Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 8, Issue 1 (June 2013), pp. 214 - 226

Applications and Applied
Mathematics:
An International Journal
(AAM)

Numerical Solution of Some Nonlinear Volterra Integral Equations of the First Kind

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Received:August 04, 2012 ; Accepted: March 29, 2013

Abstract

In this paper, the solving of a class of the nonlinear Volterra integral equations (NVIE) of the first kind is investigated. Here, we convert NVIE of the first kind to a linear equation of the second kind. Then we apply the operational Tau method to the problem and prove convergence of the presented method. Finally, some numerical examples are given to show the accuracy of the method.

Keywords: Volterra integral equation of the first kind, Tau method

MSC 2010 No.: 65R20, 97N40

1. Introduction

Consider NVIE of the first kind of the form:

$$\int_c^x K_1(x,t)G(u(t))dt = f_1(x), \quad x \in [c,b], \tag{1.1}$$

where f_1, K_1 and G are given smooth functions, G is invertible and nonlinear in u. The solution u is determined and under the assumption that $f_1(c) = 0$.

Many problems in mathematical physics and engineering are often reduced to integral equations of the first kind, which are inherently ill-posed problems, meaning that the solution is generally unstable, and small changes to the problem can make very large changes to the solutions obtained [Babolian and Delves (1979); Kythe and Puri (2002)]. Equations of the form (1.1) have been investigated in some papers. For example, in [Biazar et al. (2003)] the NVIE of the first kind has been solved using Adomian method. Babolian and Masouri (2008) proposed a simple efficient direct method for solving the Volterra integral equations of the first kind. They applied block-pulse functions and their operational matrix of integration to reduce the first kind integral equation to a linear lower triangular system. Linz (1969) applied rectangular method, trapezoidal and midpoint method for solving linear Volterra integral equations (LVIE) of the first kind.

Maleknejad (2007) solved the VIEs of the first kind by wavelet basis. Biazar (2009) applied He's homotopy perturbation method to solve systems of VIEs of the first kind. Masouri (2010) produced the approximate solution of the VIEs of the first kind via a recurrence relation. Maleknejad (2011) introduced and used a modification of block pulse functions to solve VIEs of the first kind.

In this paper, we convert the Equation (1.1) to a LVIE of the second kind and apply the operational Tau method to the LVIE. Spectral methods have been studied intensively in recent years because of their good approximation properties. The Tau method, through which the spectral methods can be described as a special case has found extensive applications in numerical solution of many operator equations. There has been considerable interest in solving integral equations using Tau methods [Hosseini and Shahmorad (2005); Shahmorad (2005); Hosseini and Shahmorad (2003); Pour-Mahmoud et al. (2005)]. Also, the Volterra-Hammerstein integral equations have been solved by the Tau method [Ghoreishi and Hadizadeh (2009)]. In recent years, the Tau method has been developed for solving the two-dimensional integral and integro-differential equations too ([Rahimi et al. (2010); Tari et al. (2009)]). The rest of this paper is organized as follows:

In Section 2, we briefly describe the Tau method. In Section 3, we formulate the problem. In Section 4, we investigate the existence and uniqueness of the solution of the problem and prove the convergence of the method. Also, in Section 5, we give some examples to show the accuracy and efficiency of the presented method. Finally, section 6 consists of a few conclusions.

2. Tau Method

In this section, we give some preliminary results about the Tau method. Complete information about this method can be found in the references [Ottiz and Samara (1981); Hosseini and Shahmorad (2002)] and specialy in [Canuto et al. (2006)].

The operational approach to the Tau method proposed by Ortiz and Samara (1981) is based on the use of three simple matrices

$$\boldsymbol{\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \boldsymbol{\eta} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \boldsymbol{\iota} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with the following properties: if $y(x) = a^t X$, where $a^t = (a_0, a_1, \dots, a_n, 0, \dots, 0)$, $X^t = (1, x, x^2, \dots)$, then

$$\frac{d}{dx} y(x) = a^t \eta X, \tag{2.1}$$

$$xy(x) = a^t \mu X, \tag{2.2}$$

and

$$\int y(x) dx = a^t \iota X. \tag{2.3}$$

In the rest of paper, we assume that μ_N, η_N and ι_N denote the matrices including the first $N+1$ rows and columns of the matrices μ, η and ι, respectively.

Lemma 2.1.

Under above assumptions, we have

$$\int_c^x x^i t^j y(t) dt = a^t \left(\mu^j \iota\, \mu^i - \xi e_{i+1}^t\right) X,$$

where e_{i+1} is the $(i+1)^{th}$ coordinate unit vector and $\xi = \mu^j \iota X|_{x=c}$.

