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ARTIN EXPONENT AND CC-SUBGROUPS
OF RATIONAL GROUPS

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A finite group whose irreducible complex characters are rational valued is called a rational group. We prove that if a rational group $G$ contains a CC-subgroup or $G$ is a direct product of Frobenius rational groups, then every rationally represented character is a generalized permutation character.

Key Words: Artin exponent; CC-subgroup; Local splitting; Rational group.

2010 Mathematics Subject Classification: 20C15.

1. INTRODUCTION AND PRELIMINARY

Let $\text{Char}_\mathbb{Q}(G)$ and $P(G)$ denote the ring of $\mathbb{Z}$-linear combination of rationally represented characters and permutation characters of a finite group $G$, respectively. It is easy to see that $P(G)$ is a subring of $\text{Char}_\mathbb{Q}(G)$. By a theorem of Artin, $|G|\chi \in P(G)$ for all $\chi \in \text{Char}_\mathbb{Q}(G)$. The minimal number $d \in \mathbb{N}$, such that $d\chi \in P(G)$ for all $\chi \in \text{Char}_\mathbb{Q}(G)$, is called the Artin exponent of $G$ and is denoted by $\gamma(G)$. Indeed, $\gamma(G)$ is the exponent of $\text{Char}_\mathbb{Q}(G)/P(G)$. If $\chi \in \text{Char}_\mathbb{Q}(G)$, then $|G|\chi = \sum_H a_H (1_H)\chi$, where $H$ runs over the cyclic subgroups of $G$, $a_H \in \mathbb{Z}$, and $(1_H)\chi$ is the identity character of $H$ induced to $G$ ([14]). Let $P(G)_{\text{cyclic}}$ denote the ring of $\mathbb{Z}$-linear combination of permutation characters of cyclic subgroups of $G$. The latter situation for finite groups was studied extensively by T. Y. Lam in [3]. He proved that $A(G) = \text{exponent} (\text{Char}_\mathbb{Q}(G)/P(G)_{\text{cyclic}}) = 1$ if and only if $G$ is cyclic. One can show that $\gamma(G)$ divides $A(G)$ and therefore divides $|G|$. There is a fundamental distinction between $\gamma(G)$ and $A(G)$. While groups satisfying $A(G) = 1$ have been characterized, there is no such characterization for groups satisfying $\gamma(G) = 1$. In this paper, we characterize all rational groups having a CC-subgroup and then prove that $\gamma(G) = 1$ for such rational groups. We also prove that if $G$ is a direct product of Frobenius rational groups, then $\gamma(G) = 1$.

The elements $x, y \in G$ are said to be $\mathbb{Q}$-conjugate if there exists $n \in \mathbb{Z}$ with $(n, |G|) = 1$, such that $x$ and $y^n$ are conjugate in $G$ (It is well-known that $x, y$ are $\mathbb{Q}$-conjugate if and only if $\langle x \rangle =_G \langle y \rangle$ where $\langle x \rangle$ is the cyclic group generated by $x$).

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by $x$, and $=g$ denotes $G$-conjugation.) Classes of $\mathbb{Q}$-conjugate elements of $G$ are called $\mathbb{Q}$-classes. For a prime number $p$, it is well-known that if $x \in G$, then $x = x_p x'_p = x_p x'_p$ where $|x_p|$ is a prime power and $|x'_p|$ is prime to $p$. We call $x_p$ the $p$-part of $x$. We say $x, y \in G$ are $p$-conjugate if $x_p$ is conjugate to $y_p$, and we write $x_p \sim y_p$. Since $p$-conjugation is an equivalent relation, one can consider the $p$-classes of $G$. Moreover, for $x \in G$ define $\mathbb{Q}$-$p$-class of $x$ by $K_p(x) = \{y \in G \mid x_p$ and $y_p$ are $\mathbb{Q}$-conjugate$\}$. Let us denote the $p$-part of a group $G$ by $G_p$, i.e., $G_p$ is a Sylow $p$-subgroup of $G$.

**Definition 1.1.** Let $G$ be a finite group and $p$ a prime number. Suppose that $K$ is a $\mathbb{Q}$-$p$-class of $G$ and $s$ is a $p$-regular element of $K$. The group $G$ is said to be locally split on $K$ if $(C_G(sl))_p$ is a direct factor of $(N_G(sl))_p$; that is, $(N_G(sl))_p = (C_G(sl))_p \times V$ for some $V \leq (N_G(sl))_p$. The group $G$ is locally split at $p$ if $G$ is locally split on every $\mathbb{Q}$-$p$-class $K$. The group $G$ is everywhere locally split if $G$ is locally split at all primes.

