Doubly warped product Finsler manifolds
with some non-Riemannian curvature properties

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Abstract. We consider doubly warped product (DWP) Finsler manifolds with some non-Riemannian curvature properties. First, we study Berwald and isotropic mean Berwald DWP-Finsler manifolds. Then we prove that every proper Douglas DWP-Finsler manifold is Riemannian. We show that a proper DWP-manifold is Landsbergian if and only if it is Berwaldian. Then we prove that every relatively isotropic Landsberg DWP-manifold is a Landsberg manifold. We show that a relatively isotropic mean Landsberg warped product manifold is a weakly Landsberg manifold. Finally, we show that there is no locally dually flat proper DWP-Finsler manifold.

1. Introduction. The study of relativity theory demands a wider class of manifolds and the idea of doubly warped products was introduced and studied by many authors. Recent studies show that the notion of doubly warped product manifolds has an important role in Riemannian geometry and its applications [A], [BEP], [BP], [G], [Mu1], [Mu2], [U]. For example, Beem–Powell studied this product for Lorentzian manifolds [BP]. Then in [A], Allison considered global hyperbolicity of doubly warped products and null pseudo convexity of Lorentzian doubly warped products.

On the other hand, Finsler geometry is dedicated to classical and generalized Finsler geometries. It studies manifolds whose tangent spaces carry a norm varying smoothly with the base point. Indeed, Finsler geometry is just Riemannian geometry without the quadratic restriction. Thus it is natural to extend the construction of warped product manifolds to Finsler geometry. In the first step, Asanov generalized the Schwarzschild metric to the Finslerian setting and obtained some models of relativity theory described through warped products of Finsler metrics [As1], [As2]. In [Koz], Kozma–Peter–Varga defined their warped product for Finsler metrics and concluded...
that completeness of a doubly warped product can be related to completeness of its components.

Let \((M_1, F_1)\) and \((M_2, F_2)\) be two Finsler manifolds and \(f_1 : M_1 \to \mathbb{R}^+\) and \(f_2 : M_2 \to \mathbb{R}^+\) be two smooth functions. Let \(\pi_1 : M_1 \times M_2 \to M_1\) and \(\pi_2 : M_1 \times M_2 \to M_2\) be the natural projection maps. The product manifold \(M_1 \times M_2\) endowed with the metric \(F : TM_1^o \times TM_2^o \to \mathbb{R}\) given by

\[
F(v_1, v_2) = \sqrt{f_2^2(\pi_2(v_2))F_1^2(v_1) + f_1^2(\pi_1(v_1))F_2^2(v_2)}
\]

is considered, where \(TM_1^o = TM_1 \setminus \{0\}\) and \(TM_2^o = TM_2 \setminus \{0\}\). The metric defined above is a Finsler metric. The product manifold \(M_1 \times M_2\) with this metric will be called the \textit{doubly warped product} of the manifolds \(M_1\) and \(M_2\), denoted \(f_2M_1 \times f_1M_2\). If either \(f_1 = 1\) or \(f_2 = 1\), then \(f_2M_1 \times f_1M_2\) becomes a warped product of Finsler manifolds \(M_1\) and \(M_2\). If \(f_1 = f_2 = 1\), then we have a product manifold. If neither \(f_1\) nor \(f_2\) is constant, then we have a proper DWP-manifold.

Let \((M, F)\) be a Finsler manifold. The second and third order derivatives of \(\frac{1}{2}F_x^2\) at \(y \in T_xM_0\) are symmetric trilinear forms \(g_y\) and \(C_y\) on \(T_xM\), called the \textit{fundamental tensor} and \textit{Cartan torsion}, respectively. The rate of change of \(C_y\) along geodesics is the \textit{Landsberg curvature} \(L_y\) on \(T_xM\) [Ba], [BCS]. The metric \(F\) is said to be a \textit{relatively isotropic Landsberg metric} if \(L + cFC = 0\), where \(c = c(x)\) is a scalar function on \(M\). Set \(I_y := \sum_{i=1}^n C_y(e_i, e_i, \cdot)\) and \(J_y := \sum_{i=1}^n L_y(e_i, e_i, \cdot)\), where \(\{e_i\}\) is an orthonormal basis for \((T_xM, g_y)\). Then \(I_y\) and \(J_y\) are called the \textit{mean Cartan torsion} and \textit{mean Landsberg curvature}, respectively. The metric \(F\) is said to be a \textit{relatively isotropic mean Landsberg metric} if \(J + cFI = 0\), where \(c = c(x)\) is a scalar function on \(M\) [ChSh].

The geodesic curves of a Finsler manifold \((M, F)\) are determined by the system of second order differential equations \(\ddot{c}^i + 2G^i(\dot{c}) = 0\), where the local functions \(G^i = G^i(x, y)\) are called the \textit{spray coefficients} of \(F\). A Finsler metric \(F\) is called a \textit{Berwald metric} if the \(G^i\) are quadratic in \(y \in T_xM\) for any \(x \in M\), and a \textit{Douglas metric} if \(G^i = \frac{1}{2}F_{jk}^i(x)y^jy^k + P(x, y)y^i\) [BM], [NST1]. Taking the trace of the Berwald curvature yields the \textit{mean Berwald curvature} \(E\). The metric \(F\) is said to be an \textit{isotropic mean Berwald metric} if \(E = \frac{n+1}{2}cF^{-1}h\), where \(h = h_{ij}dx^i \otimes dx^j\) is the angular metric and \(c = c(x)\) is a scalar function on \(M\) [NST2].

This paper is arranged as follows: In Section 2, we recall some basic concepts of Finsler manifolds. In Sections 3 and 4, we study doubly warped product Finsler metrics (DWP-Finsler metrics) with vanishing Berwald curvature and isotropic mean Berwald curvature, respectively. In Section 5, we prove that every proper Douglas DWP-Finsler manifold is Riemannian.
In Section 6, we show that a proper DWP-Finsler manifold is a Landsberg manifold if and only if it is a Berwald manifold. Then we prove that every relatively isotropic Landsberg DWP-Finsler manifold is a Landsberg manifold. In Section 7, we prove that a relatively isotropic mean Landsberg warped product manifold is a weakly Landsberg manifold. Finally in Section 8, we show that there is no locally dually flat proper DWP-Finsler manifold.

