Monitoring Variability of Multivariate-attribute Processes Using EWMA Control Charts Based on NORTA Inverse Technique

Mohammad Reza Maleki [1], Amirhossein Amiri[1], Mohsen Rasouli [2]
[1] Department of Industrial Engineering, Shahed University, Tehran, Iran
(amiri@shahed.ac.ir)
[2] Faculty of Management, Islamic Azad University, Central Tehran Branch, Tehran, Iran

Abstract - **Sometimes, the quality of a product or a process is expressed by both variable and attributes quality characteristics. The simplest approach for monitoring such processes is using separate control charts for each quality characteristic. However, this approach is not preferable in the cases that the quality characteristics are correlated. Because not only the overall probability of Type I error increases but also the correlation structure between quality characteristics is neglected. To best of our knowledge, only few methods have been proposed to monitor the mean vector of such correlated variable and attribute quality characteristics and no method is available for monitoring the variance-covariance matrix of these quality characteristics. In this paper, first we use normal distribution to anything (NORTA) inverse transformation technique to transform the distribution of the correlated variable and attribute quality characteristics to standardized multivariate normal distribution. Then, we apply another transformation to make the quality characteristics independent. Then, we develop two exponentially weighted moving average (EWMA) control charts, first proposed for monitoring the variability of multivariate processes, and apply them to monitor the variability of multivariate-attribute processes. The performance of the these EWMA-based control charts in detecting different magnitudes of step shifts in the variance of multivariate-attribute quality characteristics is evaluated in terms of average run length (ARL) criterion through simulation studies. In addition, the performance of the developed methods is compared with asymptotic normal approximation method proposed by Montgomery and Wadsworth (1972). The results of simulation studies show the adequate performance of the developed EWMA-based control charts. Moreover, the developed EWMA control charts outperform the asymptotic normal approximation method under different magnitudes of step shifts in variance of multivariate-attribute quality characteristics.**

Keywords - Multivariate-attribute process, Exponentially weighted moving average (EWMA) control chart, Average run length; NORTA inverse method, Variance-covariance matrix.

I. INTRODUCTION

In some production environments, the quality of a product or a process is expressed by only one variable or one attribute quality characteristic. In these situations, some common control charts such as $\overline{X}/R$, $\overline{X}/S$, p, c and u are proposed for monitoring variable or attribute quality characteristic. For more information about univariate control charts refer to Montgomery (2005). Woodall (1997) also reviewed some methods for monitoring attribute quality characteristics.

In some situations, more than one variable or attribute quality characteristics represent the quality of a product or a process. One approach to monitor multivariate or multi-attribute processes is to control each quality characteristic by separate control charts. However, this approach is not preferable because the quality characteristics are correlated. Since this approach does not consider the correlation structure between quality characteristics, therefore the probability of Type 1 error will increase. Hence, some multivariate and multi-attribute control charts are proposed by many authors separately. The first method for monitoring multivariate quality characteristics is proposed by Hotelling (1947). Bersimis et al. (2007) have reviewed most usual multivariate control charts. Topalidou and Psarakis (2009) have also reviewed multi-attribute control charts.

Sometimes, the quality of a product or a process is characterized by the combination of correlated variable and attributes quality characteristics. For example in the production process of LED lamps, the number of nonconformities on a product and its weight has Poisson and normal distribution, respectively. To best of our knowledge, there are only few methods for monitoring correlated variable and attribute quality characteristics. Kang and Brenneman (2010) proposed a method to determine the confidence bound of defect rate for variable and attribute quality characteristics. However, they considered the quality characteristics independent. Doroudyan and Amiri (2011) proposed a method for monitoring correlated variable and attribute quality characteristics based on the root transformation technique. Maleki et al. (2012) proposed an artificial neural network for monitoring multivariate-attribute quality characteristics. Their proposed method not only could detect different shifts in the mean vector of quality characteristics, but also diagnose the quality characteristic(s) responsible for the out-of-control signals. They also compared the performance of the proposed neural network to root transformation method proposed by Doroudyan and Amiri (2011). In this paper, we extend two EWMA-based control charts first proposed by Memar and Niaki (2011) for monitoring the variability of multivariate processes and apply them for monitoring the variability of multivariate-attribute processes. To do that, first we transform the original data to standardized

