (MLE), cumulative sum (CUSUM) and the exponentially weighted moving average (EWMA).

Most of the researchers on change point estimation are focused on one stage process with either different type of change or different type of distribution. However, few articles considered the change point estimation in multistage processes.

Hence, in this paper we develop a change point estimator for a two stage process with normal distribution using MLE in Phase II. The type of change is assumed to be a step shift in location parameter. The quality characteristics in these two stages are linked to each other through a linear regression model.

This paper is organized as follows: In the next section, a literature review in change point estimation in control charts is briefly presented. In section III a two stage process is modeled using a simple linear regression model. In section IV the maximum likelihood function is applied to estimate the change point. Simulation studies are conducted in section V. The conclusions and some future researches are given in the final section.

II. AN OVERVIEW OF CHANGE POINT ESTIMATION IN CONTROL CHARTS

The problem of the change point estimation in normal process was investigated by Samuel et al. [1] He evaluated the step change in the mean of a normal process using an MLE approach. Khoo [2] investigated a change point when the CUSUM chart used for monitoring the process. Perry and Pignatiello [3] considered linear trend change point estimation using the MLE approach. Change point estimation with a monotonic change was considered by Noorossana and Shadman [4]. The readers are referred to Amiri and Allahyari [5] for a comprehensive review on the change point estimation.

Zantec et al. [6] proposed a method to determine the effect of each stage on the quality of following stages and

identifying the important stage which is crucial for quality improvement. Xiang and Tsung [7] used a state-space model to monitor multistage processes. Yang and Yang [8] proposed a cause-selecting control chart for monitoring the mean of a multistage process. Yang and Yu [9] proposed a VSI-EWMA control chart to identify the change in process mean. Yang and Su [10] investigated adaptive sample size and sampling interval to detect small mean shift quickly. Zeng and zhou [11] applied the regression adjustment method to model and monitor variation in multistage processes.

Change point estimation is also vital in multistage processes. However, few researches have been done on this issue. Davoodi and Niaki [12] considered estimation the step change time of the mean under the auto-correlated assumption in multistage process using MLE by the state space model. Zou et al. [13] proposed change point estimation in Phase I in multistage processes. Perry and Pignatiello [14] applied a nested model in two-stage processes and proposed an MLE approach to find the real time of a change when the $\bar{x}$ control chart signals.

III. MODELING A TWO STAGE PROCESS

The process is assumed to include a two dependent stages with one quality characteristic in each stage. In two stages process, x and cause selecting control charts are used to control the first and scond stages, respectively. The simple linear regression model is as follows:

$$Y = \beta_0 + \beta_1 X + \varepsilon, \tag{1}$$

Where X and Y are the normal quality characteristics in the first and the second stages, respectively. β_0 and β_1 are the regression parameters. ε is independent random variable from a normal distribution with mean zero and variance σ^2, i.e. $N(0,\sigma^2)$.

In a two stage process the second stage is usually monitored by a cause selecting control chart. In this control chart the residuals are controlled instead of Y. The residuals are calculated as flows:

$$z_i = Y_i - \hat{Y}_i \tag{2}$$

The mean of the residuals are computed by:

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \hat{Y}_i). \tag{3}$$

When the process is in control, the residuals follow a normal distribution with mean of zero and variance of σ^2. Hence, the probability density function of the residuals is as follows:

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}. \tag{4}$$

If an assignable cause happens in the procces, the mean of the residuals shifts to μ_1 and probability density function changes as (5),

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu_1)^2}{2\sigma^2}}. \tag{5}$$

The upper and lower control limits of the cause selecting control chart are calculated as follows:

$$UCL = \mu_z + k\sigma_z \tag{6}$$
$$LCL = \mu_z - k\sigma_z \tag{7}$$

IV. THE PROPOSED CHANGE POINT ESTIMATIMATION METHOD

The process is presumed to be in-control primarily and the observations coming from normal distribution with known mean and variance. We assume that the mean changes from μ_0 to $\mu_1 = \mu_0 + \delta\sigma_0/\sqrt{n}$ after the unknown time τ where the change is described as step shift. The shift in process mean remains until the assignable causes are detected at time T. Our proposed change point estimator is the maximum likelihood estimator (MLE). The likelihood function considers both in-control and out-of-control residuals with the probability density function described in the previous section.