2. Preliminaries. Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, by $TM = \bigcup_{x \in M} T_xM$ the tangent bundle of $M$, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on $M$. A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$;
(iii) for each $y \in T_xM$, the quadratic form $g_y$ on $T_xM$ is positive definite, where $g_y(u,v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right]_{s,t=0}, \quad u,v \in T_xM.$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$C_y(u,v,w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y+tw}(u,v) \right]_{t=0}, \quad u,v,w \in T_xM.$$ 

The family $\mathcal{C} := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathcal{C} = 0$ if and only if $F$ is Riemannian. For $y \in T_xM_0$, define the mean Cartan torsion $I_y$ by $I_y(u) := I_i(y) u^i$, where $I_i := g^{jk} C_{ijk}, C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$ and $u = u^i \frac{\partial}{\partial x^i} x$. By Deicke’s Theorem, $F$ is Riemannian if and only if $I_y = 0$ [BCS], [Sh1].

Given a Finsler manifold $(M,F)$, a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0$, which in standard coordinates $(x^i, y^i)$ for $TM_0$ is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i},$ where

$$G^i := \frac{1}{4} g^{il}(y) \left\{ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right\}, \quad y \in T_xM.$$ 

$\mathbf{G}$ is called the spray associated to $(M,F)$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\dddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y : T_xM \otimes T_xM \to \mathbb{R}$ by $B_y(u,v,w) := B_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i} x$ and
\[ E_y(u, v) := E_{jk}(y)u^j v^k \]

where

\[ B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^m_{jkm}. \]

\( B \) and \( E \) are called the Berwald curvature and mean Berwald curvature, respectively. Then \( F \) is called a Berwald metric and weakly Berwald metric if \( B = 0 \) and \( E = 0 \), respectively [Sh1]. It is proved that on a Berwald space, parallel translation along any geodesic preserves the Minkowski functionals [Ich]. Thus Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

A Finsler metric \( F \) is said to be an isotropic mean Berwald metric if its mean Berwald curvature is of the form

\[ E_{ij} = \frac{1}{2} (n + 1) c F^{-1} h_{ij}, \]

where \( h_{ij} = g_{ij} - F^{-2} y_i y_j \) is the angular metric and \( c = c(x) \) is a scalar function on \( M \) [ChSh].

Define \( D_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M \) by

\[ D_y(u, v, w) := D^i_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i}. \]

where

\[ D^i_{jkl} := B^i_{jkl} - \frac{2}{n + 1} \left\{ E_{jk} \delta^i_l + E_{jl} \delta^i_k + E_{kl} \delta^i_j + \frac{E_{jk}}{\partial y^l} y^i \right\}. \]

We call \( D := \{ D_y \}_{y \in T M_0} \) the Douglas curvature. A Finsler metric with \( D = 0 \) is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó–Matsumoto as a generalization of Berwald metrics [BM].

There is another extension of Berwald curvature. For a tangent vector \( y \in T_x M_0 \), define \( L_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R} \) by \( L_y(u, v, w) := L_{ijk}(y)u^i v^j w^k \), where

\[ L_{ijk} := -\frac{1}{2} y_l B^l_{ijk}. \]

The family \( L := \{ L_y \}_{y \in T M_0} \) is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if \( L = 0 \). The quantity \( L/C \) is regarded as the relative rate of change of \( C \) along geodesics. A Finsler metric \( F \) is said to be a relatively isotropic Landsberg metric if

\[ L = c F C \]

for some scalar function \( c = c(x) \) on \( M \) [ChSh].

Taking the trace of the Landsberg curvature yields the mean Landsberg curvature \( J_y : T_x M \rightarrow \mathbb{R} \), defined by \( J_y(u) := J_i(y)u^i \), where

\[ J_i := g^{ijk} L_{ijk}. \]
A Finsler metric is called a weakly Landsberg metric if $J = 0$. The quantity $J/I$ is regarded as the relative rate of change of $I$ along geodesics. A Finsler metric $F$ is said to be a relatively isotropic mean Landsberg metric if

$$J = cFI$$

for some scalar function $c = c(x)$ on $M$ [ChSh]. It is obvious that every relatively isotropic Landsberg metric is a relatively isotropic mean Landsberg metric.

A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat if at any point there is a standard coordinate system $(x^i, y^i)$ in $TM$ that satisfies

$$\tag{2.2} (F^2)_{x^iy^j} y^k = 2(F^2)_{x^i}. $$

In this case, the coordinate system $(x^i)$ is called an adapted local coordinate system [Am], [amna]. It is easy to see that every locally Minkowskian metric satisfies (2.2), hence is locally dually flat [Sh2]. But the converse is not true, generally.

3. Berwaldian DWP-Finsler manifolds. In this section, we study DWP-Finsler manifolds with vanishing Berwald curvature.

Lemma 3.1. Every proper DWP-Finsler manifold $(f_2 M_1 \times f_1 M_2, F)$ with vanishing Berwald curvature is a Riemannian manifold.

Proof. The Berwald curvature of $(f_2 M_1 \times f_1 M_2, F)$ is as follows:

$$\tag{3.1} B^k_{ijl} = B^k_{ijl} - \frac{1}{4f_2^2} \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^l} \frac{\partial f_1^2}{\partial x^h} F^2_2, $$

$$\tag{3.2} B^k_{i\beta l} = -\frac{1}{4f_2^2} \frac{\partial^2 g^{kh}}{\partial y^i \partial y^l} \frac{\partial f_1^2}{\partial x^h} \frac{\partial F^2_2}{\partial v^\beta}, $$

$$\tag{3.3} B^k_{\alpha \beta l} = -\frac{1}{f_2} \frac{\partial f_1^2}{\partial x^h} \frac{\partial g^{kh}}{\partial y^l} g_{\alpha \beta}, $$

$$\tag{3.4} B^k_{\alpha \beta \lambda} = -\frac{1}{f_2} \frac{\partial f_1^2}{\partial x^h} g^{kh} C_{\alpha \beta \lambda}, $$

$$\tag{3.5} B^\gamma_{\alpha \beta \lambda} = B^\gamma_{\alpha \beta \lambda} - \frac{1}{4f_1^2} \frac{\partial^3 g^{\gamma \nu}}{\partial v^\beta \partial v^\alpha \partial u^\lambda} \frac{\partial f_2^2}{\partial u^\nu} F^2_1, $$

$$\tag{3.6} B^\gamma_{i \beta \lambda} = -\frac{1}{4f_1^2} \frac{\partial^2 g^{\alpha \gamma}}{\partial v^\beta \partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} \frac{\partial F^2_1}{\partial y^i}, $$

$$\tag{3.7} B^\gamma_{ij \lambda} = -\frac{1}{2f_1^2} g_{ij} \frac{\partial g^{\alpha \gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha}. $$
\begin{equation}
B_{ijk}^\gamma = -\frac{1}{f_1^2} C_{ijk} g^{\alpha \gamma} \partial f_2^2 \partial u^\alpha.
\end{equation}

If \((f_2 M_1 \times f_1 M_2, F)\) is Berwaldian, then \(B^d_{abc} = 0\). By (3.4), we get
\begin{equation}
C_{\alpha \beta \lambda} g^{k h} \partial f_2^2 \partial u^\alpha = 0.
\end{equation}

Multiplying (3.9) with \(g_{kr}\) implies that
\begin{equation}
C_{\alpha \beta \lambda} \frac{\partial f_2^2}{\partial x^r} = 0.
\end{equation}

By (3.10), if \(f_1\) is not constant then we get \(C_{\alpha \beta \lambda} = 0\), i.e., \((M_2, F_2)\) is Riemannian. In a similar way, from (3.8) we conclude that if \(f_2\) is not constant then \((M_1, F_1)\) is Riemannian.