multivariate normal distribution based on NORTA inverse method. After this transformation the marginal distribution of each quality characteristic is a standard normal distribution. Then by applying a transformation that is first proposed by Golnabi and Houshmand (1999), we transform the data to make the standardized normal quality characteristics independent. Afterwards we use two EWMA-based control charts including $MEWMS_{AS}$ and $MEWMS_{AT}$ to detect different magnitudes of step shifts in the variance(s) of multivariate-attribute of quality characteristics. We also evaluate the performance of these control charts in detecting different shifts in the variance(s) of quality characteristics and compare the results with asymptotic normal approximation method (ANA) proposed by Montgomery and Wadsworth (1972).

The structure of the paper is as follows: In section 2 the assumptions of our problem are defined. In section 3, we extend some control charts including the asymptotic normal approximation method, $MEWMS_{AS}$ and $MEWMS_{AT}$ methods and apply them for monitoring the variability of correlated multivariate attribute quality characteristics. Note that all of these methods are first proposed for monitoring the variability of multivariate quality characteristics. In section 4 we evaluate the performance of developed EWMA-based control charts in terms of out-of-control average run length criterion through a simulation study and compare the results with asymptotic normal approximation method. Finally section 5 is devoted to conclusions and feature researches.

2. ASSUMPTIONS

In this paper, we extend two EWMA-based control charts for monitoring the variability of multivariate quality characteristics and apply them for monitoring the variability of correlated multivariate-attribute quality characteristics. The extended methods are used in phase two. Hence, the mean vector and variance-covariance matrix of quality characteristics are known based on the historical data of the process. We assume that variable and attribute quality characteristics are correlated and the correlation value between the quality characteristics is stable and does not change during the process.

3. EXTENSION OF SOME MULTIVARIATE METHODS FOR MONITORING MULTIVARIATE-ATTRIBUTE PROCESSES

In this section we extend some statistical methods which are proposed for monitoring the variability of multivariate quality characteristics and apply them for monitoring the variability of multivariate-attribute quality characteristics.

3.1. EXTENDED ASYMPTOTIC NORMAL APPROXIMATION METHOD

Montgomery and Wadsworth (1972) proposed this approach for monitoring the variability of multivariate quality characteristics. In order to extend this approach for monitoring the variance-covariance matrix of correlated multivariate-attribute quality characteristics, the distribution of original data is transformed to standardized multivariate normal distribution using NORTA inverse technique. Note that NORTA inverse technique is a method for transforming the non-normal data to standardized normal data. For more information about the NORTA inverse technique, refer to Niaki and Abbasi (2009). In the asymptotic normal approximation (ANA) method the determinant of samples variance-covariance matrix ($|\mathbf{S}|$) are used as the control statistic. The l standard deviation values of upper control limits (UCL_{ANA}) and lower control limits (LCL_{ANA}) are determined as follows:

$$UCL_{\mathrm{ANA}} = \mathrm{E}\,|\mathbf{S}| + l\sqrt{\mathrm{Var}\,|\mathbf{S}|} \tag{1}$$

$$LCL_{\mathrm{ANA}} = \mathrm{E}\,|\mathbf{S}| - l\sqrt{\mathrm{Var}\,|\mathbf{S}|} \tag{2}$$

where the values of $\mathrm{E}\,|\mathbf{S}|$ and $\mathrm{Var}\,|\mathbf{S}|$ are computed as follows:

$$\mathrm{E}(|\mathbf{S}|) = b_1\,|\mathbf{\Sigma}| \tag{3}$$

$$\mathrm{Var}(|\mathbf{S}|) = b_2\,|\mathbf{\Sigma}|^2 \tag{4}$$

where $|\mathbf{\Sigma}|$ is the variance-covariance matrix of the transformed data. Hence the values of UCL_{ANA} and LCL_{ANA} are determined as:

$$UCL_{ANA} = |\mathbf{\Sigma}|\,(b_1 + l\sqrt{b_2}) \tag{5}$$

$$LCL_{ANA} = |\mathbf{\Sigma}|\,(b_1 - l\sqrt{b_2}) \tag{6}$$

Note that the value of LCL_{ANA} is considered equal to zero when it is less than zero. The values of b_1 and b_2 are computed as Equation (7) and (8), respectively.