Proof:

See [Hosseini and Shahmorad (2002)].

Theorem 2.2.

Assume that $\mathrm{K(x,t)} = \sum_{\mathrm{i=0}}^{\mathrm{N}} \sum_{\mathrm{j=0}}^{\mathrm{N}} \mathrm{d_{ij} x^i t^j}$, then we have

$$\int_c^x K(x,t) y(t) dt = a_N^t \Pi X_N, \tag{2.4}$$

where $\Pi = \sum_{i=0}^{N} \sum_{j=0}^{N} d_{ij} (\mu_N^j \iota_N \mu_N^i - \xi_N^{(ij)}(c) e_{i+1}^t$, $a_N^t = (a_0, a_1, \dots, a_N)$, $X_N = (1, x, \dots, x^N)$ and $\xi_N^{(ij)}$ is to denote $\xi_N = \mu_N^j \iota_N X_N|_{x=c}$ corresponding to the term $x^i t^j$ in the kernel.

Proof:

See [Hosseini and Shahmorad (2002)].

Remark 2.3.

In the Tau method, $f(x)$ and $K(x,t)$ are polynomials whenever $f(x)$ and $K(x,t)$ are not polynomials,

they should be approximated by suitable polynomials.
Note that in the Equation (1.1), the functions $f(x)$ and $K(x,t)$ are not required to be polynomials.

3. Formulation of the Problem

In this section, we convert the Equation (1.1) to a linear equation of the second kind, then we apply the operational Tau method to the latter equation to get an equivalent linear system of equations. Without loosing generality, we assume that $c = 0$.

We set $y(t) = G(u(t))$, so the integral Equation (1.1) can be written as

$$\int_0^x K_1(x,t)y(t)dt = f_1(x). \tag{3.1}$$

Taking the derivative with respect to x in both sides of the above equation, leads to

$$K_1(x,x)\mathrm{y}(\mathrm{x}) + \int_0^x \frac{\partial K_1(x,t)}{\partial x} y(t)dt = f_1'(x). \tag{3.2}$$

With assumption $K_1(x,x) \neq 0$, Equation (3.2) is converted to

$$y(x) + \int_0^x \left[\left(\frac{\partial K_1(x,t)}{\partial x}\right)/K_1(x,x)\right] y(t)dt = f_1'(x)/K_1(x,x). \tag{3.3}$$

By setting $K(x,t) = -\left(\frac{\partial K_1(x,t)}{\partial x}\right)/K_1(x,x)$ and $f(x) = f_1'(x)/K_1(x,x)$, Equation (3.3) can be written in the following form

$$y(x) - \int_0^x K(x,t)y(t)dt = f(x), \tag{3.4}$$

which is a LVIE of the second kind in the unknown $y(x)$.

Now we assume the computed solution of (3.4) has the following form

$$y_N(x) = \sum_{i=0}^{N} a_i x^i = a_N^t X_N, \tag{3.5}$$

which is a truncated Taylor series solution of the exact solution $y(x)$ for Equation (3.4), where $X_N^t = (1, x, x^2, \dots, x^N)$ is the standard basis of polynomials of degree N.

One can write the right hand side of the Equation (3.4) in the form

$$f(x) \simeq \sum_{i=0}^{N} f_i x^i = f^t X_N, \tag{3.6}$$

where, $f^t = (f_0, f_1, \dots, f_N)$.

Now, by substituting (2.4), (3.5) and (3.6) into (3.4), we obtain

$$a_N^t X_N - a_N^t \Pi X_N = f^t X_N,$$

or

$$(a_N^t - a_N^t \Pi - f^t) X_N = 0,$$

and since X is a standard basis of polynomials of degree N,

$$a_N^t - a_N^t \Pi = f^t,$$

or

$$a_N^t (I - \Pi) = f^t, \tag{3.7}$$

where I is the identity matrix.

By solving the linear system of Equation (3.7), the vector of unknown coefficients and hence $y_N(x)$ can be found, so that $u_N(x) = G^{-1}(y_N(x))$ is obtained, which is an approximate solution of the Equation (1.1).

4. Existence, Uniqueness and Convergence Analysis

In this section, we state some results about the existence and uniqueness of the solution of the Equation (3.4) and we prove the convergence of the method.

Theorem 4.1. [(Kress (1999, p.36)].

For each right-hand side $\mathrm{f} \in \mathrm{C}[0, \mathrm{b}]$ the LVIE of the second kind (3.4) with continuous kernel K, has a unique solution $\mathrm{y} \in \mathrm{C}[0, \mathrm{b}]$.