The mean in second stage is linked to X in previous stage as:

$$E(y_i|x_i) = \beta_0 + \beta_1 x_i\ , \tag{8}$$

which is equal to $\hat{Y}_i$.

Therefore, the likelihood function of the residuals is given by:

$$L = \prod_{i=1}^{\tau}\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-z_i^2}{2\sigma^2}} \prod_{i=\tau+1}^{T}\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(z_i-\mu_1)^2}{2\sigma^2}} = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^T e^{\sum_1^{\tau}\frac{-z_i^2}{2\sigma^2}} e^{\sum_{\tau+1}^{T}\frac{-(z_i-\mu_1)^2}{2\sigma^2}}. \tag{9}$$

Taking the logarithm of the above likelihood function, we have:

$$ln(L) = -Tln(\sigma\sqrt{2\pi}) - \sum_1^{\tau}\frac{z_i^2}{2\sigma^2} - \sum_{\tau+1}^{T}\frac{(z_i-\mu_1)^2}{2\sigma^2} \tag{10}$$

$$= -T\ ln\ (\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\left(\sum_1^{\tau} z_i^2 + \sum_{\tau+1}^{T}(z_i-\mu_1)^2\right)$$
$$= -T\ ln\ (\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\left(\sum_1^{T} z_i^2 + (T-\tau){\mu_1}^2 - 2\mu_1\sum_{\tau+1}^{T} z_i\right),$$

where the change is occurred, the mean estimator is derived as:

$$\mu_1 = \frac{\sum_{\tau+1}^{T} z_i}{T-\tau}. \quad (11)$$

Substituting (11) into (10), the MLE change point estimator is calculated as follows:

$$\hat{\tau} = argmax((T-\tau)\mu_1^2)\,. \quad (12)$$

$\hat{\tau}$ is the change point which maximizes the likelihood function. T is the time of signal by the cause selecting control chart and μ_1 is equal to $\sum_{i=\tau+1}^{T} z_i\,/(T-\tau)$. By replacing (8) into (2), z_i is calculated as follows:

$$z_i = Y_i - (\beta_0 + \beta_1 x_i), \quad (13)$$

where Y_i is random variable from normal distribution with mean $(\beta_0 + d_0) + (\beta_1 + d_1)x_i$ and variance σ^2. d_0 and d_1 are shifts in the intercept and the slope of the link function between the mean of Y and X and measured in unit of σ.

V. SIMULATION STUDIES

In this section, the performance of the MLE change point estimator for the mean are evaluated through simulation studies.

Consider a two stage process with a single quality characteristic at each stage. X comes from normal distribution with $\mu_x = 2$, $\sigma_x=1$ and Y comes from normal distribution with $\mu_y = \beta_0 + \beta_1 x$ that β_0=1, β_1=0.5 and σ =1. A cause selecting control chart is applied to monitor the quality characteristic in the second stage. The mean and variance of the residual statistics in the cause selecting control chart under in-control situation are equal to zero and 1, respectively. To obtain an in-control average run length (ARL_0) equal to 200, k in equations (6) and (7) is determined as 2.808. As a result, the upper and lower control limits of the cause selecting control chart are 2.808 and -2.808, respectively.