**Theorem 1.2** ([13], Corollary 212 A). If $G$ is rational group which is everywhere locally split, then $\gamma(G) = 1$.

Now, we recall some concepts of rational group theory. Let $G$ be a finite group and $\chi$ be a complex character of $G$. Let $\mathbb{Q}(\chi)$ denote the subfield of the complex numbers $\mathbb{C}$ generated by $\mathbb{Q}$ and all the values $\chi(x)$, $x \in G$, where $\mathbb{Q}$ denotes the field of rational numbers. By definition, $\chi$ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$. A finite group $G$ is called a rational group or a $\mathbb{Q}$-group, if all irreducible complex characters of $G$ are rational. For example, the symmetric group $S_n$ and the Weyl groups of the classical complex Lie algebras are rational groups (for more details see [13]). A comprehensive description of rational groups can be found in [1] and [13]. Throughout the paper, we use the following notations and terminology: $H : K$ stands for the semi-direct product of the groups $H$ and $K$. The symbol $\mathbb{Z}_n$ denotes a cyclic group of order $n$. The symbol $E(p^n)$, where $p$ is a prime number, denotes the elementary abelian $p$-group of order $p^n$ and $Q_8$ is employed to denote the quaternion group of order 8 with representation $\{\pm 1, \pm i, \pm j, \pm k\}$. The following basic definition and theorems can be found in [19].

**Definition 1.3.** An element $g \in G$ is said to be rational if it is conjugate to all generators of the cyclic group $(g)$.

**Theorem 1.4.** A group $G$ is rational group if and only if every $g \in G$ is rational. Equivalently, $N_G(A)/C_G(A) \cong \text{Aut}(A)$ for every cyclic subgroup $A$ of $G$.

**Theorem 1.5.** Let $p$ be a prime divisor of the order of a non-trivial rational group. If $p$ divides $|G|$, then the same is true for $p - 1$. Moreover, the order of a nontrivial $\mathbb{Q}$-group is even.

**Theorem 1.6.** Let $G$ be a rational group and $N$ be a normal subgroup of $G$, then $G/N$ is also a rational group.
\textbf{Theorem 1.7.} Let $G$ be a nontrivial rational group, and $p$ be a prime number. If $P$ is a Sylow $p$-subgroup of $G$ then $Z(P)$ is an elementary abelian $p$-group. In particular, $Z(G)$ is an elementary abelian 2-group.

The following theorem and corollary are due to Feit and Seitz [11].

\textbf{Theorem 1.8.} Let $G$ be a noncyclic finite simple group. Then $G$ is a composition factor of a rational group if and only if $G$ is isomorphic to an alternating group or one of the following groups:

(i) $\text{PSp}_4(3)$, $\text{Sp}_6(2)$, $\text{O}_8^+(2)'$;
(ii) $\text{PSL}_3(4)$, $\text{PSU}_4(3)$.

\textbf{Corollary 1.9.} Let $G$ be a noncyclic simple group. Then $G$ is a rational group if and only if $G \cong \text{Sp}_6(2)$ or $\text{O}_8^+(2)'$.

\section{CC-SUBGROUP OF RATIONAL GROUP}

In this section, we characterize rational groups containing a CC-subgroup. We need the definition of a strongly embedded subgroup.

\textbf{Definition 2.1.} A subgroup $H$ of a finite group $G$ is said to be strongly embedded in $G$ if the following two conditions are satisfied:

1. $H$ is a proper subgroup of even order;
2. For any elements $x \in G - H$, the order of $H \cap H^x$ is odd.

In the following, we will find the structure of a Sylow 2-subgroup of a rational group having a strongly embedded subgroup.

\textbf{Theorem 2.2.} Let $G$ be a rational group having a strongly embedded subgroup. Then one of the following holds:

1. $G \cong G' : \mathbb{Z}_2$, and $|G'|$ is odd;
2. $G \cong E(p^n) : Q_8$, where $p \in \{3, 5\}$ where $G'$ is the commutator subgroup of $G$.