Theorem 4.2. [(Linz (1985, p. 67)].

Assume that

(i) $K_1(x,t)$ and $\partial K_1(x,t)/\partial x$ are continuous in $0 \leq t, x \leq b$,
(ii) $K_1(x,x)$ does not vanish anywhere in $0 \leq x \leq b$,
(iii) $f_1(0) = 0$,
(iv) $f_1(x)$ and $f_1^{'}(x)$ are continuous in $0 \leq x \leq b$.

Then, LVIE of the first kind (3.1) has a unique continuous solution. This solution is identical to the continuous solution of the LVIE of the second kind (3.4).

To investigate the convergence, we define the error function as:

$$e_N(x) = y(x) - y_N(x), \tag{4.1}$$

where, $y(x)$ and $y_N(x)$ are the exact and the computed solution of the Equation (3.4), respectively.

Substituting $y_N(x)$ into Equation (3.4) leads to:

$$y_N(x) - \int_0^x K(x,t)y_N(t)dt = f(x) + p_N(x), \tag{4.2}$$

where, $p_N(x)$ is the perturbation term that can be obtained by substituting the computed solution $y_N(x)$ into Equation (3.4), i.e.,

$$p_N(x) = y_N(x) - \int_0^x K(x,t)y_N(t)dt - f(x). \tag{4.3}$$

Now, by subtracting (4.2) from (3.4) and using (4.1), the error function $e_N(x)$ satisfies:

$$e_N(x) - \int_0^x K(x,t)e_N(t)dt = -p_N(x), \tag{4.4}$$

which is similar to Equation (3.4) with a new right hand side.

Theorem 4.3.

Let assumptions of theorem 4.1 hold, i.e., f and K be continuous functions on their domains. Suppose that for some positive M, we have

$$|y^{(N+1)}(x)| \leq M\,, \quad \forall x \in [0,b]. \tag{4.5}$$

Then, $\lim_{N\to\infty} p_N = 0.$

***Proof*:**

Suppose that the solution $y(x)$ and the computed solution $y_N(x)$ of (3.4) are approximated by their Taylor expansions about zero . Then we may write

$$e_N(x) = \sum_{n=N+1}^{\infty} \frac{x^n}{n!} y^{(n)}(0), \tag{4.6}$$

which can be represented as

$$e_N(x) = \frac{x^{N+1}}{(N+1)!} y^{(N+1)}(\xi), \quad \xi \in (0,x), \tag{4.7}$$

for some $\xi \in (0,x)$ by Taylor's theorem.

Replacing $e_N(x)$ by (4.7) into (4.4) gives

$$-p_N(x) = \frac{x^{N+1}}{(N+1)!} y^{(N+1)}(\xi) - \int_0^x K(x,t) \frac{t^{N+1}}{(N+1)!} y^{(N+1)}(\xi)dt. \tag{4.8}$$

Therefore, we have

$$|p_N(x)| \le |y^{(N+1)}(\xi)| \frac{x^{N+1}}{(N+1)!} + \int_0^x |K(x,t)||y^{(N+1)}(\xi)| \frac{t^{N+1}}{(N+1)!} dt.$$

Since $K(x,t)$ is continuous on $[0, b]$, then there exists some positive real number R such that $|K(x,t)| \le R$ for all $x, t \in [0, b]$. Therefore, we have

$$|p_N(x)| \le M \frac{b^{N+1}}{(N+1)!} + RM \int_0^x \frac{t^{N+1}}{(N+1)!} dt$$

$$\le M \frac{b^{N+1}}{(N+1)!} + RM \frac{b^{N+2}}{(N+2)!} = \frac{1}{(N+1)!}\left(1 + \frac{Rb}{N+2}\right) M b^{N+1}$$

thus, the proof is complete.

Theorem 4.4.

Under the assumptions of Theorem 4.3, we have $\lim_{N\to\infty} e_N = 0$.

***Proof*:**

Let the integral operator T is given by

$$(Ty)(x) = \int_0^x K(x,t) y(t) dt,$$

then the Equation (4.4) can be rewritten as

$$(I - T)e_N = -p_N.$$

Under the assumption, $\lim_{N\to\infty} p_N(x) = 0$ and according to theorem 4.1, (I-T) is invertible. Hence, $\lim_{N\to\infty} e_N = 0$.

We conclude this section by following Remark [Hosseini and Shahmorad (2002)].

Remark 4.5.

If the solution $y(x)$ of the Equation (3.4) is a polynomial of degree m, then any Tau method approximate solution of degree ≥ m will detect it exactly. In this case we say that the Tau method is exact of degree m.