$$b_1 = \frac{1}{(n-1)^{p+q}} \prod_{i=1}^{p+q} (n-i) \tag{7}$$

$$b_2 = \frac{1}{(n-1)^{2(p+q)}} \prod_{i=1}^{p+q} (n-i) \left[\prod_{j=1}^{p+q} (n-j+2) - \prod_{j=1}^{p+q} (n-j) \right] \tag{8}$$

Equation (3) showed that $\frac{|\mathbf{S}|}{b_1}$ is an unbiased estimator of $|\mathbf{\Sigma}|$ and this estimator is used in phase two.

3.2. EXTENDED $MEWMS_{AS}$ AND $MEWMS_{AT}$ CONTROL CHARTS

As explained in the Introduction section, Memar and Niaki (2011) proposed some EWMA-based control charts based on the squared deviation of observations from target for monitoring p-dimensional multivariate normal quality characteristics. In order to apply these methods for monitoring the variability of correlated multivariate-attribute quality characteristics, first the distribution of original data should be transformed to a multivariate normal distribution. In this paper, we use NORTA inverse method to transform the distribution of multivariate-attribute quality characteristics to a multivariate normal

distribution. After using NORTA inverse transformation, the joint distribution of the quality characteristics follows a standardized multivariate normal distribution with correlated quality characteristics. After that the correlated standardized normal quality characteristics should be independent. As noted, Golnabi and Houshmand (1999) proposed a transformation method to make dependent normal quality characteristics independent. Independent standardized normal quality characteristics are computed as follows:

$$\mathbf{x}_{tk} = \mathbf{\Sigma}_0^{-1/2}(\mathbf{y}_{tk} - \boldsymbol{\mu}_0) \tag{9}$$

where $\boldsymbol{\mu}_0$ and $\mathbf{\Sigma}_0$ are the estimators of the mean vector and variance-covariance matrix of quality characteristics, respectively and obtained in Phase I analysis, $\mathbf{y_{tk}} = (y_{tk1}, y_{tk2},..., y_{tkp})'$ is the tth transformed observation (based on NORTA inverse transformation) in kth subgroup where t=1,2,... and k=1,2,…,n. Note that y_{tkj} is jth element (quality characteristic) of matrix $\mathbf{y_{tk}}$ where j=1,2,…,p.

Both $MEWMS_{AS}$ and $MEWMS_{AT}$ control charts are based on the S_t statistic that is proposed by Yeh et al. (2005). Hence, in this paper we use the $\mathbf{S_t}$ statistic with smoothing parameter of λ according to the following equation:

$$\mathbf{S_t} = (1-\lambda)\mathbf{S_{t-1}} + \frac{\lambda}{n}\sum_{k=1}^{n}\mathbf{x_{tk}x'_{tk}},\ \mathbf{S_0} = \frac{1}{n}\sum_{k=1}^{n}\mathbf{x_{1k}x'_{1k}} \tag{10}$$

The control statistic of $MEWMS_{AS}$ statistic is sum of the elements of matrix $\mathbf{S_t}$ with the following control limits:

$$LCL_{MEWMS_{AS}} = \frac{p}{\nu}\chi^2_{1-\frac{\alpha_{AS}}{2}}(\nu) \tag{11}$$

$$UCL_{MEWMS_{AS}} = \frac{p}{\nu}\chi^2_{\frac{\alpha_{AS}}{2}}(\nu) \tag{12}$$

Where $\nu = (n(2-\lambda))/\lambda$.

The $MEWMS_{AT}$ uses the trace of matrix $\mathbf{S_t}$ as a control statistic. The upper and lower control limits of the statistic are as follows, respectively:

$$LCL_{MEWMS_{AT}} = \frac{p}{\nu}\chi^2_{1-\frac{\alpha_{AT}}{2}}(\nu) \tag{13}$$

$$UCL_{MEWMS_{AT}} = \frac{p}{\nu}\chi^2_{\frac{\alpha_{AT}}{2}}(\nu) \tag{14}$$

where $\nu = (np(2-\lambda))/\lambda$.