**Theorem 3.2.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a DWP-Finsler manifold.

(i) If \(f_1\) is constant and \(f_2\) is not constant, then \((f_2 M_1 \times f_1 M_2, F)\) is a Berwald manifold if and only if \(M_1\) is Riemannian, \(M_2\) is a Berwald manifold and
\[ \frac{\partial g^{\alpha \gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial w^\alpha} = 0. \]

(ii) If \(f_2\) is constant and \(f_1\) is not constant, then \((f_2 M_1 \times f_1 M_2, F)\) is a Berwald manifold if and only if \(M_2\) is Riemannian, \(M_1\) is Berwaldian and
\[ \frac{\partial g^{ij}}{\partial y^k} \frac{\partial f_2^2}{\partial x^i} = 0. \]

**Proof.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a Berwaldian manifold with \(f_1\) constant on \(M_1\). Then from (3.8) we get \(C_{ijk} = 0\), i.e., \((M_1, F_1)\) is Riemannian. Also, (3.7) gives
\[ \frac{\partial g^{\alpha \gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial w^\alpha} = 0. \]

Differentiating this equation with respect to \(v^\beta\) implies that
\[ \frac{\partial^2 g^{\alpha \gamma}}{\partial v^\lambda \partial v^\beta} \frac{\partial f_2^2}{\partial w^\alpha} = 0, \]
and consequently
\[ \frac{\partial^3 g^{\alpha \gamma}}{\partial v^\lambda \partial v^\beta \partial v^\mu} \frac{\partial f_2^2}{\partial w^\alpha} = 0. \]

Then (3.5) reduces to \(B^\gamma_{\alpha \beta \lambda} = 0\), i.e., \((M_2, F_2)\) is Berwaldian. In a similar way, we can prove the converse of this assertion. The proof of (ii) is similar and we omit it.

By a similar argument, we obtain
Corollary 3.3. Let \((M_1 \times f_1 M_2, F)\) be a proper WP-Finsler manifold. Then \((M_1 \times f_1 M_2, F)\) is Berwaldian if and only if \(M_2\) is Riemannian, \(M_1\) is Berwaldian and
\[
C_{ij}^k \frac{\partial f_1}{\partial x^i} = 0,
\]
where \(C_{ij}^k = -2 \frac{\partial g_{ij}^k}{\partial y^k}\) is the Cartan tensor.

4. Isotropic mean Berwald DWP-manifolds. In this section, we study DWP-Finsler metrics with isotropic mean Berwald curvature. First, we compute the mean Berwald curvature of \(F\) as follows:
\[
E_{\alpha\beta} = E_{ij} = E_{i\beta} = 0.
\]

\[(4.1)\]
\[
E_{\alpha\beta} = E_{ij} = E_{i\beta} = 0.
\]

\[(4.2)\]
\[
E_{i\beta} = -\frac{1}{4f_2^2} \frac{\partial^2 g^{kh}}{\partial y^k \partial y^i} \frac{\partial f_1^2}{\partial x^h} v^\beta - \frac{1}{4f_1^2} \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\alpha} y_i,
\]
where \(E_{ij}\) and \(E_{\alpha\beta}\) are mean Berwald curvatures of \((M_1, F_1)\) and \((M_2, F_2)\), respectively.

Theorem 4.2. Let \((f_2 M_1 \times f_1 M_2, F)\) be a proper DWP-Finsler manifold. Then \(F\) is a weakly Berwald metric if and only if \(F_1\) and \(F_2\) are weakly Berwald metrics and
\[
\frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_1^2}{\partial x^h} = \frac{\partial g^{\gamma\nu}}{\partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

Proof. Let \((f_2 M_1 \times f_1 M_2, F)\) be a weakly Berwald manifold. Then we have \(E_{\alpha\beta} = E_{ij} = E_{i\beta} = 0\). Using (4.3), we get
\[
\frac{1}{f_2^2} \frac{\partial^2 g^{kh}}{\partial y^k \partial y^i} \frac{\partial f_1^2}{\partial x^h} v^\beta = -\frac{1}{f_1^2} \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\alpha} y_i.
\]
Contracting (4.5) with \(y^j\) gives
\[
(4.6)\]
\[
\frac{1}{f_2^2} \frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_1^2}{\partial x^h} v^\beta = \frac{1}{f_1^2} \frac{\partial g^{\nu\gamma}}{\partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} F_1^2.
\]
Differentiating (4.6) with respect \(v^\alpha\) implies that
\[
(4.7)\]
\[
\frac{1}{f_2^2} \frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_1^2}{\partial x^h} g_{\alpha\beta} = \frac{1}{f_1^2} \frac{\partial g^{\nu\gamma}}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} F_1^2.
\]
In a similar way, one can obtain

\[
\frac{1}{f_1^2} \frac{\partial g^\alpha \gamma}{\partial u^\gamma} \frac{\partial f_2^2}{\partial u^\gamma} g_{ij} = \frac{1}{f_2^2} \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2.
\]

Substituting (4.7) into (4.1) and plugging (4.8) into (4.2), we have

\[
E_{\alpha \beta} = \frac{3}{8f_1^2} \frac{\partial^3 g^{\nu \gamma}}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} F_1^2,
\]

\[
E_{ij} = \frac{3}{8f_2^2} \frac{\partial^3 g^{\nu \gamma}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2.
\]

Since \(E_{\alpha \beta}\) is a function of \((u^\alpha, v^\alpha)\), by differentiating (4.9) with respect \(y^h\) we deduce that

\[
\frac{\partial^3 g^{\nu \gamma}}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} y_h = 0,
\]

and consequently

\[
\frac{\partial^3 g^{\nu \gamma}}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

Putting (4.11) into (4.9) gives \(E_{\alpha \beta} = 0\). A similar argument yields \(E_{ij} = 0\). Further, from (4.11) and (4.7) we derive that

\[
\frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_1^2}{\partial x^h} = 0.
\]

Also, contracting (4.11) with \(v^\alpha v^\beta\) implies that

\[
\frac{\partial g^{\nu \gamma}}{\partial v^\nu} \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

Thus we have (4.4).

