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Numerical solution of the parabolic integro-differential equations with an integral boundary condition

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Abstract

In this paper, the solving of the parabolic integro-differential equations with an integral boundary condition is investigated. Then we develop the operational Tau method with standard basis for obtaining a numerical solution. Finally, the accuracy of the method is verified by presenting some numerical computations.

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1 Introduction

Consider the parabolic integro-differential equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = Ku(x,t) + f(x,t), \qquad (x,t) \in (0,1) \times [0,T] \quad (1)$$

with the supplementary conditions

$$u(x,0) = u_0(x), \qquad x \in [0,1] \quad (2)$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad t \in [0,T] \quad (3)$$

and an integral boundary condition

$$\int_0^1 u(x,t)dx = 0, \qquad (x,t) \in (0,1) \times [0,T] \quad (4)$$

where K has the following form

$$Ku(x,t) = \int_0^t k(t-s)u(x,s)ds \tag{5}$$

in which k is a continuous function and f is defined from $(0,1) \times [0,T]$ into R.
The motivation behind such problems lies in different fields such as physics, rheology and especially the theory of heat conduction in materials where the inner heat sources are of a special type. Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems [1].
In 1963, nonlocal boundary condition was presented by Cannon and Batten. Then, parabolic initial-boundary problems with nonlocal integral conditions for parabolic equations were investigated by Kamynin, (1964) and Ionkin, (1977).
For the boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to the papers [2-8]. Many methods have been used to solve such problems as the functional methods, methods of approximation, a priori estimates and etc.
Cannon and Van Der Hoek [2] studied the existence, uniqueness and continuous dependence on the data in the heat diffusion equation with an integral boundary condition and drove an equivalent Volterra integral equation of the second kind and treat the problem numerically. The authors of [3] used the Galerkin method and analyzed the numerical solution of the heat equation with an integral condition. In Ref. [6, 7], a linear heat equation and a semilinear heat equation with two integral boundary conditions was studeid by N. Merazga and A. Bouziani. Guezane-Lakoud and Chaoui [9], interested in the study of hyperbolic integro-differential equation with initial and integral conditions by using Rothes method. A numerical technique was developed for the one-dimensional convection-diffusion equation with classical and integral boundary conditions by Soltanalizadeh in Ref. [10].
Existence and uniqueness of the solution of the parabolic integro-differential equation with an integral boundary condition was investigated in Ref. [1].
In this work, we develop the operational Tau method to present the numerical solution of Eq. (1). To this end, we formulate a matrix Tau method for parabolic integro-differential equation with an integral boundary condition.

The rest of this paper is organized as follows:
In Section 2, we brifly describe Tau method. In Section 3, we convert the

problem to a system of algebraic equation. In Section 4, we give some examples to show the accuracy and efficiency of the presented method.

2 Some preliminary results of the Tau method

We recall from [11] that the Tau method is based on the following three simple matrices

$$\iota = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with the following properties:

if $y(x) = \mathbf{a}^t\mathbf{X}$, where $\mathbf{a}^t = (a_0, a_1, \ldots, a_n, 0, \ldots, 0)$, $\mathbf{X}^t = (1, x, x^2, \ldots)$, then

$$\frac{d}{dx} y(x) = \mathbf{a}^t \eta \mathbf{X}, \tag{6}$$

$$x y(x) = \mathbf{a}^t \mu \mathbf{X}, \tag{7}$$

and

$$\int y(x) dx = \mathbf{a}^t \iota \mathbf{X}. \tag{8}$$

We consider the approximate solution of the above problem in the following form:

$$u_N(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} C_{ij} x^i t^j = \underline{X}^T C \underline{T} \tag{9}$$

where $\underline{X} = (1, x, x^2, \cdots, x^N)^T$, $\underline{T} = (1, t, t^2, \cdots, t^N)^T$ are basis vectors

and C is an $(N+1) \times (N+1)$ matrix containing the unknown values

C_{ij} :

$$C = \begin{pmatrix} C_{00} & C_{01} & \cdots & C_{0N} \\ C_{10} & C_{11} & \cdots & C_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ C_{N0} & C_{N1} & \cdots & C_{NN} \end{pmatrix} \tag{10}$$

3 Formulation of the problem

To convert the problem to the matrix form, we replace the differential and integral parts of Eq. (1) by their matrix representation and then convert it to a corresponding system of algebraic equations. In a similar manner, we transform the supplementary conditions to an algebraic system of equations. Finally, by combining these two systems of algebraic equations, we obtain a system of algebraic equations and solve it to obtain an approximate solution of the problem.
To convert the differential part of the problem to the matrix form, we recall the following lemma from [12].