4. NUMERICAL EXAMPLE

In this section, we evaluate the performance of the extended EWMA-based methods under different shifts in the variance of quality characteristics in terms of out-of-control average run length criterion. We also compare the obtained results to asymptotic normal approximation method proposed first by Montgomery and Wadsworth (1972). Consider a process whose quality is characterized by the combination of correlated attribute and variable quality characteristics with correlation of 0.357. The attribute quality characteristic (X_1) follows a Poisson distribution with parameter 4 and the variable quality characteristic (X_2) follows a normal distribution with mean 3 and variance 4. We use the sample size and smoothing parameter of n=10 and λ=0.2, respectively. We also assume the correlation between X_1 and X_2 is constant and does not change under different magnitudes of step shifts during the process.

In order to compare the performance of the extended methods in terms out-of-control average run length (ARL_1) criterion, the in-control average run length value (ARL_0) of all methods should be roughly equal. We set the parameters of all methods to obtain the value of ARL_0 equal to 200 for all the methods. As noted in section 3.1, in order to apply asymptotic normal approximation method for monitoring the variability of multivariate-attribute processes, first the distribution of data is transformed to standardized multivariate normal distribution using NORTA inverse technique. After that using the sample size of n=10, the coefficient of control limits in asymptotic normal approximation method is set such that the in-control average run length equal to 200 is achieved. To do that, the control limits of 3.55 standard deviation is used and consequently the upper control limit (UCL_{ANA}) and lower control limit (LCL_{ANA}) are determined as 2.90 and -1.28, respectively. Consequently, the value of in-control average run length of this method is obtained as 202.22 based on 10000 replicates. Note that in this method, if the lower control limit is less than zero, it will be replaced with zero.

As explained in section 3.2 in order to monitor the variability of correlated multivariate-attribute quality characteristics with $MEWMS_{AS}$ and $MEWMS_{AT}$ control charts, first the distribution of data is transformed to standardized bivariate normal distribution using NORTA inverse technique. Then using Equation (9) the quality characteristics become independent. Using the sample size of n=10 and smoothing parameter of λ=0.2, the degrees of freedom in chi-square distribution used in control limits of the $MEWMS_{AS}$ control chart is computed as ν=90 and consequently the upper control limit and lower control limit of this method are determined as 2.94 and 1.26, respectively. Hence considering α_{AS}=0.005 the value of ARL_0 of this method based on 10000 replications is determined as 218.39.

The value of ARL_0 in $MEWMS_{AT}$ control chart is determined like $MEWMS_{AS}$ control chart. Hence, the distribution of original data is transformed to independent standardized bivariate normal distribution by applying NORTA inverse technique and transformation in Equation (9). Hence, the degrees of freedom in chi-square distribution used in control limits of $MEWMS_{AT}$ control

chart is computed as ν=180 Using the sample size of n=10 and smoothing parameter of λ=0.2, The values of upper control limit and lower control limit in this control chart based on α_{AT}=0.005 is computed as 2.64 and 1.46, respectively. Finally based on 10000 replicates the value of ARL_0 of this method is computed as 205.16.

The performance of the extended EWMA-based control charts is compared with asymptotic normal approximation method (ANA) in terms of out-of-control average run length criterion under different shifts based on 10000 replicates in Table 1.

The results of Table 1 show that the values of out-of control average run length of both $MEWMS_{AS}$ and $MEWMS_{AT}$ control charts are roughly small. However, the performance of second approach is more satisfactory. We can also conclude that the performance of the asymptotic normal approximation method (ANA) in detecting different shifts in the variance of qulity characteristics is not preferable.

5. CONCLUSION

In this paper, we extended two EWMA-based control charts including $MEWMS_{AS}$ and $MEWMS_{AT}$ for monitoring the variability of correlated multivariate-attribute quality characteristics. For this purpose, first the distribution of original data was transformed to standardized multivariate normal distribution using NORTA inverse technique. Then, applying another transformation, we made the transformed quality characteristics independent. In addition, we compared the performance of the extended EWMA-based control charts with the asymptotic normal approximation method in terms of out-of-control average run length criterion under different magnitude of shifts. The results showed that the performance of both extended $MEWMS_{AS}$ and $MEWMS_{AT}$ in detecting different shifts in the variance of correlated quality characteristics is satisfactory. However, the performance of the asymptotic normal approximation ethod in detecting shifts in variance of quality characteristics is not preferable. As a future research, monitoring the variability of correlated multivariate-attribute quality characteristics using other transformation techniques such as root transformation and Q transformation is recommended.