Conversely, suppose \((M_1, F_1)\) and \((M_2, F_2)\) are weakly Berwald manifolds and (4.4) holds. Then \(E_{ij} = E_{\alpha \beta} = 0\). Equation (4.4) gives

\[
\frac{\partial^2 g^{kh}}{\partial y^i \partial y^j} \frac{\partial f_1^2}{\partial x^h} = \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} = \frac{\partial^2 g^{\nu \gamma}}{\partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = \frac{\partial^3 g^{\nu \gamma}}{\partial v^\alpha \partial v^\beta \partial v^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

By plugging \(E_{ij} = E_{\alpha \beta} = 0\) and the above equation into (4.1)–(4.3), we obtain \(E_{\alpha \beta} = E_{ij} = E_{i\beta} = 0\). This means that \((f_2M_1 \times f_1M_2, F)\) is a weakly Berwald manifold.

Now, if \(f_2\) is a constant function on \(M_2\), then (4.7) implies that \(E_{\alpha \beta} = 0\). Thus we have

**Corollary 4.3.** Let \((f_2M_1 \times f_1M_2, F)\) be a DWP-Finsler manifold and \(f_1\) be constant on \(M_1\) (resp. \(f_2\) be constant on \(M_2\)). Then \((f_2M_1 \times f_1M_2, F)\) is weakly Berwald if and only if \((M_1, F_1)\) and \((M_2, F_2)\) are weakly Berwald.
Corollary 4.4. Let $(M_1 \times f_1 M_2, F)$ be a WP-Finsler manifold. Then $(M_1 \times f_1 M_2, F)$ is weakly Berwald if and only if $(M_1, F_1)$ and $(M_2, F_2)$ are weakly Berwald manifolds and

\[ \frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_1^2}{\partial x^h} = 0. \]

Now, we consider DWP-Finsler manifolds with isotropic mean Berwald curvature. First, as a consequence of Lemma 4.1, we have

Lemma 4.5. A DWP-Finsler manifold $(f_2 M_1 \times f_1 M_2, F)$ has isotropic mean Berwald curvature if and only if

\begin{align*}
E_{\alpha\beta} & = \frac{1}{8 f_1^2} \frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial v^\gamma} \frac{\partial f_1^2}{\partial u^\nu} F_1^2 - \frac{1}{4 f_2^2} g_{\alpha\beta} \frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_2^2}{\partial x^h} \\
& \quad - \frac{n+1}{2} c f_1^2 F^{-1} \left( g_{\alpha\beta} - \frac{f_1^2}{F_1^2} v^\alpha v^\beta \right) = 0, \\
E_{ij} & = \frac{1}{8 f_2^2} \frac{\partial^3 g^{\gamma\nu}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_2^2}{\partial x^h} F_2^2 - \frac{1}{4 f_2^2} g_{ij} \frac{\partial g^{\alpha\gamma}}{\partial v^\gamma} \frac{\partial f_2^2}{\partial u^\alpha} \\
& \quad - \frac{n+1}{2} c f_2^2 F^{-1} \left( g_{ij} - \frac{f_2^2}{F_2^2} y^i y^j \right) = 0, \\
(n+1) c f_1^2 f_2^2 F^3 y_i v_\beta & = \frac{1}{2 f_1^2} \frac{\partial^2 g^{kh}}{\partial y^k \partial y^i} \frac{\partial f_1^2}{\partial x^h} v_\beta - \frac{1}{2 f_1^2} \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma} \frac{\partial f_1^2}{\partial u^\alpha} y_i = 0,
\end{align*}

where $c = c(x)$ is a scalar function on $M$.

Theorem 4.6. A DWP-Finsler manifold $(f_2 M_1 \times f_1 M_2, F)$ with isotropic mean Berwald curvature is a weakly Berwald manifold provided that

\[ \frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_1}{\partial x^h} = 0 \quad \text{or} \quad \frac{\partial g^{\gamma\nu}}{\partial v^\gamma} \frac{\partial f_2}{\partial u^\nu} = 0. \]

Proof. Suppose that $\frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_1}{\partial x^h} = 0$ and $F$ is an isotropic mean Berwald DWP-Finsler metric. Then by using (4.14), we obtain

\[ (n+1) c f_1^2 f_2^2 F^3 v_\beta = 1 \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\gamma} \frac{\partial f_1^2}{\partial u^\alpha}. \]

Differentiating the above equation with respect $y^j$ gives

\[ \frac{n+1}{F^5} c f_1^2 f_2^4 v_\beta y_j = 0. \]

Thus, $c = 0$, so $F$ is a weakly Berwald metric.  \[\blacksquare\]
5. Douglas DWP-Finsler manifolds. In this section, we study DWP-Finsler manifolds with vanishing Douglas curvature. We prove that every Douglas proper DWP-Finsler manifold is Riemannian. To do this, we need

**Lemma 5.1.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a DWP-Finsler manifold. Then the Douglas curvature of \(F\) is as follows:

\[
D^k_{ijl} = B^k_{ijl} - \frac{1}{4 f_2^2} \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^l} \frac{\partial f_1^2}{\partial x^h} F_2^2
- \frac{2}{n+1} \left\{ E_{ij} \delta^k_l - \frac{1}{8 f_2^2} \frac{\partial^3 g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} F_2^2 \delta^k_l \right\},
\]

\[
D^k_{i\beta l} = -\frac{1}{4 f_2^2} \frac{\partial^2 g^{kh}}{\partial y^i \partial y^j} \frac{\partial f_1^2}{\partial x^h} F_2^2 + \frac{2}{n+1} \left\{ \frac{1}{4 f_2^2} \frac{\partial^2 g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} \delta^k \delta^l v_{\beta} \right\},
\]

\[
D^k_{\alpha \beta l} = -\frac{1}{2 f_2} g_{\alpha \beta} \frac{\partial g^{kh}}{\partial y^i} \frac{\partial f_1^2}{\partial x^h} - \frac{2}{n+1} \left\{ E_{\alpha \beta} \delta^k_l - \frac{1}{4 f_2} \frac{\partial^2 g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} \delta^k_l \right\},
\]