5. Numerical Examples

Here we give some examples to show the simplicity and accuracy of the presented method. In the following examples, we approximate nonpolynomial parts of functions $f(x)$ and $K(x,t)$ by Taylor polynomials.

We made use of the Maple 13 package to perform all computations.

Example 5.1. Consider the following Volterra equation of the first kind [Babolian and Masouri (2008)]

$$\int_0^x cos(x-t)u''(t)dt = 2sinx, \quad x \in [0,1] \tag{5.1}$$

with initial conditions $u(0) = 0$ and $u'(0) = 0$, whose exact solution is $u(x) = x^2$.

To solve this example, as mentioned before, first we convert Equation (5.1) to LVIE of the second kind. To do this, we set $y(t) = u''(t)$ so the integral Equation (5.1) can be written as

$$\int_0^x cos(x-t)y(t)dt = 2sinx, \quad x \in [0,1]. \tag{5.2}$$

Taking the derivative with respect to x on both sides of the above equation, leads to

$$y(x) - \int_0^x sin(x-t)y(t)dt = 2cos(x), \quad x \in [0,1], \tag{5.3}$$

which is a LVIE of the second kind in the unknown $y(x)$.

To solve Equation (5.3), first we expand $\sin(x - t)$ and $\cos(x)$ in Taylor series on $x_0 = 0$ and $t_0 = 0$. By solving LVIE of the second kind (5.3) using the Tau method, the vector of unknown coefficients and hence $y_N(x)$ can be found. Therefore, using initial conditions and two times integration of $u_N''(x)$, concludes the exact solution.

Comparison of the proposed method and the direct method in [Babolian and Masouri (2008)] shows that with assumption $h = b/N$ and $x_i = ih$ where $i = 0,1, \ldots, N$ for $N = 8$, the proposed method gives the solution $y_N(x) = u_N''(x) = 2$ with the computed error $(1/N \sum_{i=0}^{N} e_N^2(x_i))^{1/2} = 0$ but the direct method in [Babolian and Masouri (2008)] gives the solution $y_N(x) = u_N''(x) = 2$ with the computed error $(1/N \sum_{i=0}^{N} e_N^2(x_i))^{1/2} \simeq 1.7E-14$. Therefore, the proposed method is very powerful in comparison the numerical results of [Babolian and Masouri (2008)].

Example 5.2. Consider the NVIE of the first kind [Babolian and Salimi (2008)]

$$\int_0^x e^{(x-t)}u^2(t)dt = e^{2x} - e^x, \quad x \in [0,1], \tag{5.4}$$

whose exact solution is $u(x) = e^x$.

Similar to Example 5.1, we convert NVIE of the first kind (5.4) to LVIE of the second kind. For this purpose, we set $y(t) = u^2(t)$. So the integral Equation (5.4) convert to

$$\int_0^x e^{(x-t)}y(t)dt = e^{2x} - e^x, \quad x \in [0,1]. \tag{5.5}$$

Taking derivative with respect to x on both sides of the above equation, leads to

$$y(x) + \int_0^x e^{(x-t)} y(t)dt = 2e^{2x} - e^x, \quad x \in [0,1], \tag{5.6}$$

which is a LVIE of the second kind in the unknown $y(x)$.

To solve Equation (5.6), first we expand $e^{(x-t)}$, e^{2x} and e^x in Taylor series on $x_0 = 0$ and $t_0 = 0$. By solving LVIE of the second kind (5.6) using the Tau method, the vector of unknown coefficients and hence $y_N(x)$ can be found, therefore $u_N(x) = \sqrt{y_N(x)}$.

Computational results in Table 1 show that high accuracy is obtained for $N = 16$ in comparison to the absolute error at the points $x_i = i/N$ where $i = 0,1,\dots,N$ for $N = 16$, in [Babolian and Salimi (2008)].

Example 5.3. Consider the third example as

$$\int_0^x e^{x-t} ln\big(u(t)\big)dt = e^x - x - 1, \quad x \in [0,1], \tag{5.7}$$

which has the exact solution $u(x) = e^x$.

Similar to previous examples, we convert NVIE of the first kind (5.7) to LVIE of the second kind. By setting $y(t) = ln(u(t))$ the integral Equation (5.7) can be written as

$$\int_0^x e^{x-t} y(t)dt = e^x - x - 1, \quad x \in [0,1]. \tag{5.8}$$

Taking derivative with respect to x on both sides of the above equation, leads to

$$y(x) + \int_0^x e^{x-t} y(t)dt = e^x - 1, \quad x \in [0,1], \tag{5.9}$$

which is a LVIE of the second kind in the unknown $y(x)$.

