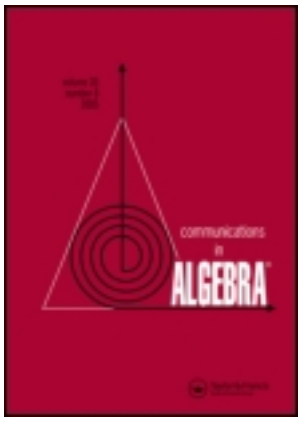

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ARTIN EXPONENT AND CC-SUBGROUPS OF RATIONAL GROUPS

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A finite group whose irreducible complex characters are rational valued is called a rational group. We prove that if a rational group G contains a CC-subgroup or G is a direct product of Frobenius rational groups, then every rationally represented character is a generalized permutation character.

Key Words: Artin exponent; CC-subgroup; Local splitting; Rational group.

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1. INTRODUCTION AND PRELIMINARY

Let $Char_{\mathbb{Q}}(G)$ and $P(G)$ denote the ring of $\mathbb{Z}$-linear combination of rationally represented characters and permutation characters of a finite group G, respectively. It is easy to see that $P(G)$ is a subring of $Char_{\mathbb{Q}}(G)$. By a theorem of Artin, $|G|\chi \in P(G)$ for all $\chi \in Char_{\mathbb{Q}}(G)$. The minimal number $d \in \mathbb{N}$, such that $d\chi \in P(G)$ for all $\chi \in Char_{\mathbb{Q}}(G)$, is called the Artin exponent of G and is denoted by $\gamma(G)$. Indeed, $\gamma(G)$ is the exponent of $Char_{\mathbb{Q}}(G)/P(G)$. If $\chi \in Char_{\mathbb{Q}}(G)$, then $|G|\chi = \sum_H a_H(1_H)^G$, where H runs over the cyclic subgroups of G, $a_H \in \mathbb{Z}$, and $(1_H)^G$ is the identity character of H induced to G ([14]). Let $P(G)_{cyclic}$ denote the ring of $\mathbb{Z}$-linear combination of permutation characters of cyclic subgroups of G. The latter situation for finite groups was studied extensively by T. Y. Lam in [3]. He proved that $A(G) =$ exponent $(\frac{Char_{\mathbb{Q}}(G)}{P(G)_{cyclic}})=1$ if and only if G is cyclic. One can show that $\gamma(G)$ divides $A(G)$ and therefore divides $|\ G\ |$. There is a fundamental distinction between $\gamma(G)$ and $A(G)$. While groups satisfying $A(G) = 1$ have been characterized, there is no such characterization for groups satisfying $\gamma(G) = 1$. In this paper, we characterize all rational groups having a CC-subgroup and then prove that $\gamma(G) = 1$ for such rational groups. We also prove that if G is a direct product of Frobenius rational groups, then $\gamma(G) = 1$.

The elements $x, y \in G$ are said to be $\mathbb{Q}$-conjugate if there exists $n \in \mathbb{Z}$ with $(n, |G|) = 1$, such that x and y^n are conjugate in G (It is well-known that x, y are $\mathbb{Q}$-conjugate if and only if $\langle x\rangle =_G \langle y\rangle$ where $\langle x\rangle$ is the cyclic group generated

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by x, and $=_G$ denotes G-conjugation.) Classes of $\mathbb{Q}$-conjugate elements of G are called $\mathbb{Q}$-classes. For a prime number p, it is well-known that if $x \in G$, then $x = x_p x_{p'} = x_{p'} x_p$ where $|x_p|$ is a prime power and $|x_{p'}|$ is prime to p. We call x_p the p-part of x. We say $x, y \in G$ are p-conjugate if x_p is conjugate to y_p, and we write $x_p \sim y_p$. Since p-conjugation is an equivalent relation, one can consider the p-classes of G. Moreover, for $x \in G$ define $\mathbb{Q}$-p-class of x by $K_p(x) = \{y \in G \mid x_p \text{ and } y_p \text{ are } \mathbb{Q}\text{-conjugate}\}$. Let us denote the p-part of a group G by G_p, i.e., G_p is a Sylow p-subgroup of G.