\textbf{Proof.} By ([7]) every Sylow 2-subgroup of $G$ contains exactly one element of order 2 or there exists a proper normal subgroup $L$ such that $G/L$ is of odd order. But the second case is impossible because of Theorems 1.5 and 1.6. Now if every Sylow 2-subgroup of $G$ contains exactly one element of order 2, then by ([19], satz 8.2) the Sylow 2-subgroup is isomorphic either to a cyclic group or a generalized quaternion group. If it is cyclic isomorphic to $\mathbb{Z}_2$, since by Theorem 1.7, the center a Sylow 2-subgroup of rational group is an elementary 2-group, then by Theorem 1.4 $n = 1$ and it is isomorphic to $\mathbb{Z}_2$. If it is a quaternion group, then it has a presentation of the form $\langle x, y \mid x^{2n} = 1, y^2 = x^{2n-1}, y^{-1}xy = x^{-1} \rangle$ and consequently has order $2^{n+1}$ where $n \geq 2$. Let $A = \langle x \rangle$ be the cyclic subgroup order $2^n$. Then we must have $\frac{N_G(A)}{C_G(A)} \cong \text{Aut}(A)$. Hence $|N_G(A)| = |C_G(A)| \times 2^{n-1}$, where $|n|_2$ denotes the 2-part of the positive integer $n$. Therefore, we have the inequality $2^{n+1} \geq 2^n \times...
2^{n-1} which implies \( n \leq 2 \). Hence \( n = 2 \) and Sylow 2-subgroup is quaternion group of 8 order, \( Q_8 \). In [4] Hayashi classified rational group with abelian Sylow 2-subgroup; therefore, if the Sylow 2-subgroup is isomorphic to \( \mathbb{Z}_2 \), then by the same theorem, we conclude that \( G \cong G' : \mathbb{Z}_2 \). Similarly, by the same theorem, if the Sylow 2-subgroup is isomorphic to \( Q_8 \), then by ([13]) \( G \cong E(p^n) : Q_8 \). This completes the proof. □

**Definition 2.3.** A proper nontrivial subgroup \( H \) of \( G \) is called a CC-subgroup of \( G \) if \( C_G(x) \leq H \) for any \( x \in H \).

Here, we use strongly embedded subgroups rather than the classification of CC-subgroup.

**Theorem 2.4** ([7], p. 407). The following conditions on a group \( G \) are equivalent:

(i) The group \( G \) contains a strongly embedded subgroup;
(ii) The group \( G \) contains a subgroup \( K \) of even order such that \( \text{Inv}(G) \not\subset K \) and \( C_G(t) \subset K \) for any \( t \in \text{Inv}(K) \), where \( \text{Inv}(G) \) is the set of involutions of \( G \).

**Theorem 2.5.** Let \( H \) be a CC-subgroup of even order of a rational group. Then a Sylow 2-subgroup of \( H \) is isomorphic to either \( \mathbb{Z}_2 \) or \( Q_8 \).

**Proof.** If \( \text{Inv}(G) \subset H \), then we show that \( H = G \). In this case the subgroup \( K \) of \( G \) generated by \( \text{Inv}(G) \) is normal in \( G \) and hence \( K \leq H \). If we let \( C = \cap_{x \in G} H^x \), then \( K \leq C \leq G \). Hence \( C \) is a proper nontrivial subgroup of \( G \). But since \( C \) is also a CC-subgroup of \( G \), and hence by ([7]), \( G \) would be a Frobenius group with kernel \( C \). But in a Frobenius group orders of complement and kernel are coprime, therefore \( (|\bar{C}|, |C|) = 1 \), since \( C \) has even order we deduce that \( \bar{C} \) is a rational group of odd order. Therefore, \( G = C \) implying that \( G \leq H \) a contradiction to the fact that \( H \) is a proper subgroup of \( G \).

Hence we must have \( \text{Inv}(G) \not\subset H \). Now by Theorem 2.4, \( G \) contains a strongly embedded subgroup, and hence by Theorem 2.2 every Sylow 2-subgroup of \( G \) contains exactly one involution. If \( t \in H \) is an involution, then we may assume \( t \) is in the center of a Sylow 2-subgroup \( P \) of \( G \). Since \( H \) is a CC-subgroup of \( G \), \( P \leq C_G(t) \leq H \), which implies that \( H \) contains a Sylow 2-subgroup of \( G \) and the Theorem is proved. □

Now, we use the classification of CC-subgroup of odd order for finding the structure of rational groups with such a CC-subgroup.