\[
D^k_{\alpha \beta \lambda} = -\frac{1}{f_2} C_{\alpha \beta \lambda} g^{kh} \frac{\partial f_1^2}{\partial x^h} - \frac{2}{n+1} \left\{ \frac{\partial E_{\alpha \beta}}{\partial y^k} \right\},
\]

\[
-\frac{1}{8 f_2} \frac{\partial^4 g^{\gamma \nu}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} F_1^2 g^k_i F^k_j - \frac{1}{4 f_2} \frac{\partial g_{\alpha \beta}}{\partial y^k} \frac{\partial g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} y_i y_j \delta^k_l \right\},
\]

\[
-\frac{1}{4 f_2} \frac{\partial^2 g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} y_i y_j \delta^k_l \right\},
\]

\[
-\frac{1}{8 f_2} \frac{\partial^4 g^{\gamma \nu}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} F_1^2 g^k_i F^k_j - \frac{1}{4 f_2} \frac{\partial g_{\alpha \beta}}{\partial y^k} \frac{\partial g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} y_i y_j \delta^k_l \right\},
\]
\[
D^\gamma_{\alpha\beta\lambda} = B^\gamma_{\alpha\beta\lambda} - \frac{1}{4f_1^2} \frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial u^\lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 + \frac{2}{n+1} \left\{ \frac{1}{8f_1^2} \frac{\partial^3 g^{\gamma\nu}}{\partial v^\beta \partial v^\alpha \partial u^\mu} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 \delta_\lambda^\gamma \right. \\
- E_{\alpha\beta} \delta_\lambda^\gamma + \frac{1}{4f_2^2} \frac{\partial g^{kh}}{\partial y^k} \frac{\partial f_2^1}{\partial x^h} \delta_\lambda^\gamma + \frac{1}{8f_1^2} \frac{\partial^3 g^{\mu\nu}}{\partial v^\lambda \partial v^\alpha \partial u^\nu} \frac{\partial f_2^2}{\partial u^\mu} F_1^2 \delta_\beta^\gamma \\
- E_{\alpha\lambda} \delta_\beta^\gamma + \frac{1}{4f_2^2} \frac{g_{\alpha\lambda}}{\partial y^k} \frac{\partial f_2^1}{\partial x^h} \delta_\beta^\gamma + \frac{1}{8f_1^2} \frac{\partial^3 g^{\mu\nu}}{\partial v^\lambda \partial v^\beta \partial u^\mu} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 \delta_\alpha^\gamma \\
- E_{\beta\lambda} \delta_\alpha^\gamma + \frac{1}{4f_2^2} \frac{g_{\beta\lambda}}{\partial y^k} \frac{\partial f_2^1}{\partial x^h} \delta_\alpha^\gamma + \frac{1}{8f_1^2} \frac{\partial^4 g^{\mu\nu}}{\partial v^\lambda \partial v^\beta \partial v^\alpha \partial u^\mu} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 v^\gamma \\
+ \frac{1}{4f_2^2} \frac{g_{\alpha\beta}}{\partial v^\gamma} \frac{\partial g^{kh}}{\partial v^\gamma} \frac{\partial f_2^1}{\partial x^h} v^\gamma - \frac{\partial E_{\alpha\beta}}{\partial v^\lambda} v^\gamma \right\},
\]

\[
D^\gamma_{i\beta\lambda} = -\frac{1}{4f_1^2} \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\alpha \partial u^\lambda} \frac{\partial f_2^2}{\partial u^\gamma} F_1^2 + \frac{2}{n+1} \left\{ \frac{1}{4f_2^2} \frac{\partial^2 g^{kh}}{\partial v^\beta \partial v^\alpha \partial u^\mu} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 \delta_\lambda^\gamma \right. \\
+ \frac{1}{4f_1^2} \frac{\partial^2 g^{\mu\nu}}{\partial u^\alpha \partial u^\beta} \frac{\partial f_2^2}{\partial u^\alpha} \delta_\lambda^\gamma + \frac{1}{4f_2^2} \frac{\partial^2 g^{kh}}{\partial y^k \partial y^i} \frac{\partial f_2^1}{\partial x^h} \delta_\beta^\gamma v^\lambda \\
+ \frac{1}{4f_1^2} \frac{\partial^2 g^{\mu\nu}}{\partial v^\alpha \partial u^\beta} \frac{\partial f_2^2}{\partial u^\alpha} \delta_\beta^\gamma y_i + \frac{1}{4f_2^2} \frac{\partial^2 g^{kh}}{\partial y^k \partial y^i} \frac{\partial f_2^1}{\partial x^h} g_{\beta\lambda} v^\gamma \\
+ \frac{1}{4f_1^2} \frac{\partial^3 g^{\mu\nu}}{\partial v^\alpha \partial v^\beta \partial u^\mu} \frac{\partial f_2^2}{\partial u^\alpha} y_i \right\},
\]

\[
D^\gamma_{ij\lambda} = -\frac{1}{2f_1^2} g_{ij} \frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} - \frac{2}{n+1} \left\{ E_{ij} \delta_\lambda^\gamma - \frac{1}{8f_2^2} \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_2^2}{\partial x^h} F_1^2 \delta_\lambda^\gamma \\
- \frac{1}{4f_1^2} g_{ij} \frac{\partial g^{\mu\nu}}{\partial v^\mu} \frac{\partial f_2^2}{\partial u^\nu} \delta_\lambda^\gamma - \frac{1}{8f_2^2} \frac{\partial^3 g^{kh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_2^2}{\partial x^h} v^\gamma \\
- \frac{1}{4f_1^2} g_{ij} \frac{\partial^2 g^{\mu\nu}}{\partial v^\alpha \partial v^\beta} \frac{\partial f_2^2}{\partial u^\alpha} v^\gamma \right\},
\]

\[
D^\gamma_{ijk} = -\frac{1}{f_1^2} C_{ijk} g^{\alpha\gamma} \frac{\partial f_2^2}{\partial u^\alpha} - \frac{2}{n+1} \left\{ \frac{\partial E_{ijk}}{\partial y^k} v^\gamma \\
- \frac{1}{8f_2^2} \frac{\partial^4 g^{sh}}{\partial y^k \partial y^j \partial y^i \partial y^s} \frac{\partial f_2^2}{\partial x^h} F_1^2 v^\gamma - \frac{1}{4f_1^2} \frac{\partial g_{ij}}{\partial v^\mu} \frac{\partial f_2^2}{\partial u^\alpha} v^\gamma \right\}.
\]