Definition 1.1. Let G be a finite group and p a prime number. Suppose that K is a $\mathbb{Q}$-p-class of G and s is a p-regular element of K. The group G is said to be locally split on K if $(C_G(\langle s \rangle))_p$ is a direct factor of $(N_G(\langle s \rangle))_p$; that is, $(N_G(\langle s \rangle))_p = (C_G(\langle s \rangle))_p \times V$ for some $V \leqslant (N_G(\langle s \rangle))_p$. The group G is locally split at p if G is locally split on every $\mathbb{Q}$-p-class K. The group G is everywhere locally split if G is locally split at all primes.

Theorem 1.2 ([13], Corollary 212 A). *If G is rational group which is everywhere locally split, then $\gamma(G) = 1$.*

Now, we recall some concepts of rational group theory. Let G be a finite group and χ be a complex character of G. Let $\mathbb{Q}(\chi)$ denote the subfield of the complex numbers $\mathbb{C}$ generated by $\mathbb{Q}$ and all the values $\chi(x)$, $x \in G$, where $\mathbb{Q}$ denotes the field of rational numbers. By definition, χ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$. A finite group G is called a rational group or a $\mathbb{Q}$-group, if all irreducible complex characters of G are rational. For example, the symmetric group S_n and the Weyl groups of the classical complex Lie algebras are rational groups (for more details see [13]). A comprehensive description of rational groups can be found in [1] and [13]. Throughout the paper, we use the following notations and terminology: $H : K$ stands for the semi-direct product of the groups H and K. The symbol $\mathbb{Z}_n$ denotes a cyclic group of order n. The symbol $E(p^n)$, where p is a prime number, denotes the elementary abelian p-group of order p^n and Q_8 is employed to denote the quaternion group of order 8 with representation $\{\pm 1, \pm i, \pm j, \pm k\}$. The following basic definition and theorems can be found in [19].

Definition 1.3. An element $g \in G$ is said to be rational if it is conjugate to all generators of the cyclic group $\langle g \rangle$.

Theorem 1.4. *A group G is rational group if and only if every $g \in G$ is rational. Equivalently, $N_G(A)/C_G(A) \cong Aut(A)$ for every cyclic subgroup A of G.*

Theorem 1.5. *Let p be a prime divisor of the order of a non-trivial rational group. If p is a divides of $|G|$, then the same is true for $p - 1$. Moreover, the order of a nontrivial $\mathbb{Q}$-group is even.*

Theorem 1.6. *Let G be a rational group and N be a normal subgroup of G, then G/N is also a rational group.*

Theorem 1.7. *Let G be a nontrivial rational group, and p be a prime number. If P is a Sylow p-subgroup of G then $Z(P)$ is an elementary abelian p-group. In particular, $Z(G)$ is an elementary abelian 2-group.*

The following theorem and corollary are due to Feit and Seitz [11].

Theorem 1.8. *Let G be a noncyclic finite simple group. Then G is a composition factor of a rational group if and only if G is isomorphic to an alternating group or one of the following groups*:

(i) $PSp_4(3)$, $Sp_6(2)$, $O_8^+(2)'$;
(ii) $PSL_3(4)$, $PSU_4(3)$.

Corollary 1.9. *Let G be a noncyclic simple group. Then G is a rational group if and only if $G \cong Sp_6(2)$ or $O_8^+(2)'$.*

2. CC-SUBGROUP OF RATIONAL GROUP

In this section, we characterize rational groups containing a CC-subgroup. We need the definition of a strongly embedded subgroup.

Definition 2.1. A subgroup H of a finite group G is said to be strongly embedded in G if the following two conditions are satisfied:

(1) H is a proper subgroup of even order;
(2) For any elements $x \in G - H$, the order of $H \cap H^x$ is odd.

In the following, we will find the structure of a Sylow 2-subgroup of a rational group having a strongly embedded subgroup.