According to [9] a 2-Frobenius group \( G \) is a group containing a normal series \( K < F < G \) such that \( F \) and \( G/K \) are Frobenius groups having kernels \( K \) and \( F/K \), respectively. For a group \( G \), we denote the Fitting of \( G \) by \( F(G) \). Then, we have the following theorem.

**Theorem 2.6** (Theorem A in ([10]). Let \( G \) be a finite group containing a CC-subgroup \( A \) of odd order. Then we have one of the following cases:

(1) \( A \) is non-nilpotent and one of the following holds:
   (i) \( G \) is a solvable Frobenius group with complement \( A \);
   (ii) \( G \cong \text{PSL}(2, q), q \equiv 3(\mod 4) \) and \( A \) is solvable of odd order \( |A| = \frac{q(q-1)}{2} \).
(2) A is nilpotent and one of the following holds:

(i) \( G \) is a Frobenius group with \( A \) either kernel or complement;  
(ii) \( G \) is a simple non-abelian group which is classified in ([10], Tables 1–6);  
(iii) \( G \) is a not simple and if \( H \) is the normal closure of \( A \) in \( G \), then \( F(H) = F(G) \) and \( H/F(G) \) is a simple group of type (2) (i);  
(iv) \( G \) is a 2-Frobenius group.

**Theorem 2.7.** Let \( H \) be a CC-subgroup of odd order in a rational group \( G \). Then \( H \cong E(3^s) \) or \( E(5^s) \).

**Proof.** We consider two cases.

**Case (a).** \( H \) is not a normal subgroup of \( G \).

By Theorems 2.6 and 1.6, \( G \) cannot be a Frobenius group with complement \( H \). By [12], \( PSL(2, 3) \) is not a rational group furthermore \( PSL(2, q) \) is a simple group. But the only noncyclic simple rational group are \( S_p(2) \) or \( O_s(2) \) (see Corollary 1.9). Therefore, case (1) and case (2)(ii) in Theorem 2.6 are impossible.

By Theorem 1.8, a noncyclic finite simple group is a composition factor of a rational group if and only if it is isomorphic to an alternating group or one of the following groups: \( PSL(3, 2) \), \( PSL(2, 2) \), \( O_s(2) \), \( PSL(4, 2) \), or \( PSU(3, 2) \). But none of these groups occur in tables of simple groups in ([10]). Thus case (2)(iii) in Theorem 2.6 is impossible. By the main theorem of [15] and the assumption that \( H \) is not a normal subgroup of \( G \) and Theorems 1.5 and 1.6 case (2) (iv) in Theorem 2.6 is not possible.

**Case (b).** \( H \) is a normal subgroup of \( G \).

By (Theorem 2.6, (2)(ii)) \( G \) is a Frobenius group with kernel \( H \) and complement \( K \). Because \( G/H \) is of even order, the \( K \) must be a CC-subgroup of even order. But by argument used in Theorem 2.5, a Sylow 2-subgroup of \( G \) is normal in \( K \), from the other hand rationality of \( K \) force \( K \cong \mathbb{Z}_2 \) or \( \mathbb{Q}_s \). Since \( K \) is of even order, \( H \) is abelian ([14] Lemma (7. 21)). Because \( G/H \) is solvable then by ([20], Corollary 2) \( \pi(H) \subseteq \{3, 5\} \), where \( \pi(H) \) denotes the set of all prime divisors of the order of \( H \). If \( g \in H \), then we may put \( o(g) = 3^r 5^s \) where \( r \) and \( s \) are non-negative integers. Since \( H \) is a CC-subgroup of a rational group \( G \) we must have \( C_G(g) \subseteq H \) and \( N_G((g))/C_H((g)) \cong \text{Aut}(\langle g \rangle) \). But \( |\text{Aut}(\langle g \rangle)| = \phi(o(g)) \) must be a power of 2 because it must divide \( |K| \), from which we obtain \( r, s \leq 1 \). Therefore, the order of every non-identity element of \( H \) is either 3, 5, or 15. If \( H \) has an element of order 15, say \( g \), then \( N_G((g))/H \) has order \( \phi(15) = 8 \), hence it must be isomorphic to \( \mathbb{Q}_8 \). But this is a contradiction because \( \text{Aut}(\langle g \rangle) \) is an abelian group. Therefore, \( H \) is an elementary abelian 3 or 5-group and the Theorem is proved.