Lemma 3.1 *For $u(x,t)$ defned by (9); we have*

a) $\dfrac{\partial^2 u(x,t)}{\partial x^2} = \underline{X}^T(\eta^T)^2 C\underline{T}$

b) $\dfrac{\partial^2 u(x,t)}{\partial t^2} = \underline{X}^T C\eta^2\underline{T}$

c) $\dfrac{\partial^2 u(x,t)}{\partial x\partial t} = \underline{X}^T\eta^T C\eta\underline{T}$

d) $\dfrac{\partial u(x,t)}{\partial x} = \underline{X}^T\eta^T C\underline{T}$

e) $\dfrac{\partial u(x,t)}{\partial t} = \underline{X}^T C\eta\underline{T}.$

Also to convert the integral part of the problem (1) to the matrix form, we have

$$\begin{aligned} \int_0^t (t-s)u(x,s)ds &= \int_0^t tu(x,s)ds - \int_0^t su(x,s)ds \\ &= \int_0^t t\underline{X}^T CSds - \int_0^t s\underline{X}^T CSds \\ &= t\underline{X}^T C\iota T - \underline{X}^T C\int_0^t \mu Sds \\ &= \underline{X}^T C\iota\mu T - \underline{X}^T C\mu\iota T \end{aligned} \tag{11}$$

We can also write the function $f(x,t)$ in the following form:

$$f(x,t) = \underline{X}^T F \underline{T} \tag{12}$$

where $F = [f_{ij}]$ is an $(N+1) \times (N+1)$ matrix.
By substituting from Lemma 3.1.a, b, relations (11) and (12) into (1) we obtain

$$\underline{X}^T C\eta\underline{T} - \underline{X}^T(\eta^T)^2 C\underline{T} - (\underline{X}^T C\iota\mu T - \underline{X}^T C\mu\iota T) = \underline{X}^T F\underline{T} \tag{13}$$

since $\underline{X}^T$ and $\underline{T}$ are basis vectors, we have

$$C\eta - (\eta^T)^2 C - (C\iota\mu - C\mu\iota) = F. \tag{14}$$

Using the given relations in refrence [12] to convert the supplementary conditions (2) and (3) to the matrix form, we do the following.
Assume that the $u_0(x)$ has the following form

$$u_0(x) = \sum_{i=0}^{N} x^i \alpha_i = \underline{X}^T \alpha \tag{15}$$

therefore, Eq. (2) can be written as the following matrix form

$$\underline{X}^T C\underline{T_0} = \underline{X}^T \alpha$$

and since $\underline{X}^T$ is basis vector, we have

$$C\underline{T_0} = \alpha \tag{16}$$

where $\underline{T_0} = \underline{T} \mid_{t=0}$.
To convert the supplementary condition (3) to the matrix form, we have

$$\underline{X_0}^T \eta^T C\underline{T} = 0$$

and since $\underline{T}$ is a basis vector, we have

$$\underline{X_0}^T \eta^T C = 0 \tag{17}$$

where $\underline{X_0} = \underline{X} \mid_{x=0}$.
To convert the integral condition (4) to the matrix form, we have

$$\int_0^1 \underline{X}^T C\underline{T}dx = \left(\int_0^1 \underline{X}^T dx\right) C\underline{T} = \underline{X}^T \iota^T|_0^1 C\underline{T} = (\underline{X_1}^T \iota^T - \underline{X_0}^T \iota^T)CT$$

$$(\underline{X_1}^T \iota^T - \underline{X_0}^T \iota^T)CT = 0$$

and since $\underline{T}$ is a basis vector, we have

$$(\underline{X_1}^T \iota^T - \underline{X_0}^T \iota^T)C = 0 \tag{18}$$

By solving the system of equations (16), (17) and (18) simultaneously, we obtain $u_N(x,t)$ from (9).

Remark 3.1 *Note that the final system must include $(N+1)^2$ equations to find $C_{ij}s$. To form these equations, we consider all of the equations obtained from the conditions and select the remainder from (14).*

Remark 3.2 *Note that in Tau method, $k(t,s)$ and $f(x,t)$ should be polynomials, whenever $k(t,s)$ or $f(x,t)$ are not polynomyals, they should be approximated by suitable polynomyals.*

4 Numerical examples

Here are some examples to show the simplicity and accuracy of the presented method. In the following examples, we approximate non-polynomial parts of the known function by Taylor polynomyals.