Theorem 2.2. *Let G be a rational group having a strongly embedded subgroup. Then one of the following holds*:

(1) $G \cong G' : \mathbb{Z}_2$, *and* $|G'|$ *is odd*;
(2) $G \cong E(p^n) : Q_8$, *where* $p \in \{3, 5\}$ *where* G' *is the commutator subgroup of* G.

Proof. By ([7]) every Sylow 2-subgroup of G contains exactly one element of order 2 or there exists a proper normal subgroup L such that G/L is of odd order. But the second case is impossible because of Theorems 1.5 and 1.6. Now if every Sylow 2-subgroup of G contains exactly one element of order 2, then by ([19], satz 8.2) the Sylow 2-subgroup is isomorphic either to a cyclic group or a generalized quaternion group. If it is cyclic isomorphic to $\mathbb{Z}_{2^n}$, since by Theorem 1.7, the center a Sylow 2-subgroup of rational group is an elementary 2-group, then by Theorem 1.4 $n = 1$ and it is isomorphic to $\mathbb{Z}_2$. If it is a quaternion group, then it has a presentation of the form $\langle x, y \mid x^{2n} = 1, y^2 = x^{2^n-1}, y^{-1}xy = x^{-1}\rangle$ and consequently has order 2^{n+1} where $n \geq 2$. Let $A = \langle x \rangle$ be the cyclic subgroup order 2^n. Then we must have $\frac{N_G(A)}{C_G(A)} \cong Aut(A)$. Hence $|N_G(A)|_2 = |C_G(A)|_2 \times 2^{n-1}$, where $|n|_2$ denotes the 2-part of the positive integer n. Therefore, we have the inequality $2^{n+1} \geq 2^n \times$

2^{n-1} which implies $n \leq 2$. Hence $n = 2$ and Sylow 2-subgroup is quaternion group of 8 order, Q_8. In [4] Hayashi classified rational group with abelian Sylow 2-subgroup; therefore, if the Sylow 2-subgroup is isomorphic $\mathbb{Z}_2$, then by the same theorem, we conclude that $G \cong G' : \mathbb{Z}_2$. Similarly, by the same theorem, if the Sylow 2-subgroup is isomorphic to Q_8, then by ([13]) $G \cong E(p^n) : Q_8$. This completes the proof. □

Definition 2.3. A proper nontrivial subgroup H of G is called a CC-subgroup of G if $C_G(x) \leq H$ for any $x \in H - 1$.

Here, we use strongly embedded subgroups rather than the classification of CC-subgroup.

Theorem 2.4 ([7], p. 407). *The following conditions on a group G are equivalent*:

(i) *The group G contains a strongly embedded subgroup*;
(ii) *The group G contains a subgroup K of even order such that $Inv(G) \nsubseteq K$ and $C_G(t) \subset K$ for any $t \in Inv(K)$, where $Inv(G)$ is the set of involutions of G.*

Theorem 2.5. *Let H be a CC-subgroup of even order of a rational group. Then a Sylow 2-subgroup of H is isomorphic to either $\cong \mathbb{Z}_2$ or Q_8.*

Proof. If $Inv(G) \subset H$, then we show that $H = G$. In this case the subgroup K of G generated by $Inv(G)$ is normal in G and hence $K \trianglelefteq H$. If we let $C = \cap_{x \in G} H^x$, then $K \leq C \lhd G$. Hence C is a proper nontrivial subgroup of G. But since C is also a CC-subgroup of G, and hence by ([7]), G would be a Frobenius group with kernel C. But in a Frobenius group orders of complement and kernel are coprime, therefore $(|\frac{G}{C}|, |C|) = 1$, since C has even order we deduce that $\frac{G}{C}$ is a rational group of odd order. Therefore, $G = C$ implying that $G \leq H$ a contradiction to the fact that H is a proper subgroup of G.