\( \square \)

3. **CALCULATION \( \gamma(G) \) FOR SOME FINITE GROUPS**

First, note that, the notions \( \mathbb{Q}-p \)-class and \( p \)-class are equivalent on rational groups.

**Theorem 3.1.** Let \( G \) be a rational group with a CC-subgroup, then \( \gamma(G) = 1 \).
**Proof.** By proof Theorem 2.5, if a rational group $G$ has a CC-subgroup, then $G$ is a Frobenius rational group, but in [15], we classify this type of groups. Therefore, $G \cong E(3^n) : \mathbb{Z}_2, E(3^{2m}) : Q_8$ or $E(5^2) : Q_8$. Calculate Artin exponent for each of them and consider the following three cases:

1. Let $G$ be $E(3^n) : \mathbb{Z}_2$. Then we consider two cases:
   
   (i) $G$ is locally split at $p = 2$. Let $V$ be a 2-class of $G$ and let $g$ be a 2-regular element of $V$. Then a Sylow 2-subgroup of $C_G(\langle g \rangle)$ is a subgroup of a Sylow 2-subgroup of $N_G(\langle g \rangle)$ which is an elementary abelian 2-group. Therefore, $(N_G(\langle g \rangle), 3)$ is a direct factor of $N_G(\langle g \rangle)$.
   
   (ii) Let $G$ be locally split at $p = 3$. Let $V$ be a 3-class of $G$ and $g$ be a 3-regular element of $V$. Then $g$ is an element of 2 power order and, hence $o(g) \leq 2$. Hence $N_G(\langle g \rangle) = C_G(\langle g \rangle)$ and it follows that $C(\langle g \rangle)$ is a direct factor of $N(\langle g \rangle)$.

2. Let $G$ be $E(3^{2m}) : Q_8$. First, we show that $Q_8$ on $E(3^{2m})$ is fixed-point free. The fact that $Q_8$ acts without fixed-points can be seen from the matrix representation of $Q_8$ over $\mathbb{F}_{3^m}$:

$$
i \mapsto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\quad k \mapsto \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},
\quad -1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore $C_G(\langle g \rangle) = 1$ and it follows that $G$ is locally split at $p$. Hence $\gamma(G) = 1$.

3. Let $G$ be $E(5^2) : Q_8$. Again, $Q_8$ acts without fixed point, according to the following matrix representation of $Q_8$ over $\mathbb{F}_5$:

$$
i \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},
\quad j \mapsto \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix},
\quad k \mapsto \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix},
\quad -1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, $G$ is everywhere locally split. So, by Theorem 1.2, $\gamma(G) = 1$. □

**Corollary 3.2.** Let $G$ be a rational group having abelian Sylow 2-subgroups. Then $\gamma(G) = 1$. 
Proof. By Theorem 2.2 $G$ is isomorphic with $E(3^n) : \mathbb{Z}_2$. Now the result following as proof of Theorem 3.1.

Theorem 3.3. Let $G$ be a finite group which is not a 2-group and such that the number of distinct kernels of irreducible characters of $G$ is equal to the number of conjugacy classes of $G$. Then $\gamma(G/O_2(G)) = 1$.

Proof. A group $G$ with the above property is a rational group, we refer the reader to [2, 17]. But according to [16] any Frobenius group satisfying the assumption of the theorem is isomorphic to $E(3^n) : \mathbb{Z}_2$ or $E(3^2) : Q_8$. Again by [16], $G/O_2(G)$ is direct product of groups of above types. Obviously, by Theorem 1.2 the proof is completed.

Remark 3.4. We notice that Theorem 3.3 can not be generalized to nilpotent groups. G. Segal, J. Ritter, and J. Rasmussen in 1972 independently showed that $\gamma(G) = 1$ for any $p$-group $G$ by using methods of algebraic K-theory [18]. Ritter and Rasmussen proved the same result using group-theoretic methods and tried to extend the result to nilpotent groups. However, J. P. Serre [5] remarked that $\gamma(\mathbb{Z}_2 \times Q_8) = 2$, thus showed that the extension of Segal’s result to nilpotent groups required some modification. Rasmussen did this by showing that $\gamma(G) = 1$ or 2 when $G$ is nilpotent and providing necessary and sufficient condition for when each case occurs [6].

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