Table 1: THE ARL VALUES UNDER DIFFERENT SHIFTS

shift	*$(1.5\sigma_1,0)$*	*$(0,1.5\sigma_2)$*	*$(1.5\sigma_1, 1.5\sigma_2)$*	*$(2\sigma_1,0)$*	*$(0,2\sigma_2)$*	*$(2\sigma_1, 2\sigma_2)$*
$MEWMS_{AS}$	2.25	5.59	2.05	1.65	2.80	1.64
$MEWMS_{AT}$	1.37	3.36	1.45	1.35	2.46	1.29
ANA	150.63	201.34	151.15	139.43	200.43	138.30
shift	$(2.5\sigma_1,0)$	$(0,2.5\sigma_2)$	$(2.5\sigma_1, 2.5\sigma_2)$	$(3\sigma_1,0)$	$(0,3\sigma_2)$	$(3\sigma_1, 3\sigma_2)$
$MEWMS_{AS}$	1.64	2.16	1.65	1.65	1.90	1.64
$MEWMS_{AT}$	1.28	2.09	1.28	1.26	1.98	1.22
ANA	135.09	201.87	135.04	134.22	199.97	133.48

REFERENCES

[1] S. Bersimits, S. Psarakis, and J. Panaretos, "Multivariate statistical process control charts: an overview", *Quality and Reliability Engineering International*, vol. 23, pp. 517–543, 2007.

[2] M. H. Doroudian, and A. Amiri, "Root transformation method for monitoring correlated variable and attribute quality characteristics", Proceedings of 11th Islamic countries conference on statistical sciences, Lahore, Pakistan, vol. 21, pp. 195–200, 2011.

[3] S. Golnabi, and A. A. Houshmand, "Multivariate Shewhart x-bar chart", Inter Stat, No. 4, A web based journal. http://www.interstat.stat.vt.edu/interstat/index/Sep99.html, 1999.

[4] H. Hotelling, *Multivariate Quality Control-Illustrated by the Air Testing of Sample bombsight in Techniques of Statistical Analysis*. Eds. Eisenhart, C., Hastay, M.W., Wallis, W.A., New York: McGraw-Hill, pp. 11–184, 1947.

[5] L. Kang, and W. A. Brenneman, "Product defect rate confidence bound with attribute and variable data", *Quality and Reliability Engineering International*, vol. 27, pp. 353–368, 2011.

[6] M. R. Maleki, A. Amiri, and M. H. Doroudyan, Monitoring multivariate-attribute processes using artificial neural network, Proceedings of the 42th conference on Computers and Industrial Engineering, Cape Town, South Africa, 2012.

[7] A. O. Memar, and S. T. A. Niaki, "Multivariate variability monitoring using EWMA control charts based on squared deviation of observations from target", *Quality and Reliability Engineering International*, vol. 27, pp. 1069–1086, 2011.

[8] D. C. Montgomery, *Introduction to Statistical Quality Control*. 5thed. New York: John Wiley & Sons, 2005.

[9] S. T. A. Niaki, and B. Abbasi, "Monitoring Multi-attribute processes based on NORTA inverse transformed vectors", *Communications in Statistics - Theory and Methods*, vol. 38, pp. 946–979.

[10] E. Topalidou, and S. Psarakis, "Review of multinomial and multiattribute quality control charts", *Quality and Reliability Engineering International*, 25, 773–804, 2009.

[11] W. H. Woodall, "Control charts based on attribute data: bibliography and review", *Journal of Quality Technology*, vol. 29, pp. 172–183, 1997.

[12] A. B. Yeh, L. Huwang, C. V. Wu, "A multivariate EWMA control chart for monitoring multivariate process variability with individual observations" *IIE Transactions on Quality and Reliability Engineering*, vol. 37, pp. 1023–1035, 2005.