Hence we must be have $Inv(G) \nsubseteq H$. Now by Theorem 2.4, G contains a strongly embedded subgroup, and hence by Theorem 2.2 every Sylow 2-subgroup of G contains exactly one involution. If $t \in H$ is an involution, then we may assume t is in the center of a Sylow 2-subgroup P of G. Since H is a CC-subgroup of G, $P \subseteq C_G(t) \subseteq H$, which implies that H contains a Sylow 2-subgroup of G and the Theorem is proved. □

Now, we use the classification of CC-subgroup of odd order for finding the structure of rational groups with such a CC-subgroup.

According to [9] a 2-Frobenius group G is a group containing a normal series $K < F < G$ such that F and G/K are Frobenius groups having kernels K and F/K, respectively. For a group G, we denote the Fitting of G by $F(G)$. Then, we have the following theorem.

Theorem 2.6 (Theorem A in ([10]). *Let G be a finite group containing a CC-subgroup A of odd order. Then we have one of the following cases*:

(1) *A is non-nilpotent and one of the following holds*:

(i) *G is a solvable Frobenius group with complement A*;
(ii) *$G \cong PSL(2, q)$, $q \equiv 3 (mod\ 4)$ and A is solvable of odd order $| A |= \frac{q(q-1)}{2}$.*

(2) *A is nilpotent and one of the following holds*:

(i) *G is a Frobenius group with A either kernel or complement*;
(ii) *G is a simple non-abelian group which is classified in* ([10], *Tables* 1–6);
(iii) *G is a not simple and if H is the normal closure of A in G, then* $F(H) = F(G)$ *and* $H/F(G)$ *is a simple group of type* (2) (*i*);
(iv) *G is a* 2*-Frobenius group.*

Theorem 2.7. *Let H be a CC-subgroup of odd order in a rational group G. Then* $H \cong E(3^n) or E(5^n)$.

Proof. We consider two cases.

Case (a). H is not a normal subgroup of G.

By Theorems 2.6 and 1.6, G cannot be a Frobenius group with complement H. By [12], $PSL(2, 3)$ is not a rational group furthermore $PSL(2, q)$ is a simple group. But the only noncyclic simple rational group are $Sp_6(2)$ or $O_8^+(2)'$ (see Corollary 1.9). Therefore, case (1) and case (2)(ii) in Theorem 2.6 are impossible.

By Theorem 1.8, a noncyclic finite simple group is a composition factor of a rational group if and only if it is isomorphic to an alternating group or one of the following groups: $PSp_4(3)$, $Sp_6(2)$, $O_8^+(2)'$, $PSL_3(4)$, or $PSU_4(3)$. But none of these groups occur in tables of simple groups in ([10]). Thus case (2)(iii) in Theorem 2.6 is impossible. By the main theorem of [15] and the assumption that H is not a normal subgroup of G and Theorems 1.5 and 1.6 case (2) (iv) in Theorem 2.6 is not possible.

Case (b). H is a normal subgroup of G.

By (Theorem 2.6, (2)(i)) G is a Frobenius group with kernel H and complement K. Because G/H is of even order, the K must be a CC-subgroup of even order. But by argument used in Theorem 2.5, a Sylow 2-subgroup of G is normal in K, from the other hand rationality of K force $K \cong \mathbb{Z}_2$ or $\mathbb{Q}_8$. Since K is of even order, H is abelian ([14] Lemma (7. 21)). Because G/H is solvable then by ([20], Corollary 2) $\pi(H) \subseteq \{3, 5\}$, where $\pi(H)$ denotes the set of all prime divisors of the order of H. If $g \in H$, then we may put $o(g) = 3^r 5^s$ where r and s are non-negative integers. Since H is a CC-subgroup of a rational group G we must have $C_G(g) \leqslant H$ and $N_G(\langle g \rangle)/C_G(\langle g \rangle) \cong Aut(\langle g \rangle)$. But $|Aut(\langle g \rangle)| = \phi(o(g))$ must be a power of 2 because it must divide $|K|$, from which we obtain $r, s \leq 1$. Therefore, the order of every non-identity element of H is either 3, 5, or 15. If H has an element of order 15, say g, then $N_G(\langle g \rangle)/H$ has order $\phi(15) = 8$, hence it must be isomorphic to Q_8. But this is a contradiction because $Aut(\langle g \rangle)$ is an abelian group. Therefore, H is an elementary abelian 3 or 5-group and the Theorem is proved.