We generated 100 in-control observations under the above assumptions. At the time 101, we apply a step shift in the parameter of μ_y by the shift sizes d_0 and d_1 equal to 0.5, 1, 1.5 and 2. The mean of subsequent residuals changes from μ_0 = 0 to μ_1 as mentioned in previous section. The proposed procedure is applied on all possible combination of shifts in the regression parameters (d_0 represents shift in β_0 and d_1 represents shift in β_1 in unit of σ). Under the all combination of shifts in the regression parameters, 10000 simulation runs are applied to obtain the mean and standard error of the change point estimator.

The results of the average and standard error of the change point estimator under different shifts considered are summarized in Table I. The results show that as the magnitude of the shifts increases, the accuracy of the change point estimator will improve. In addition, the results show the perfect performance of the change point estimator in identifying the real time of the change.

Table II contains the various intervals represent the precision of the change point estimator versus real time of the change. It indicates the precision of the change point estimator under different shift sizes and approves the results in Table I.

TABLE I.
ESTIMATING THE CHANGE POINT ($\bar{\hat{\tau}}$)
WITH REAL TIME OF THE CHANGE IN τ =100

d_0	d_1	$\bar{\hat{\tau}}$	Std.error ($\bar{\hat{\tau}}$)
0.5	0	103.62	0.023
1	0	100.14	0.007
1.5	0	99.95	0.004
2	0	99.43	0.005
0	0.5	100.57	0.010
0	1	100.08	0.005
0	1.5	100.08	0.002
0	2	100.07	0.003
0.5	0.5	99.7	0.007
1	1	99.77	0.004
1.5	1.5	99.92	0.002
2	2	99.97	0.002

TABLE II
VARIOUS INTERVAL OF THE PRECISION OF CHANGE POINT ESTIMATOR WITH τ =100

	d_0	d_1	probability
$P(\lvert\bar{\hat{\tau}} - \tau\rvert = 0)$	0.5	0	0.070
	1	0	0.225
	1.5	0	0.461
	2	0	0.548
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 1)$	0.5	0	0.170
	1	0	0.394
	1.5	0	0.697
	2	0	0.760
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 2)$	0.5	0	0.259
	1	0	0.500
	1.5	0	0.805
	2	0	0.851
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 3)$	0.5	0	0.310
	1	0	0.597
	1.5	0	0.867
	2	0	0.890

TABLE II (CONTINUE)
VARIOUS INTERVAL OF THE PRECISION OF CHANGE POINT ESTIMATOR WITH τ =100

	d_0	d_1	probability
$P(\lvert\bar{\hat{\tau}} - \tau\rvert = 0)$	0	0.5	0.236
	0	1	0.425
	0	1.5	0.630
	0	2	0.687
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 1)$	0	0.5	0.420
	0	1	0.675
	0	1.5	0.816
	0	2	0.879
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 2)$	0	0.5	0.502
	0	1	0.803
	0	1.5	0.882
	0	2	0.921
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 3)$	0	0.5	0.593
	0	1	0.865
	0	1.5	0.915
	0	2	0.938
$P(\lvert\bar{\hat{\tau}} - \tau\rvert = 0)$	0.5	0.5	0.394
	1	1	0.746
	1.5	1.5	0.898
	2	2	0.916
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 1)$	0.5	0.5	0.602
	1	1	0.855
	1.5	1.5	0.955
	2	2	0.987
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 2)$	0.5	0.5	0.729
	1	1	0.920
	1.5	1.5	0.981
	2	2	0.994
$P(\lvert\bar{\hat{\tau}} - \tau\rvert \le 3)$	0.5	0.5	0.813
	1	1	0.940
	1.5	1.5	0.987
	2	2	0.995

V. CONCLUSION AND FUTURE RESEARCH

In this paper, we proposed a residual approach by using MLE estimator to estimate the change point at a two stages process in Phase II. The performance of the proposed estimator was investigated through simulation studies. The results indicated the proposed estimator performs well. Developing this approach for processes with more than two stages or more than one quality characteristic in each stage can be investigated as future research. Also considering other types of shifts such as drift or monotonic change could be fruitful areas for research.

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