To solve Equation (5.9), applying the proposed method to the problem leads to $y_N(x) = x$ which is the exact solution of (5.9). Note that this confirms the remark 4.5. Therefore $u_N(x) = e^{y_N(x)} = e^x$ which is the exact solution of Equation (5.7), too.

Example 5.4. Consider the integral equation

$$\int_0^x (sin(x-t) + 1)cos(u(t))dt = \frac{xsin(x)}{2} + sin(x), \quad x \in [0,1] \tag{5.10}$$

with the exact solution $u(x) = x$.

As mentioned before, we convert NVIE of the first kind (5.10) to LVIE of the second kind. To do this, we set $y(t) = cos(u(t))$ so the integral Equation (5.10) is converted to

$$\int_0^x (sin(x-t)+1)y(t)dt = \frac{xsin(x)}{2} + sin(x), \quad x \in [0,1]. \tag{5.11}$$

Taking derivative with respect to x on both sides of the above equation, leads to

$$y(x) + \int_0^x cos(x-t)y(t)dt = \frac{1}{2}xcos(x) + \frac{1}{2}sin(x) + cos(x), \quad x \in [0,1], \tag{5.12}$$

which is a LVIE of the second kind in the unknown $y(x)$.

We proceed as in previous examples and obtain the results of Table 2 for the absolute error at some nodes with $N = 8, 16$.

Table 1. Computational results of example 5.2 for different N at some nodes. In [Babolian and Salimi (2008)] this problem was solved just for $N = 16$.

$N = 8$				
x	$exact$	Tau	$error$	
0.00	1.000000	1.000000	0.00	
1/16	1.064494	1.064494	0.97E-14	
3/16	1.206230	1.206230	0.17E-9	
3/8	1.454991	1.454991	0.76E-7	
1/2	1.648721	1.648720	0.92E-6	
5/8	1.868245	1.868239	0.62E-5	
3/4	2.117000	2.116970	0.29E-4	
13/16	2.253534	2.253477	0.57E-4	
15/16	2.553589	2.553400	0.18E-3	
1	2.718281	2.717959	0.32E-3	
$N = 16$				error [Babolian and Salimi(2008)]
0.00	1.000000	1.000000	0.00	---
1/16	1.064494	1.064494	0.59E-30	0.0328
3/16	1.206230	1.206230	0.68E-22	0.0374
3/8	1.454991	1.454991	0.75E-17	0.0454
1/2	1.648721	1.648721	0.90E-15	0.0517
5/8	1.868245	1.868245	0.35E-13	0.0589
3/4	2.117000	2.117000	0.71E-12	0.0671
13/16	2.253534	2.253534	0.26E-11	0.0716
15/16	2.553589	2.553589	0.26E-10	0.0815
1	2.718281	2.718281	0.76E-10	0.0870

Table 2. Computational result of example 5.4 for different N at some nodes

$N = 8$			
x	$exact$	Tau	$error$
0.00	0.000000	0.000000	0.00
0.10	0.100000	0.099999	0.27E-15
0.20	0.200000	0.199999	0.14E-12
0.30	0.300000	0.299999	0.55E-11
0.40	0.400000	0.399999	0.74E-10
0.50	0.500000	0.499999	0.56E-9
0.60	0.600000	0.599999	0.29E-8
0.70	0.700000	0.699999	0.12E-7
0.80	0.800000	0.799999	0.41 E-7
0.90	0.900000	0.899999	0.12E-6
1.00	1.000000	0.999999	0.32E-6
$N = 16$			
0.00	0.000000	0.000000	0.00
0.10	0.100000	0.199999	0.15E-32
0.20	0.200000	0.199999	0.20E-27
0.30	0.300000	0.299999	0.20E-24
0.40	0.400000	0.399999	0.27 E-22
0.50	0.500000	0.499999	0.12E-20
0.60	0.600000	0.599999	0.28E-19
0.70	0.700000	0.699999	0.39E-18
0.80	0.800000	0.799999	0.39E-17
0.90	0.900000	0.899999	0.29E-16
1.00	1.000000	0.999999	0.18E-15

6. Conclusion

In this paper we proposed a simple technique for solving the NVIE of the first kind. In this method, we transformed NVIE of the first kind to LVIE of the second kind. Then we converted LVIE of the second kind to an equivalent linear system of equations by the operational Tau method. Comparison with available literature shows that the proposed method gives results of high accuracy.

Acknowledgement

The authors would like to thank the referees for critical reading of the manuscript and valuable comments that helped the authors to improve the manuscript.

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