**Proof.** By lengthy calculations using Lemmas 3.1 and 4.1. 

**Theorem 5.2.** Every proper DWP-Finsler manifold \((f_1 M_1 \times f_2 M_2, F)\) with vanishing Douglas curvature is a Riemannian manifold.
Proof. Suppose that the Douglas curvature of \((f_2M_1 \times f_1M_2, F)\) vanishes, i.e., \(D^d_{abc} = 0\). Then by contracting (5.8) with \(y^k\) we obtain

\[
E_{ij} = \frac{3}{8f_2^2} \frac{\partial^3 g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} F^2_2.
\]

Since \(E_{ij}\) is a function of \((x, y)\), by differentiating the above equation with respect to \(v^\alpha\), we get

\[
\frac{\partial^3 g^{sh}}{\partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} = 0.
\]

Putting the above into (5.9) gives \(E_{ij} = 0\). Further, (5.10) implies

\[
\frac{\partial^4 g^{sh}}{\partial y^k \partial y^i \partial y^j \partial y^s} \frac{\partial f_1^2}{\partial x^h} = 0.
\]

In a similar way, we conclude that \(E_{ij} = 0\) and

\[
\frac{\partial^4 g^{\gamma \nu}}{\partial \nu^\lambda \partial \nu^2 \partial \nu^\alpha \partial \nu^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = \frac{\partial^3 g^{\gamma \nu}}{\partial \nu^\lambda \partial \nu^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = \frac{\partial^2 g^{\gamma \nu}}{\partial \nu^\alpha} \frac{\partial f_2^2}{\partial u^\nu} = \frac{\partial g^{\gamma \nu}}{\partial \nu^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

Inserting (5.10)–(5.12) and \(E_{ij} = E_{ij} = 0\) into (5.4) and (5.8) implies that \(C_{ijk} = C_{\alpha \beta \lambda} = 0\). Therefore \((M_1, F_1)\) and \((M_2, F_2)\) are Riemannian, and consequently \((f_2M_1 \times f_1M_2, F)\) is Riemannian.

From Theorem 5.2, we obtain

**Corollary 5.3.** Let \((f_2M_1 \times f_1M_2, F)\) be a DWP-Finsler manifold.

(i) If \(f_2\) is constant on \(M_2\), then \(F\) is a Douglas metric if and only if \(F_2\) is a Riemannian metric, \(F_1\) is a Berwald metric and \(\frac{\partial g^{sh}}{\partial y^i} \frac{\partial f_1^2}{\partial x^h} = 0\).

(ii) If \(f_1\) is constant on \(M_1\), then \(F\) is a Douglas metric if and only if \(F_1\) is a Riemannian metric, \(F_2\) is a Berwald metric and \(\frac{\partial g^{\gamma \nu}}{\partial \nu^\gamma} \frac{\partial f_2^2}{\partial u^\nu} = 0\).

Finally, we consider warped product Finsler manifolds with vanishing Douglas curvature:

**Corollary 5.4.** The WP-Finsler manifold \((M_1 \times f_1M_2, F)\) is a Douglas manifold if and only if \(F_2\) is a Riemannian metric, \(F_1\) is a Berwald metric and \(\frac{\partial g^{sh}}{\partial y^i} \frac{\partial f_1^2}{\partial x^h} = 0\).

Proof. By Lemma 5.1.

**6. Relatively isotropic Landsberg DWP-Finsler manifolds.** In this section, we prove that for a proper DWP-Finsler manifold the notions of being a Landsberg manifold and of being a Berwald manifold are equivalent. Then we study DWP-Finsler metrics with relatively isotropic Landsberg curvature.
**Lemma 6.1.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a DWP-Finsler manifold. Then the Landsberg curvature of \(F\) is as follows:

\[
L_{ijk} = f_2^2 L_{ijk} + \frac{1}{8} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2 + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha},
\]

\[
L_{ij\lambda} = \frac{1}{4} y_l \frac{\partial^2 g^{lh}}{\partial y^i \partial y^j} \frac{\partial f_1^2}{\partial x^h} v_\lambda + \frac{1}{4} g_{ij} v_\gamma \frac{\partial g^{\alpha\gamma}}{\partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha},
\]

\[
L_{i\beta\lambda} = \frac{1}{4} y_l \frac{\partial g^{lh}}{\partial y^i} \frac{\partial f_1^2}{\partial x^h} g_{\beta\lambda} + \frac{1}{4} v_\gamma \frac{\partial^2 g^{\alpha\gamma}}{\partial v^\beta \partial v^\lambda} \frac{\partial f_2^2}{\partial u^\alpha} y_i,
\]

\[
L_{\alpha\beta\lambda} = f_1^2 L_{\alpha\beta\lambda} + \frac{1}{8} f_1^2 v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial u^\alpha \partial v^\beta \partial v^\lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 + \frac{1}{2} C_{\alpha\beta\lambda} y^h \frac{\partial f_2^2}{\partial x^h}.
\]

**Proof.** Use the definition of Landsberg curvature and (3.1)–(3.8).

**Proposition 6.2.** Every proper DWP-Finsler manifold \((f_2 M_1 \times f_1 M_2, F)\) with vanishing Landsberg curvature is Riemannian.

**Proof.** Let the Landsberg curvature tensor of \((f_2 M_1 \times f_1 M_2, F)\) be zero. Then by using (6.1) we obtain

\[
f_2^2 L_{ijk} + \frac{1}{8} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2 + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha} = 0.
\]

Differentiating (6.5) with respect to \(v^\gamma\) implies that

\[
\frac{1}{4} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} v_\gamma + \frac{1}{2} C_{ijk} \frac{\partial f_2^2}{\partial v^\gamma} = 0.
\]

By differentiating (6.6) with respect to \(v^\lambda\), we have

\[
y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} g_{\gamma\lambda} = 0,
\]

and consequently

\[
y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} = 0.
\]

Thus (6.5) reduces to

\[
f_2^2 L_{ijk} + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha} = 0.
\]

Differentiating (6.7) with respect to \(v^\beta\) gives

\[
C_{ijk} \frac{\partial f_2^2}{\partial u^\beta} = 0,
\]

and consequently \(C_{ijk} = 0\). Thus \((M_1, F_1)\) is a Riemannian manifold. In a similar way, we can conclude that \((M_2, F_2)\) is Riemannian.

Using Proposition 6.2 and Lemma 3.1, we get
Theorem 6.3. A proper DWP-Finsler manifold is Landsbergian if and only if it is Berwaldian.