Example 4.1 *Consider the following integro-differential parabolic equation*

$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \int_0^1 (t-s)u(x,s)ds + (-sin(\pi t)\pi^3 + 3sin(\pi t)\pi^3 x^2 + 6\pi^2 cos(\pi t) - 1 + 3x^2 + cos(\pi t) - 3cos(\pi t)x^2)/\pi^2 \qquad (x,t) \in (0,1) \times [0,T]$

with conditions

$u(x,0) = (1-3x^2) \qquad x \in [0,1]$

$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t \in [0,T]$

$\int_0^1 u(x,t)dx = 0 \qquad (x,t) \in (0,1) \times [0,T]$

As mentioned in the previous Section, first we approximated nonpolynomial parts by a Taylor series. Then the integro-differential parabolic equation with given conditions was given in the matrix form using the Tau method.
Computational results in Table 1 show that high accuracy is achieved for $N = 8, 12$ at some arbitrary points.

Example 4.2 *As the second example consider the following integro-differential parabolic problem*

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \int_0^1 (t-s)u(x,s)ds + 6e^t + t + 1 - 3x^2 t - 3x^2 \quad (x,t) \in (0,1) \times [0,T]$$

with supplementary conditions as

$$u(x,0) = (1-3x^2) \qquad x \in [0,1]$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t \in [0,T]$$

$$\int_0^1 u(x,t)dx = 0 \qquad (x,t) \in (0,1) \times [0,T]$$

We procced as in previous example and obtain the results of Table 2 at some nodes with $N = 8, 12$.

$N=8$			
(x,t)	$u(x,t)$	$u_N(x,t)$	*error*
$(0.2, 0.2)$	0.7119349550	0.7119349573	$-0.231857e-8$
$(0.3, 0.3)$	0.4290832341	0.4290833446	$-0.110497e-6$
$(0.4, 0.4)$	0.1606888370	0.1606902275	$-0.139045e-5$
$(0.7, 0.4)$	-0.1452379873	-0.1452392441	$0.125675e-5$
$(0.8, 0.9)$	0.8749719949	0.8749719949	$7.79837e-3$
$(1, 1)$	1.952044425	2.00	$4.79555e-2$
$N=12$			
$(0.2, 0.2)$	0.7119349550	0.7119349550	$-0.150629e-13$
$(0.3, 0.3)$	0.4290832341	0.4290832341	$-0.364028e-11$
$(0.4, 0.4)$	0.1606888370	0.160688837	$-0.145116e-9$
$(0.7, 0.4)$	-0.14523798	-0.145237987	$0.131104e-9$
$(0.8, 0.9)$	0.8749719949	0.8749506871	$2.13077e-5$
$(1, 1)$	1.99979905166	2.00	$2.00948e-4$

Table 1: Computational results of example 4.1 for $N = 8, 12$ at some nodes.

$N=8$			
(x,t)	$u(x,t)$	$u_N(x,t)$	$error$
$(0.0,0.0)$	1.0	1.0	0.0
$(0.2,0.2)$	1.074834427	1.074834427	$0.268293e-11$
$(0.3,0.4)$	1.089032029	1.089032027	$0.135198e-8$
$(0.4,0.4)$	0.7757488427	0.7757488408	$0.188629e-8$
$(0.7,0.4)$	-0.7011576078	-0.7011576109	$0.306633e-8$
$(0.8,0.9)$	-2.262834862	-2.262834317	$0.544889e-6$
$(1,1)$	-5.436563656	-5.436535591	$2.806528e-5$
$N=12$			
$(0.0,0.0)$	1.0	1.0	0.0
$(0.2,0.2)$	1.074834427	1.074834427	$0.8e-18$
$(0.3,0.4)$	1.089032029	1.089032029	$0.58086e-14$
$(0.4,0.4)$	0.775748842	0.775748842	$0.960522e-14$
$(0.7,0.4)$	-0.701157607	-0.701157607	$0.361206e-13$
$(0.8,0.9)$	-2.262834862	-2.262834862	$0.173072e-9$
$(1,1)$	-5.436563656	-5.436563649	$-0.733893e-8$

Table 2: Computational results of example 4.2 for $N=8,12$ at some nodes.

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