□

3. CALCULATION $\gamma(G)$ FOR SOME FINITE GROUPS

First, note that, the notions $\mathbb{Q}$-p-class and p-class are equivalent on rational groups.

Theorem 3.1. *Let G be a rational group with a CC-subgroup, then* $\gamma(G) = 1$.

Proof. By proof Theorem 2.5, if a rational group G has a CC-subgroup, then G is a Frobenius rational group, but in [15], we classify this type of groups. Therefore, $G \cong E(3^n) : \mathbb{Z}_2$, $E(3^{2m}) : Q_8$ or $E(5^2) : Q_8$. Calculate Artin exponent for each of them and consider the following three cases:

(1) Let G be $E(3^n) : \mathbb{Z}_2$. Then we consider two cases:

(i) G is locally split at $p = 2$. Let V be a 2-class of G and let g be a 2-regular element of V. Then a Sylow 2-subgroup of $C_G(\langle g \rangle)$ is a subgroup of a Sylow 2-subgroup of $N_G(\langle g \rangle)$ which is an elementary abelian 2-group. Therefore, $(N_G(\langle g \rangle))_2$ is a direct factor of $N_G(\langle g \rangle)$.

(ii) Let G be locally split at $p = 3$. Let V be a 3-class of G and g be a 3-regular element of V. Then g is an element of 2 power order and, hence $o(g) \leq 2$. Hence $N_G(\langle g \rangle) = C_G(\langle g \rangle)$ and it follows that $C(\langle g \rangle)_3$ is a direct factor of $N(g)_3$.

(2) Let G be $E(3^{2m}) : Q_8$. First, we show that Q_8 on $E(3^{2m})$ is fixed-point free. The fact that Q_8 acts without fixed-points can be seen from the matrix representation of Q_8 over $\mathbb{F}_{3^m}$:

$$i \mapsto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$k \mapsto \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$-1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore $C_G(g)_3 = 1$ and it follows that G is locally split at p. Hence $\gamma(G) = 1$.

(3) Let G be $E(5^2) : Q_8$. Again, Q_8 acts without fixed point, according to the following matrix representation of Q_8 over $\mathbb{F}_5$: 10

$$i \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

$$k \mapsto \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$$

$$-1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, G is everywhere locally split. So, by Theorem 1.2, $\gamma(G) = 1$. □

Corollary 3.2. *Let G be a rational group having abelian Sylow 2-subgroups. Then $\gamma(G) = 1$.*

Proof. By Theorem 2.2 G is isomorphic with $E(3^n) : \mathbb{Z}_2$. Now the result following as proof of Theorem 3.1. □

Theorem 3.3. *Let G be a finite group which is not a 2-group and such that the number of distinct kernels of irreducible characters of G is equal to the number of conjugacy classes of G. Then $\gamma(G/O_2(G)) = 1$.*

Proof. A group G with the above property is a rational group, we refer the reader to [2, 17]. But according to [16] any Frobenius group satisfying the assumption of the theorem is isomorphic to $E(3^n) : \mathbb{Z}_2$ or $E(3^2) : Q_8$. Again by [16], $G/O_2(G)$ is direct product of groups of above types. Obviously, by Theorem 1.2 the proof is completed. □

Remark 3.4. We notice that Theorem 3.3 can not be generalized to nilpotent groups. G. Segal, J. Ritter, and J. Rasmussen in 1972 independently showed that $\gamma(G) = 1$ for any p-group G by using methods of algebraic K-theory [18]. Ritter and Rasmussen proved the same result using group-theoretic methods and tried to extend the result to nilpotent groups. However, J. P. Serre [5] remarked that $\gamma(\mathbb{Z}_3 \times Q_8) = 2$, thus showed that the extension of Segal's result to nilpotent groups required some modification. Rasmussen did this by showing that $\gamma(G) = 1$ or 2 when G is nilpotent and providing necessary and sufficient condition for when each case occurs [6].

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