Now, let $f_1$ be non-constant on $M_1$ and $f_2$ be constant on $M_2$. Then as in the proof of Proposition 6.2, we conclude that $(M_2, F_2)$ is a Riemannian manifold. Also, from (6.7) we conclude $L_{ijk} = 0$, because $f_2$ is constant. Thus we obtain

**Theorem 6.4.** Let $(f_2M_1 \times f_1M_2, F)$ be a proper DWP-Finsler manifold.

(i) If $f_2$ is constant and $f_1$ is not constant, then $(f_2M_1 \times f_1M_2, F)$ is a Landsberg manifold if and only if $(M_1, F_1)$ is a Landsberg manifold, $(M_2, F_2)$ is Riemannian and

$$y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1}{\partial x^h} = 0. \quad (6.8)$$

(ii) If $f_1$ is constant and $f_2$ is not constant, then $(f_2M_1 \times f_1M_2, F)$ is a Landsberg manifold if and only if $(M_2, F_2)$ is a Landsberg manifold, $(M_1, F_1)$ is Riemannian and

$$v_\gamma \frac{\partial^3 g^{\gamma\nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \frac{\partial f_2}{\partial u^\nu} = 0. \quad (6.9)$$

Theorem 6.4 yields

**Corollary 6.5.** A WP-Finsler manifold $(M_1 \times f_1M_2, F)$ is a Landsberg manifold if and only if $(M_1, F_1)$ is Landsberg, $(M_2, F_2)$ is Riemannian and

$$C_{hij} \frac{\partial f_1}{\partial x^h} = 0. \quad (6.10)$$

**Proof.** It suffices to show that (6.4) implies (6.10). Multiplying (6.4) with $y^i$ implies that

$$y_l \frac{\partial C_{hij}^{\ell}}{\partial y^k} \frac{\partial f_1}{\partial x^h} = 0. \quad (6.11)$$

Using $y_l C_{hij}^{\ell} = 0$ and $\frac{\partial y^l}{\partial y^k} = g_{lk}$, one can obtain (6.10).

Now, we deal with DWP-Finsler manifolds with relatively isotropic Landsberg metric.

**Theorem 6.6.** Let $(f_2M_1 \times f_1M_2, F)$ be a DWP-Finsler manifold. Suppose that $F$ is a relatively isotropic Landsberg metric. Then $F$ is a Landsberg metric.

**Proof.** Let $(f_2M_1 \times f_1M_2, F)$ be a relatively isotropic Landsberg manifold. Then by (6.1), we have

$$f_2^2 L_{ijk} + \frac{1}{8} f_2^2 y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2^2 + \frac{1}{2} C_{ijk} v^\alpha \frac{\partial f_2^2}{\partial u^\alpha} = c F f_2^2 C_{ijk}. \quad (6.12)$$
By differentiating \((6.12)\) with respect to \(v^\gamma\) and \(v^\lambda\), one obtains
\[
(6.13) \quad \frac{1}{4} y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} g_{\gamma \lambda} = cF^{-1} h_{\gamma \lambda} C_{ijk}.
\]
Contracting \((6.13)\) with \(g^{\gamma \lambda}\) implies that
\[
(6.14) \quad \frac{n}{4} y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} = (n - 1)cF^{-1} C_{ijk}.
\]
Differentiating \((6.14)\) with respect to \(v^\beta\) gives
\[
(n - 1)c \left( F^{-1} \right) v^\beta C_{ijk} = 0,
\]
and so \(c = 0\). Thus \(F\) reduces to a Landsberg metric. \(\blacksquare\)

From Proposition 6.2 and Theorem 6.6, we deduce

**Corollary 6.7.** Every proper DWP-Finsler manifold with relatively isotropic Landsberg curvature is Riemannian.

**7. Relatively isotropic mean Landsberg DWP-Finsler manifolds.**

In this section, we consider DWP-Finsler metrics with relatively isotropic mean Landsberg curvature. First, by the definition of mean Landsberg curvature and Lemma 6.1, we get

**Lemma 7.1.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a DWP-Finsler manifold. Then the mean Landsberg curvature of \(F\) is as follows:
\[
(7.1) \quad J_i = \frac{1}{f_2} g^{jk} L_{ijk} + \frac{1}{f_1} g^{\beta \lambda} L_{i \beta \lambda} = J_i + \frac{y_l g^{jk}}{8} \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} \frac{\partial f_1^2}{\partial x^h} F_2 + \frac{I_i v^\nu \frac{\partial f_1^2}{\partial x^h}}{f_2^2} \frac{\partial^2 g^{\nu \gamma}}{\partial y^i \partial y^j} \frac{\partial f_1^2}{\partial x^h} y_i,
\]
\[
(7.2) \quad J_\alpha = \frac{1}{f_2} g^{jk} L_{\alpha jk} + \frac{1}{f_1} g^{\beta \lambda} L_{\alpha \beta \lambda} = J_\alpha + \frac{v_\gamma g^{\beta \lambda}}{8} \frac{\partial^3 g^{\nu \gamma}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \frac{\partial f_1^2}{\partial x^h} F_2 + \frac{I_\alpha v^h \frac{\partial f_1^2}{\partial x^h}}{f_2^2} \frac{\partial^2 g^{h \gamma}}{\partial y^i \partial y^j} \frac{\partial f_1^2}{\partial x^h} v_\alpha.
\]

**Theorem 7.2.** Let \((f_2 M_1 \times f_1 M_2, F)\) be a DWP-Finsler manifold.

(i) If \(f_1\) is constant and \(f_2\) is not constant, then \((f_2 M_1 \times f_1 M_2, F)\) is a weakly Landsberg manifold if and only if \((M_1, F_1)\) is Riemannian, \((M_2, F_2)\) is weakly Landsberian and
\[
v_\gamma \frac{\partial^3 g^{\nu \gamma}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta \lambda} \frac{\partial f_1^2}{\partial v^\nu} = 0.
\]

(ii) If \(f_2\) is constant and \(f_1\) is not constant, then \((f_2 M_1 \times f_1 M_2, F)\) is a weakly Landsberg manifold if and only if \((M_1, F_1)\) is weakly Landsberian, \((M_2, F_2)\) is Riemannian and
\[
y_l \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} g^{ik} \frac{\partial f_1^2}{\partial x^h} = 0.
\]
Proof. Let \((f_2M_1 \times f_1M_2, F)\) be a weakly Landsberg manifold and \(f_1\) be constant on \(M_1\). Then by (7.1) and (7.2), we have

\[
J_i + \frac{1}{2f_2^2} I_i v^\nu \frac{\partial f_2^2}{\partial u^\nu} + \frac{1}{4} v_\gamma \frac{\partial^2 g^{\gamma \nu}}{\partial v^\beta \partial v^\lambda} g^{\beta \lambda} \frac{\partial f_2^2}{\partial u^\nu} y_i = 0,
\]

(7.3)

\[
J_\alpha + \frac{1}{8} v_\gamma \frac{\partial^3 g^{\gamma \nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta \lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 = 0.
\]

(7.4)

By differentiating (7.4) with respect to \(y^i\), we get

\[
v_\gamma \frac{\partial^3 g^{\gamma \nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta \lambda} \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

(7.5)

Contracting (7.5) with \(v^\alpha\) gives

\[
v_\gamma \frac{\partial^2 g^{\gamma \nu}}{\partial v^\beta \partial v^\lambda} g^{\beta \lambda} \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

(7.6)

Inserting (7.6) into (7.3) implies that

\[
J_i + \frac{1}{2f_2^2} I_i v^\nu \frac{\partial f_2^2}{\partial u^\nu} = 0.
\]

(7.7)

By differentiating (7.7) with respect \(v^\beta\), we conclude that \(I_i = 0\), i.e., \((M_1, F_1)\) is a Riemannian manifold. By inserting (7.5) into (7.2), we get \(J_\alpha = 0\), i.e., \((M_2, F_2)\) is a weakly Landsberg manifold.

From Theorem 7.2, we deduce

**Corollary 7.3.** A proper WP-Finsler manifold \((M_1 \times f_1M_2, F)\) is a weakly Landsberg manifold if and only if \((M_1, F_1)\) is weakly Landsberg, \((M_2, F_2)\) is Riemannian and

\[
y_i \frac{\partial^3 g^{lh}}{\partial y^i \partial y^j \partial y^k} g^{jk} \frac{\partial f_1^2}{\partial x^h} = 0.
\]

Now, we consider DWP-Finsler manifolds with relatively isotropic mean Landsberg curvature.

**Theorem 7.4.** Let \((f_2M_1 \times f_1M_2, F)\) be a DWP-Finsler manifold with relatively isotropic mean Landsberg curvature. If \(f_1\) is constant on \(M_1\) (resp. \(f_2\) is constant on \(M_2\)), then the DWP-Finsler manifold is a weakly Landsberg manifold.

**Proof.** Let \((f_2M_1 \times f_1M_2, F)\) be a relatively isotropic mean Landsberg manifold and \(f_1\) be constant on \(M_1\). Then by (7.4) we have

\[
J_\alpha + \frac{1}{8} v_\gamma \frac{\partial^3 g^{\gamma \nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta \lambda} \frac{\partial f_2^2}{\partial u^\nu} F_1^2 = cFI_\alpha.
\]

(7.8)

Differentiating (7.8) with respect to \(y^k\) implies that

\[
\frac{1}{4} v_\gamma \frac{\partial^3 g^{\gamma \nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} g^{\beta \lambda} \frac{\partial f_2^2}{\partial u^\nu} y_k = c \frac{f_2^2 y_k}{F} I_\alpha.
\]

(7.9)
Contracting (7.9) with $y^k$ gives

\begin{equation}
\frac{1}{4} v_\gamma \frac{\partial^3 g^{\gamma \nu}}{\partial v^\alpha \partial v^\beta \partial v^\lambda} y^\beta y^\alpha \frac{\partial f^2_2}{\partial u^\nu} F^2_1 = c \frac{f^2_2 F^2_1}{F} I_\alpha.
\end{equation}

By inserting (7.10) into (7.8), it follows that

\begin{equation}
2J_\alpha + c \left( \frac{f^2_2 F^2_1}{F} - 2F \right) I_\alpha = 0.
\end{equation}

By differentiating (7.11) with respect to $y^k$, we obtain $c f^2_2 F^2_1 y^k I_\alpha = 0$. Therefore, $c = 0$ and $F$ reduces to a weakly Landsberg metric.

**Corollary 7.5.** Every WP-manifold $(M_1 \times f_1 M_2, F)$ with relatively isotropic mean Landsberg curvature is a weakly Landsberg manifold.

**8. Locally dually flat DWP-Finsler manifolds.** Dually flat Finsler metrics form a special and useful class of Finsler metrics in Finsler information geometry, which play an important role in studying flat Finsler information structure. In this section, we study locally dually flat DWP-Finsler metrics. We recall that a Finsler metric $F = F(x, y)$ on a manifold $M$ is locally dually flat if at any point there is a standard coordinate system $(x^i, y^j)$ in $TM$ such that

\begin{equation}
\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k = 2 \frac{\partial F^2}{\partial x^l}.
\end{equation}

In this case, the coordinate system $(x^i)$ is called adapted.

**Theorem 8.1.** Let $(f_2 M_1 \times f_1 M_2, F)$ be a DWP-Finsler manifold. Then $F$ is locally dually flat if and only if $F_1$ and $F_2$ are locally dually flat and $f_1$ and $f_2$ are constant.

**Proof.** Let $(f_1 M_1 \times f_2 M_2, F)$ be a locally dually flat doubly DWP-Finsler manifold. Then

\begin{equation}
f^2_2 \frac{\partial^2 F^2_1}{\partial x^k \partial y^l} y^k + \frac{\partial f^2_2}{\partial u^\alpha} \frac{\partial F^2_1}{\partial y^l} v^\alpha = 2 f^2_2 \frac{\partial F^2_1}{\partial x^l},
\end{equation}

\begin{equation}
\frac{\partial f^2_2}{\partial x^k} \frac{\partial F^2_2}{\partial v^\beta} y^k + f^2_1 \frac{\partial^2 F^2_2}{\partial u^\alpha \partial v^\beta} v^\alpha = 2 f^2_2 \frac{\partial F^2_2}{\partial u^\beta} + 2 \frac{\partial f^2_2}{\partial u^\beta} F^2_1.
\end{equation}

Differentiating (8.2) with respect to $v^\gamma$ and then with respect to $y^k$ and using non-singularity of $g_{ij}$ yields

$$\frac{\partial f_2}{\partial u^\gamma} = 0.$$
which means that $f_2$ is constant. Similarly, $f_1$ is constant. In this case, (8.2) and (8.3) reduce to

\[
\frac{\partial^2 F_1^2}{\partial x^k \partial y^l} y^k = \frac{\partial F_1^2}{\partial x^l},
\]

(8.4)

\[
\frac{\partial^2 F_2^2}{\partial u^\alpha \partial u^\beta} v^\alpha = 2 \frac{\partial F_2^2}{\partial u^\beta}.
\]

(8.5)

Hence $F_1$ and $F_2$ are locally dually flat.

From Theorem 8.1, we deduce

**Corollary 8.2.** There is no locally dually flat proper DWP-Finsler manifold.

**References**


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